Bandwidth Expansion in N-fold Frequency Multiplier: Is it N or $\sqrt{N}$?

Ahmet Alperen Oznam, Chouaieb Farhati, Mohaned Chraiti, Ali Ghrayeb, Hasari Celebi, Fatma Abdelkefi and Korkut Kaan Tokgoz

***Abstract*—Frequency up-conversion is an essential step in wireless communication systems. Meanwhile, frequency multipliers are increasingly becoming an integral part of communication chains operating at millimeter-wave (mmWave) and sub-terahertz (sub-THz) frequencies. They offer a viable alternative to traditional mixers, which are highly constrained by their instability and high phase noise at such high frequencies. Despite the frequency multipliers' advantages and the fact that they are commonly recommended by hardware designers, there is a lack of comprehensive theoretical guidelines detailing their impact on digital communications. One key aspect is their tendency to induce bandwidth expansion, a phenomenon not observed with mixers. When an N-fold frequency multiplier is used, a common practice among hardware designers is to allocate a bandwidth scaled by a multiple of N, which is deemed large enough to keep signals integrity. We provide an analytical framework to quantify the exact scaling factor of the bandwidth expansion. Contrary to common belief, we show that the bandwidth expansion scales as $\sqrt{N}$, using three counterexamples, namely Gaussian, Sinc, and raised cosine pulses. These findings can help conserve radio resources and further decrease the noise effect.**

***Index Terms*—Bandwidth expansion, frequency multiplier, mmWave, sub-THz, and up-conversion.**

I. Introduction

The demand for Gigabit/s wireless links and massive connectivity is rapidly increasing. Seeking solutions, both the wireless industry and academia are continuously exploring millimeter-wave (mmWave) and Sub-Terahertz (Sub-THz) bands, which offer bandwidths in the gigahertz range.

Up-converting the signal to mmWave and Sub-THz bands is deemed to be a challenging task due to hardware limitations. In the sub-6 GHz band, a sequence of mixers is typically used for up-conversion, where the input signal is sequentially multiplied by single-tone waveforms generated from local oscillators. However, at higher frequencies, mixers suffer from conversion losses, high noise, and increased cost. Circuit designers are compelled to rely more on passive components for signal up-conversion and develop novel architectures involving frequency multipliers [1]–[4]. The proposed architecture involves passing an intermediate frequency's modulated signal, centered around low-frequency carrier, through a frequency multiplier, rather than a sequence of mixers. Leveraging electronic components with non-linear behavior, it is possible to generate multiple harmonics from an input signal, which can then be selectively filtered to obtain the desired high-frequency output [1], [3]. The study in [1] demonstrates the feasibility of achieving 16 Gbps transmission at 240 GHz, highlighting the efficacy of frequency multipliers for up-conversion.

A phenomenon known as bandwidth expansion has been reported when frequency multipliers are used [3]–[5]. This effect arises from the inherent non-linearity of frequency multipliers, which can introduce distortions to the data carrier pulse [3]–[5], resulting in an increased output signal bandwidth compared to the input (equivalently, the output signal when only mixers are used). When N-fold frequency multiplier is used, a common practice among the hardware designers is to allocate a multiple of N-fold bandwidth, which is deemed large enough to keep signal integrity [3]–[5]. However, no formal proof or comprehensive empirical study quantifies the bandwidth expansion. The work in [3] presents a proof-of-concept for a mmWave transmitter that involves a frequency multiplier for up-conversion. While bandwidth expansion is evident in multiple figures, the analysis primarily focuses on Adjacent Channel Power Ratio and Error Vector Magnitude, without explicitly quantifying the bandwidth expansion factor. In [4], the authors emphasize bandwidth expansion, estimating it to be on the order of N. However, their estimation relies on visual observations rather than on formal mathematical proofs or precise measurements of the expansion factor.

The phenomenon of bandwidth expansion has been reported in multiple studies on high-frequency transmitter design involving frequency multipliers [1]–[4]. However, no formal framework or comprehensive experimental study has quantified the exact bandwidth expansion. Some assume—without proof—that an N-frequency multiplier leads to an N-fold bandwidth expansion [4]. Given the scarcity of bandwidth and its direct impact on noise variance and, consequently, the signal-to-noise ratio, accurately determining the scaling factor of bandwidth expansion and validating or refuting existing conjectures is of interest. To the best of our knowledge, this paper is the first that develops a closed-form expression for bandwidth expansion, demonstrating through two counterexamples that it can be of the order of $\sqrt{N}$, which contradicts the prevailing belief. Specifically, this work pertains to two types

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Ahmet Alperen Oznam, Chouaieb Farhati, Mohaned Chraiti and Korkut Kaan Tokgoz are with the Faculty of Engineering and Natural Sciences, Sabanci University, Istanbul, Turkey (e-mails: aoznam@sabanciuniv.edu, chouaib.farhati@supcom.tn, mohaned.chraiti@sabanciuniv.edu, and korkut.tokgoz@sabanciuniv.edu).

Ali Ghrayeb is with College of Science and Engineering, Hamad Bin Khalifa University, Doha, Qatar (e-mail: aghrayeb@hbku.edu.qa).

Hasari Celebi is with Department of Computer Engineering, Gebze Technical University, Kocaeli, Turkey (email: hcelebi@gtu.edu.tr).

Fatma Abdelkefi is with Higher School of Communications of Tunis, Tunisia (email: fatma.abdelkefi@supcom.tn).

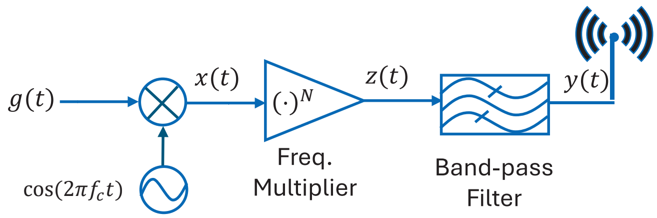

Fig. 1: Communication chain.

of pulses: Gaussian and cardinal sine (Sinc) pulses. We have also conducted a numerical investigation on the raised-cosine pulse, where the results align with our findings.

The paper is organized as follows: Sec. II introduces the system model. In Sec. III, we derive the bandwidth expansion factor. Sec. IV discusses the numerical results. Finally, Sec. V concludes our work.

II. System model

A. Frequency multiplier input-output relation

The transmitter communications chain incorporating a frequency multiplier is depicted in Fig. 1. Initially, the signal is up-converted to a low and intermediate frequency using a mixer. The output then passes through an N-fold frequency multiplier (instead of a sequence of mixers typically used at lower frequencies). Finally, a bandpass filter is applied to suppress unwanted harmonics, keeping the signal around the intended carrier frequency.

The input to the frequency multiplier $x(t)$ is a low-frequency modulated signal, with bandwidth β_1. The input signal over a symbol period can be expressed as:

$$x(t) = g(t)\cos(2\pi f_c t). \tag{1}$$

Here, $g(t)$ denotes the pulse shape or expression, and f_c refers to the carrier frequency before the multiplier. Given the input signal $x(t)$, the response of the nonlinear circuit can be expressed as:

$$z(t) = \sum_{n=1}^{N} a_n x^n(t) = \sum_{n=1}^{N} a_n g^n(t)\cos^n(2\pi f_c t), \tag{2}$$

where $\{a_n, n = 1, 2, \cdots, N\}$ are the coefficients of the nonlinear circuit's response. By expanding and then grouping the terms according to their harmonics, the output signal expression becomes:

$$\begin{aligned} z(t) &= \sum_{n=0}^{N} b_n(t)\cos(2\pi n f_c t) \\ &= \sum_{n=0}^{N-1} b_n(t)\cos(2\pi n f_c t) + c\, g^N(t)\cos(2\pi N f_c t). \end{aligned} \tag{3}$$

The term involving the harmonic $\cos(2\pi N f_c t)$ is a scalar multiple of $g^N(t)$, representing the N-th harmonic contribution to the signal. The remaining terms, denoted by $\{b_n(t), n \neq N\}$, are polynomial functions of $g(t)$, capturing the contributions from other harmonics. Here, c refers to a constant scaling factor. To elaborate, consider the case of a frequency quadruple, i.e., $N = 4$. In this case, the output expression becomes:

$$\begin{aligned} z(t) = &\left(\frac{a_2 g^2(t)}{2} + \frac{3a_4 g^4(t)}{8}\right) \\ &+ b_1(t)\cos(2\pi f_c t) + b_2(t)\cos(2\pi(2f_c)t) \\ &+ b_3(t)\cos(2\pi(3f_c)t) + b_4(t)\cos(2\pi(4f_c)t), \end{aligned} \tag{4}$$

where:

$$b_1(t) = a_1 g(t) + \frac{3a_3}{4}g^3(t), b_2(t) = \frac{a_2}{2}g^2(t) + \frac{a_4}{2}g^4(t),$$
$$b_3(t) = \frac{a_3}{4}g^3(t), \quad b_4(t) = \frac{a_4}{8}g^4(t).$$

The last term in (4) is the dominant contribution from the fourth harmonic, and the sum represents the additional harmonic, each modulated by polynomial functions of $g(t)$.

The output of the nonlinear device is a linear combination of modulated harmonics. Since the coefficients are polynomial functions of $g(t)$, the bandwidth associated with each harmonic is no longer equal to the original bandwidth β_1. This effect is known as *bandwidth expansion*. For the rest of this paper, we denote the bandwidth of the n-th harmonic as β_n. It is important to note that the bandwidth expansion can lead to *intermodulation distortion* when the modulated signals of different harmonics overlap or interfere. To mitigate the inter-modulation, the harmonic frequencies must be spaced sufficiently apart, higher than the expanded signal's bandwdith, to avoid intermodulation. Hence, an exact value, or at least an upper bound estimate, of the bandwidth expansion must be first determined. Mathematically, we need to ensure that the following condition holds:

$$(n+1)f_c - n f_c = f_c \geq \max_{n=1\ldots N} \beta_n = \beta_N \tag{5}$$

For the rest of this paper, we consider the case of practical scenario where the zero intermodulation condition is satisfied. Furthermore, we focus on the highest harmonic modulated signal. In this case, the filtered signal is expressed as:

$$y(t) = c\, g^N(t)\cos(2\pi N f_c t). \tag{6}$$

B. Output of frequency multiplier: the case of the Gaussian and Sinc pulses

For the case of Gaussian and Sinc pulses, the input signal expressions are given respectively by [6]:

$$x_g(t) = Ae^{-\frac{t^2}{2\sigma^2}}\cos(2\pi f_c t), \tag{7}$$

$$x_s(t) = A\frac{sin(\pi t)}{\pi t}cos(2\pi f_c t), \tag{8}$$

where σ is the standard deviation of the Gaussian pulse, with a mean of 0, and A is the amplitude. Accordingly, the output signals after filtering are respectively:

$$y_g(t) = cA^N e^{-\frac{Nt^2}{2\sigma^2}}cos(2\pi N f_c t), \tag{9}$$

$$y_s(t) = cA^N\frac{(sin(\pi t))^N}{(\pi t)^N}cos(2\pi N f_c t), \tag{10}$$

where c is a constant. For simplicity, we will omit the factor $c\,A^N$ in the rest of the paper, as it has no impact on the bandwidth calculations.

III. Bandwidth expansion for Gaussian and Sinc pulses

In this section, we assess the bandwidth expansion resulting from the frequency multiplier, defined as the ratio of the bandwidth after the multiplier to the bandwidth before the multiplier, i.e., $\frac{\beta_N}{\beta_1}$. Whether we consider the signal before or after the frequency multiplier, the bandwidth can theoretically be infinite, as seen in the case of Gaussian pulses; however, this does not accurately reflect the effective bandwidth. In practice, the system can only operate within a limited bandwidth (finite bandwidth), and hence the signal amplitude typically decreases around the carrier frequency. Therefore, the essential bandwidth is typically considered, which reflects the minimum bandwidth containing at least $\gamma_{\rm th}$ percent of the energy of the signal.

A. Scaling factor of the bandwidth of the Gaussian pulse

We start by deriving the essential bandwidth at the input of the frequency multiplier. Consider $x_{\rm g}(t)$ given in (7). Its Fourier transform is given by

$$\begin{aligned} X_{\rm g}(f) &= \sigma\sqrt{2\pi} e^{-2\pi^2\sigma^2 f^2} * \frac{\delta\left(f-f_{\rm c}\right)+\delta\left(f+f_{\rm c}\right)}{2} \\ &= \sigma\sqrt{\pi/2}\left[e^{-2\pi^2\sigma^2(f-f_{\rm c})^2}+e^{-2\pi^2\sigma^2(f+f_{\rm c})^2}\right], \end{aligned} \tag{11}$$

where $(*)$ denotes the convolution operator. Accordingly, the total energy of the signal is:

$$E_1 = \int_{-\infty}^{+\infty} |X_{\rm g}(f)|^2 df = \pi\sigma^2 \int_{-\infty}^{+\infty} e^{-4\pi^2\sigma^2 f^2} df. \tag{12}$$

Given that

$$\int_{-\infty}^{+\infty} e^{-af^2} df = \sqrt{\frac{\pi}{a}}, \tag{13}$$

the total energy is:

$$E_1 = \frac{1}{2}\sigma\sqrt{\pi}. \tag{14}$$

The essential bandwidth is defined as the frequency range that contains γ_{th} of the total energy. Therefore, the essential bandwidth can be obtained by solving the following equation:

$$\begin{aligned} \gamma_{\rm th} E_1 &= \sigma^2 \frac{\pi}{2}\left[\int_{f_{\rm c}-\frac{B_{\rm g}^1}{2}}^{f_{\rm c}+\frac{B_{\rm g}^1}{2}} e^{-4\pi^2\sigma^2(f-f_{\rm c})^2} df \right. \\ &\quad \left. + \int_{-f_{\rm c}-\frac{B_{\rm g}^1}{2}}^{-f_{\rm c}+\frac{B_{\rm g}^1}{2}} e^{-4\pi^2\sigma^2(f+f_{\rm c})^2}\right] df \\ &= \sigma^2\pi \int_{-\frac{B_{\rm g}^1}{2}}^{+\frac{B_{\rm g}^1}{2}} e^{-4\pi^2\sigma^2 f^2} df = \frac{1}{2}\sigma\sqrt{\pi}\,\mathrm{erf}\left(\pi\sigma\beta_1^{\rm g}\right), \end{aligned} \tag{15}$$

where erf(.) is the error function. Solving the equality, we obtain:

$$\beta_1^{\rm g} = \mathrm{erf}^{-1}\left(\frac{\gamma_{\rm th}E_1}{\frac{1}{2}\sigma\sqrt{\pi}}\right) \overset{(a)}{=} \frac{\mathrm{erf}^{-1}(\gamma_{\rm th})}{\sigma\pi}. \tag{16}$$

The operator $(\cdot)^{-1}$ is used to denote the inverse of a function $(\cdot)$. The equality in (a) is obtained by incorporating the expression of E_1 in (14).

In the next, we derive the expression for the essential bandwidth of the output signal $y_{\rm g}(t)$, given by (10), starting with the derivation of the total energy and subsequently determining the bandwidth that contains at least $\gamma_{\rm th}$ percent of the total energy. We have

$$\begin{aligned} Y_{\rm g}(f) &= \sigma\sqrt{\frac{2\pi}{N}} e^{-2\pi^2\sigma^2\frac{f^2}{N}} * \frac{\delta\left(f-Nf_{\rm c}\right)+\delta\left(f+Nf_{\rm c}\right)}{2} \\ &= \sigma\sqrt{\frac{\pi}{2N}}\left[e^{-2\pi^2\sigma^2\frac{(f+Nf_{\rm c})^2}{N}}+e^{-2\pi^2\sigma^2\frac{(f-Nf_{\rm c})^2}{N}}\right]. \end{aligned} \tag{17}$$

The total energy expression hence is equal to:

$$\begin{aligned} E_N &= \int_{-\infty}^{+\infty} |Y_{\rm g}(f)|^2 df \\ &= \frac{\pi\sigma^2}{N}\int_{-\infty}^{+\infty} e^{-\frac{4\pi^2\sigma^2 f^2}{N}} df = \frac{\sigma}{2}\sqrt{\frac{\pi}{N}}. \end{aligned} \tag{18}$$

The essential bandwidth of the output signal can be obtained by solving the following equation:

$$\begin{aligned} \gamma_{\rm th} E_N &= \sigma^2 \frac{\pi}{2N}\left[\int_{f_{\rm c}-\frac{\beta_N^{\rm g}}{2}}^{f_{\rm c}+\frac{\beta_N^{\rm g}}{2}} e^{-\frac{4\pi^2\sigma^2(f-f_{\rm c})^2}{N}} df \right. \\ &\quad \left. + \int_{-f_{\rm c}-\frac{\beta_N^{\rm g}}{2}}^{-f_{\rm c}+\frac{\beta_N^{\rm g}}{2}} e^{-\frac{4\pi^2\sigma^2(f+f_{\rm c})^2}{N}}\right] df \\ &= \sigma^2 \frac{\pi}{2N}\int_{-\frac{\beta_N^{\rm g}}{2}}^{+\frac{\beta_N^{\rm g}}{2}} e^{-\frac{4\pi^2\sigma^2 f^2}{N}} df = \frac{\sigma}{2}\sqrt{\frac{\pi}{N}}\,\mathrm{erf}\left(\frac{\pi\sigma\beta_N^{\rm g}}{\sqrt{N}}\right). \end{aligned} \tag{19}$$

Solving the equality, we obtain:

$$\beta_N^{\rm g} = \mathrm{erf}^{-1}\left(\frac{\gamma_{\rm th}E_N}{\frac{\sigma}{2}\sqrt{\frac{\pi}{N}}}\right) \overset{(b)}{=} \frac{\mathrm{erf}^{-1}(\gamma_{\rm th})}{\sigma\pi}\sqrt{N}. \tag{20}$$

The equality in (b) is obtained by incorporating the expression of E_N in (32).

Using the results in (16) and (20), and the definition of the bandwidth expansion as the ratio of the essential bandwidths at the input and output of the frequency multiplier, we can express the bandwidth expansion as:

$$\frac{\beta_N^{\rm g}}{\beta_1^{\rm g}} = \frac{\frac{\mathrm{erf}^{-1}(\gamma_{\rm th})}{\sigma\pi}\sqrt{N}}{\frac{\mathrm{erf}^{-1}(\gamma_{\rm th})}{\sigma\pi}} = \sqrt{N}. \tag{21}$$

Accordingly, the bandwidth expansion scales with a factor of $\sqrt{N}$.

B. Sinc pulse

The expression of Sinc pulse with period T_c is given by

$$\operatorname{sinc}\left(\frac{t}{T_c}\right) = \frac{\sin(\frac{\pi t}{T_c})}{\frac{\pi t}{T_c}}. \tag{22}$$

Its Fourier transform gives a rectangular function with a width equal to $\frac{1}{T_c}$:

$$T_{\mathrm{c}}\mathrm{rect}_{\frac{1}{T_{\mathrm{c}}}}(f) = \begin{cases} T_{\mathrm{c}}, & \text{if} \quad |f| \leq \dfrac{1}{2T_c} \\ 0, & \text{else where.} \end{cases} \tag{23}$$

Accordingly, the expression of the input signal in the frequency domain can be written as:

$$\begin{aligned} X_s(f) &= T_{\mathrm{c}}\mathrm{rect}_{\frac{1}{T_{\mathrm{c}}}}(f) * \frac{\delta(f-f_c)+\delta(f+f_c)}{2} \\ &= T_{\mathrm{c}}\frac{\mathrm{rect}_{\frac{1}{T_{\mathrm{c}}}}(f-f_{\mathrm{c}}) + \mathrm{rect}_{\frac{1}{T_{\mathrm{c}}}}(f+f_{\mathrm{c}})}{2}. \end{aligned} \tag{24}$$

Consequently, the total energy of the input signal is :

$$\begin{aligned} E_1 &= \int_{-\infty}^{+\infty} |X_{\mathrm{s}}(f)|^2 df \\ &= \frac{T_{\mathrm{c}}^2}{4}\int_{-\infty}^{+\infty}\left[\mathrm{rect}_{\frac{1}{T_{\mathrm{c}}}}(f-f_c) + \mathrm{rect}_{\frac{1}{T_{\mathrm{c}}}}(f+f_c)\right] df \\ &= \frac{T_{\mathrm{c}}^2}{4}\left(\int_{f_c-\frac{1}{2T_c}}^{f_c+\frac{1}{2T_c}} 1df + \int_{-f_c-\frac{1}{2T_c}}^{-f_c+\frac{1}{2T_c}} 1df\right) = \frac{T_{\mathrm{c}}}{2}. \end{aligned} \tag{25}$$

The essential bandwidth is defined as the frequency range that contains γ_{th} of the total energy. Therefore, the essential bandwidth can be obtained by solving the following equation:

$$\begin{aligned} \gamma_{\mathrm{th}}E_1 = \frac{T_{\mathrm{c}}^2}{4}&\left(\int_{f_c-\frac{\beta_1^{\mathrm{s}}}{2}}^{f_c+\frac{\beta_1^{\mathrm{s}}}{2}} \mathrm{rect}_{\frac{1}{T_{\mathrm{c}}}}(f-f_c)df\right. \\ &\left. + \int_{-f_c-\frac{\beta_1^{\mathrm{s}}}{2}}^{-f_c+\frac{\beta_1^{\mathrm{s}}}{2}} \mathrm{rect}_{\frac{1}{T_{\mathrm{c}}}}(f+f_c)df\right) \\ = \frac{T_{\mathrm{c}}^2}{2}&\beta_1^{\mathrm{s}}. \end{aligned} \tag{26}$$

The above gives:

$$\beta_1^{\mathrm{s}} = \gamma_{\mathrm{th}}\frac{1}{T_c}. \tag{27}$$

We next derive an approximation of the bandwidth of the output signal of the frequency multiplier. We have

$$Y_{\mathrm{s}}(f) = \mathrm{rect}^{*N}_{\frac{1}{T_{\mathrm{c}}}}(f) * \frac{\delta(f-Nf_{\mathrm{c}}) + \delta(f+Nf_{\mathrm{c}})}{2}. \tag{28}$$

Here $\mathrm{rect}^{*N}(f)_{\frac{1}{T_c}}$ denotes the N-times convolution of the rectangular function. The N-fold convolution of rectangular functions yields a piecewise polynomial function of degree $N-1$. It is common knowledge that when N increases, the exact expression for the latter convolution becomes increasingly complex [7]. To generalize the result for any order N, we employ an approximation. Notably, as N increases, the convolution increasingly resembles a bell-shaped curve, and hence can be approximated with a Gaussian function. By approximating each pairwise convolution of rectangular functions with a Gaussian [7], we can closely estimate the bandwidth expansion for any order N (see Sec. IV).

Consider the case of $N = 2K$, being an even number.[1] The frequency domain N-times convolutions will be grouped into k pairs of convoluted functions, i.e., k pairs of triangular functions. Each pair of convolution is approximated with a Gaussian with zero mean and variance σ^2 [7]. While both the triangular function and its Gaussian approximation have their amplitudes set equal, we determine the variance by aligning the intersection point of their slopes. Specifically, this intersection is set at the midpoint of the triangular function's slope on each side [7]. Accordingly, the variance can be obtained through solving the following equation:

$$T_{\mathrm{c}}e^{-\frac{(1/2T_{\mathrm{c}})^2}{2\sigma^2}} = \frac{T_{\mathrm{c}}}{2}, \tag{29}$$

which gives

$$\sigma = \frac{1}{2T_{\mathrm{c}}}\frac{1}{\sqrt{2\log(2)}}. \tag{30}$$

Armed with the above results, the $N = 2K$ convolutions of the rectangular function can be approximated by the K-time convolution of Gaussian functions. Incorporating the results in (17), the output signal $Y_s(f)$ can be written as follows.

$$\begin{aligned} Y_{\mathrm{s}}(f) &\simeq T_{\mathrm{c}}^N e^{-\frac{f^2}{2\sigma^2 K}} * \frac{\delta(f-Nf_{\mathrm{c}}) + \delta(f+Nf_{\mathrm{c}})}{2} \\ &= \frac{T_{\mathrm{c}}^N}{2}\left[e^{-\frac{(f+Nf_{\mathrm{c}})^2}{2\sigma^2 K}} + e^{-\frac{(f-Nf_{\mathrm{c}})^2}{2\sigma^2 K}}\right]. \end{aligned} \tag{31}$$

Following the same steps as in (32), the total energy can be expressed as

$$E_N = \frac{T_{\mathrm{c}}^{2K}}{2}\sqrt{\pi\sigma^2 K}. \tag{32}$$

The effective bandwidth can be identified by solving the following expression:

$$\begin{aligned} \gamma_{\mathrm{th}}E_N = \frac{T_{\mathrm{c}}^{2K}}{4}&\left[\int_{f_{\mathrm{c}}-\frac{\beta_N^{\mathrm{s}}}{2}}^{f_{\mathrm{c}}+\frac{\beta_N^{\mathrm{s}}}{2}} e^{-\frac{(f-f_{\mathrm{c}})^2}{\sigma^2 K}} df\right. \\ &\left. + \int_{-f_{\mathrm{c}}-\frac{\beta_N^{\mathrm{s}}}{2}}^{-f_{\mathrm{c}}+\frac{\beta_N^{\mathrm{s}}}{2}} e^{-\frac{(f+f_{\mathrm{c}})^2}{\sigma^2 K}}\right] df \\ = \frac{T_{\mathrm{c}}^{2K}}{2}&\sigma\sqrt{\pi K}\,\mathrm{erf}\left(\frac{\beta_N^{\mathrm{s}}}{2\sqrt{\sigma^2 K}}\right). \end{aligned} \tag{33}$$

Incorporating the results in (32), we obtain:

$$\begin{aligned} \beta_N^{\mathrm{s}} &= 2\sigma\,\mathrm{erf}^{-1}(\gamma_{\mathrm{th}})\sqrt{K} \\ &= \frac{1}{2T_{\mathrm{c}}}\frac{2\mathrm{erf}^{-1}(\gamma_{\mathrm{th}})}{\sqrt{2\log(2)}}\sqrt{K} = \frac{\mathrm{erf}^{-1}(\gamma_{th})}{T_{\mathrm{c}}\sqrt{2\log(2)}}\sqrt{K}. \end{aligned} \tag{34}$$

Accordingly, the bandwidth expansion expression is as follows:

$$\frac{\beta_N^{\mathrm{s}}}{\beta_1^{\mathrm{s}}} = \frac{\mathrm{erf}^{-1}(\gamma_{\mathrm{th}})}{\gamma_{\mathrm{th}}\sqrt{2\log(2)}}\sqrt{\frac{N}{2}}. \tag{35}$$

[1] When $N = 2k+1$ is an odd number, the bandwidth expansion can be bounded by considering frequency multipliers of different orders: a frequency multiplier of order $2k+2$ serves as an upper bound, while a multiplier of order $2k$ provides a lower bound.

Thus, the bandwidth expansion is on the order of $\sqrt{\frac{N}{2}}$.

IV. Simulations results

We validate the derived analytical expressions, through a comparison with the numerical results. The analysis is conducted for the both Gaussian and Sinc pulse cases. In the simulation, we assume an initial carrier frequency $f_c = 2.5\,\text{GHz}$ and an input signal with an essential bandwidth of 20 MHz. The output signal is obtained following the application of a frequency multiplier and a bandpass filter with a bandwidth of 50 MHz. We consider the essential bandwidth with $\gamma_{th} = 0.9$.

Table I depicts the bandwidth expansion values for the Gaussian Pulse obtained from simulations, compared with those derived from experimental results. The findings indicate a match between the theoretical values obtained in equation (21) and the simulation results related to bandwidth expansion.

TABLE I: Bandwidth expansion (BW-E) for the Gaussian pulse.

Order N	BW-E	Numerical value	Theoretical value
2	$\beta_2^{\mathrm{g}}/\beta_1^{\mathrm{g}}$	$\frac{1.4109}{0.9966} = 1.4157$	$\sqrt{2} \simeq 1.4142$
3	$\beta_3^{\mathrm{g}}/\beta_1^{\mathrm{g}}$	$\frac{1.7287}{0.9966} = 1.7345$	$\sqrt{3} \simeq 1.7321$
4	$\beta_4^{\mathrm{g}}/\beta_1^{\mathrm{g}}$	$\frac{1.9967}{0.9966} = 2.0035$	$\sqrt{4} = 2$

Unlike the case of the Gaussian pulse, the theoretical expression for bandwidth expansion in (35) is an approximation. Table III presents the bandwidth expansion values for the Sinc pulse obtained from simulations alongside those derived from experimental results. The findings reveal a minor discrepancy, within one significant digit, between the theoretical and simulation results across different values of the frequency multiplier order N.

TABLE II: Bandwidth expansion for Sinc pulse.

Order N	BW-E	Numerical value	Theoretical value
2	$\beta_2^{\mathrm{s}}/\beta_1^{\mathrm{s}}$	$\frac{1.068}{0.8955} = 1.1926$	1.0974
4	$\beta_4^{\mathrm{s}}/\beta_1^{\mathrm{s}}$	$\frac{1.3980}{0.8955} = 1.5611$	1.5519
6	$\beta_6^{\mathrm{s}}/\beta_1^{\mathrm{s}}$	$\frac{1.6871}{0.8955} = 1.8839$	1.9014

For $N = 2$ (with $K = 1$), the theoretical results suggest a bandwidth expansion of $1.0974\sqrt{1} = 1.0974$, which differs from the simulation result by approximately 8%. For $N = 4$, the derivations suggest a bandwidth expansion with a factor $1.0974\sqrt{2} = 1.5519$. In comparison, with the simulation results, there is a discrepancy of less than 1%. Similarly, for $N = 6$, the theory and simulation results match with a gap of 1%. Collectively, the results confirm that the bandwidth expansion scales with a factor of $\sqrt{N/2}$.

TABLE III: Bandwidth expansion for raised-cosine pulse

Order N	BW-E	Numerical value	$\sqrt{N}$
2	$\beta_2^{\mathrm{r}}/\beta_1^{\mathrm{r}}$	1.38	1.41
3	$\beta_3^{\mathrm{r}}/\beta_1^{\mathrm{r}}$	1.69	1.73
4	$\beta_4^{\mathrm{r}}/\beta_1^{\mathrm{r}}$	1.96	2
5	$\beta_5^{\mathrm{r}}/\beta_1^{\mathrm{r}}$	2.2	2.23

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In Tab. III, we provide numerical results for bandwidth expansion when a raised-cosine pulse is used, with a roll-off factor of one. The results clearly indicate that the scaling factor for the bandwidth is significantly smaller than N; rather, it scales with a factor on the order of $\sqrt{N}$, further supporting our findings and refuting the prevailing conjecture.

Bandwidth is a scarce resource that must be allocated efficiently to preserve signal integrity and prevent spectral truncation, inter-symbol interference, and waveform distortion. Overestimating bandwidth, if left unaddressed, not only does it lead to inefficient spectrum utilization (limiting the wireless system capacity) but also degrades the signal-to-noise ratio. In fact, the noise variance is proportional to the product of bandwidth and noise power spectral density. Overestimating the bandwidth unnecessarily incorporates additional noise components, increasing overall noise power and further degrading the signal-to-noise ratio. By demonstrating that the bandwidth expansion follows the order of $\sqrt{N}$ rather than N, it becomes possible to conserve $(N - \sqrt{N})\beta_1$ spectral resources while simultaneously reducing the noise power by a factor of $N/\sqrt{N}$. Besides, such a work will provide insight on the matched filter design at the receiver, which serves as a low-pass filter. Overall, this finding supports the feasibility of high-frequency communication, as the impact of bandwidth expansion is less severe than previously assumed.

V. Conclusion

In this paper, we demonstrated that the bandwidth expansion in frequency multipliers scales as $\sqrt{N}$, contrary to the common belief that it is on the order of N. This finding directly impacts spectral efficiency and signal-to-noise ratio improvements. While frequency multipliers offer power and cost advantages, the $\sqrt{N}$-order bandwidth expansion remains significant. This raises questions about their spectral efficiency compared to mixers combined with low-rate channel coding and signaling overhead for error correction. Future work will focus on real-world experiments to further investigate bandwidth expansion and the trade-off between performance and spectral efficiency.

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