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GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

ON CHARACTER DEGREES OF FINITE GROUPS AND SOME ASSOCIATED GRAPHS

NOUR ALNAJJARINE

A THESIS SUBMITTED FOR THE DEGREE OF

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FEN BİLİMLERİ ENSTİTÜSÜ

SONLU GRUPLARIN KARAKTER DERECELERİ VE BAZI İLİŞKİLİ ÇİZGELER

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DR. ÖĞR. ÜYESİ ROGHAYEH HAFEZIEH

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GTÜ Fen Bilimleri Enstitüsü Yönetim Kurulu'nun 30/05/2018 tarih ve 2018/29 sayılı kararıyla oluşturulan jüri tarafından 13/06/2018 tarihinde tez savunma sınavı yapılan Nour ALNAJJARİNE'nin tez çalışması Matematik Anabilim Dalında YÜKSEK LİSANS tezi olarak kabul edilmiştir.

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SUMMARY

Graphs associated with group structures have been actively investigated and many fascinating results have been obtained. Let G be a finite group and $Irr(G)$ be the set of irreducible characters of G. The set $cd(G) = \{\chi(1): \chi \in Irr(G)\}$ is called the character degree of G. We associate with $cd(G)$ three undirected graphs which are the prime vertex graph $\Delta(G)$, the common divisor degree graph $\Gamma(G)$ and the bipartite divisor graph $B(G)$. These graphs are strongly related and share many combinatorial properties. In this thesis, we discuss the strong interplay between the structure of a finite group G and the graphs associated with its character degree set. In particular, we claim that there does not exist any nonsolvable group G whose prime vertex graph is a path of length three or a cycle of length four. Furthermore, we give a classification of all graphs with four vertices that can arise as the common divisor degree graph $\Gamma(G)$ when G is nonsolvable. Moreover, we consider finite groups whose prime vertex graphs have no triangles and obtain a classification of finite graphs with five vertices and no triangles that can occur as the prime graph for a finite group G. We follow in this thesis paper [Hafezieh, 2017] and we focus on studying some group theoretical properties of G when $B(G)$ is a path, a union of paths or a cycle.

Keywords: Prime Vertex Graph, Common Divisor Degree Graph, Bipartite Divisor Graph, Path, Cycle.

ÖZET

Grup yapıları ile ilişkili çizgeler aktif olarak araştırılmış ve pek çok ilginç sonuç elde edilmiştir. G sonlu bir grup ve $Irr(G)$, G'nin indirgenemez karakterlerinin kümesi olsun. $cd(G) = \{\chi(1) : \chi \in Irr(G)\}$ kümesi, G'nin karakter derecesi olarak adlandırılır. Biz, asal köşe çizgesi, ortak bölen derecesi çizgesi ve iki-parçalı bölen çizgesi şeklindeki üç yönsüz çizgeyi $cd(G)$ ile ilişkilendiririz. Bu çizgeler güçlü bir şekilde ilişkilidir ve birçok birleştirici özelliği paylaşır. Bu tezde, bir sonlu grup olan G'nin yapısı ve onun karakter derece kümesi ile ilişkili çizgeler arasındaki etkileşimi tartışacağız. Özellikle, asal köşe çizgesi dört uzunluklu bir döngü ya da üç uzunluklu bir patika olan çözülemez herhangi bir G grubun var olmadığını iddia ediyoruz. Ayrıca, G çözülemez olduğu zaman, ortak bölen derece çizgesi $\Gamma(G)$ olarak ortaya çıkabilen dört köşe ile tüm çizgelerin bir sınıflandırmasını veririz. Dahası, üçgen içermeyen asal köşe çizgelerin sonlu gruplarını ele alıp ve beş köşeli ve sonlu bir G grubu için asal köşe çizgesi olarak ortaya çıkan üçgen içermeyen sonlu çizgelerin bir sınıflandırmasını elde ediyoruz. Bu tez yazısında [Hafezieh, 2017]'u takip ediyoruz ve $B(G)$ bir patika, patikaların birleşimi ya da bir döngü olduğu zaman, G'nin bazı kuramsal grup özeliklerini çalışmaya odaklanıyoruz.

Anahtar Kelimeler: Asal Köşe Çizgesi, Ortak Bölen Derecesi Çizgesi, İki-Partçalı Bölen Çizgesi, Patika, Döngü.

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LIST of ABBREVIATIONS and ACRONYMS

Abbreviations and Acronyms		**Explanations**
(m, n)	:	The greatest common divisor of the integers m and n.
$\|G\|$	:	The order of the group G.
$\|x\|$	:	The order of an element x in a group G.
$\|G: N\|$	:	The index of the normal subgroup N in the group G.
m_p	:	The largest prime power p^i that divides m.
$Irr(G)$	:	The set of irreducible characters of G.
$cd(G)$	:	The character degree set of G, Definition 2.47.
X^*	:	$X \setminus \{1\}$
$\rho(G)$	:	The set of prime divisors of elements of $cd(G)^*$.
$Z(G)$	:	The center of G.
$F(G)$	:	The Fitting subgroup of G.
$\pi(G)$	:	The set of prime divisors of $\|G\|$.
$\pi(m)$	:	The set of prime divisors of the integer m.
$\Delta(G)$	:	The prime vertex graph = The prime degree graph = The prime graph, Definition 2.63.
$\Gamma(G)$	:	The common divisor degree graph, Definition 2.63.
$B(G)$	:	The bipartite divisor graph, Definition 2.63.
P_n	:	A path of length n, Definition 2.52.
C_n	:	A cycle of length n, Definition 2.52.
$H \times K$	:	The direct product of H and K, Definition 2.19.
$H \rtimes K$	:	The semidirect product of H and K, Definition 2.19.
$C_G(H)$	:	The centralizer of H in G, where $H \subseteq G$.
$Aut(G)$	:	The set of automorphisms of G.
$tr(A)$	:	The trace of a square matrix A.
I_n	:	The identity matrix of order $n \times n$.
$I_G(\theta)$	:	The inertia group of θ in G, Definition 2.42.
$dim_F(V)$	:	The dimension of the vector space V over the field F.
χ_N	:	The restriction character of χ to N, Definition 2.42.

θ^G	:	The induced character of θ in G, Definition 2.41.
$ker\ \chi$	:	The kernel of the character χ, Definition 2.38.
O_χ	:	The determinal order of χ which is equal to $\lvert G: ker\chi \rvert$.
$dl(G)$	:	The derived length of G, Definition 2.22.
$h(G)$	:	The Fitting height of G, Definition 2.26.
$[x, y]$	:	$x^{-1}y^{-1}x\, y$, the commutator of x and y.
$G' = [G, G]$	:	The derived subgroup which is generated by all commutators of G.
π'	:	The set of primes that does not belong to the prime set π.
$a_{\pi'}$	:	${p_{i_1}}^{\alpha_{i_1}} \times \ldots \times {p_{i_k}}^{\alpha_{i_k}}$, where $p_{i_s} \notin \pi$ and ${p_{i_s}}^{\alpha_{i_s}+1} \nmid a$ for all $s \leq k$.

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1. INTRODUCTION

Over the past few decades, there have been many papers that discussed the strong bond between the structure of a finite group G and its character degree set $cd(G)$. For instance, the celebrated Ito-Michler's Theorem states that if a prime p divides no character degree of a finite group G, then G has an abelian normal Sylow p-subgroup. Another interesting result is due to J. Thompson who proved that if a prime p divides all the nontrivial character degrees of a finite group G, then G has a normal p-complement.

One of the key tools for studying the character degree set of a finite group G is by attaching some graph structure on $cd(G)$. Mark Lewis provided in [Lewis, 2008] an overview of various graphs associated with groups. Surprisingly, it was shown that strong combinatorial information about these graphs can lead to structural information about the groups and their representations. Lewis unified many results concerning these graphs by first defining the prime vertex graph $\Delta(X)$ and the common divisor degree graph $\Gamma(X)$ for an arbitrary subset of positive integers X. He showed that for any $X \subseteq \mathbb{N}$, the graphs $\Delta(X)$ and $\Gamma(X)$ share many similar combinatorial properties. For example, they have the same number of connected components and equal diameters. Inspired by Lewis's paper [Lewis, 2008], M. Iranmanesh and C. Praeger introduced in [Iranmanesh and Praeger, 2010] the bipartite divisor graph $B(X)$, for a subset of positive integers X. Moreover, they discussed some basic invariants of the three graphs such as the number of connected components, diameters and girths (the girth of a graph is the length of the shortest cycle in the graph).

Studying group structures by using the arithmetical properties of their character degree sets has been an active area in the recent years. In this field of study, there are three main questions that arise naturally. Which sets of positive integers can occur as $cd(G)$ for some group G, what can be said about the structure of G and what graphs can occur as $\Delta(G)$, $\Gamma(G)$ and $B(G)$ for a finite group G? Where by $\Delta(G)$, $\Gamma(G)$ and $B(G)$ we mean the prime vertex graph, the common divisor degree graph and the bipartite divisor graph associated with $cd(G)$. One of the most extensive result regarding which graphs can occur as $\Delta(G)$ when G is solvable is Pálfy's Theorem (Theorem 18.7 in [Manz and Wolf, 1993]). Pálfy claimed that among any three vertices in $\Delta(G)$, two of them must be adjacent, which eliminated many graphs as possibilities for $\Delta(G)$

when G is solvable. Manz, Staszewski and Willems bounded the number of connected components of $\Delta(G)$ by 3 (Theorem 6.4 in [Lewis, 2008]). In particular, Manz proved that $\Delta(G)$ can have at most two connected components when G is solvable (Corollary 4.2 in [Lewis, 2008]).

In this study, we investigate the relationship between the structure of a finite group G and the graphs associated with its character degree set. We start by giving some preliminaries in group and graph theories. Then we mention some basic definitions and results that will be used throughout the thesis.

In chapter 3, we claim that there does not exist any nonsolvable group G whose $\Delta(G)$ is a path of length three or a cycle of length four. We follow in this chapter paper [Lewis and White, 2013].

In chapter 4, we discuss classifications of all graphs with three vertices that can arise as the common divisor degree graph $\Gamma(G)$ for a finite group G. Furthermore, we classify all graphs with four vertices that can occur as $\Gamma(G)$ when G is nonsolvable. We follow in this chapter paper [LiGuo and GuoHua, 2015].

In chapter 5, we consider finite groups whose prime vertex graphs have no triangles. We claim first that prime graphs of such groups have at most five vertices. Then, we give a classification of finite graphs with five vertices and no triangles that can occur as prime graphs of finite groups. Finally, we claim that the prime graph of any finite group cannot be a cycle or a tree with at least five vertices. We follow in this chapter Tong-Viet's paper [Tong-Viet, 2013].

In chapter 6, we study finite groups whose bipartite divisor graphs are paths. In particular, we see that any group satisfying this property is solvable and the length of its bipartite divisor graph is at most 6. Furthermore, we discuss some group theoretical properties of such groups.

In chapter 7, we investigate nonsolvable groups whose bipartite divisor graphs are union of paths.

In the last chapter, chapter 8, we consider finite groups whose bipartite divisor graphs are cycles. We claim that if $B(G)$ is a cycle, then G is solvable and the length of $B(G)$ is either four or six. In addition, we discuss some group theoretical properties of G when $B(G)$ is a cycle of length four. We follow in chapters 6, 7 and 8 R. Hafezieh's paper [Hafezieh, 2017], which is our main paper.

2. PRELIMINARIES AND BASIC RESULTS

2.1. Preliminary Definitions and Results in Group Theory

Definition 2.1. *A subgroup N of a group G is normal if and only if for every $g \in G$ we have: $gN = Ng$.*

Definition 2.2. *A characteristic subgroup H of a group G is a subgroup that is mapped to itself by each automorphism of G. It is denoted by H char G.*

It is not hard to see that if H *char* G, then $H \trianglelefteq G$.

Remark 2.3. *If G is a finite group of order $p^{\alpha}n$ where p is a prime, $\{\alpha, n\} \subset \mathbb{N}^*$ and $p \nmid n$. Then for each $1 \leq i \leq \alpha$, G has a subgroup of order p^i. In particular, every subgroup of order p^{α} is called a Sylow p-subgroup of G and we denote by $Syl_p(G)$ the set of all Sylow p-subgroups of G.*

It is well-known that if $P \in Syl_p(G)$,then P char G if and only if $P \trianglelefteq G$ if and only if P is the unique Sylow p-subgroup of G.

Definition 2.4. *Let G be a group and $K,\ L \trianglelefteq G$ such that $L \leq K$. Then K/L is a chief factor of G if and only if there exists no $M \trianglelefteq G$ such that $L < M < K$.*

Definition 2.5. *Let G be a nontrivial group and $S \subseteq G$, we define the centralizer of S in G as follows: $C_G(S) = \{x \in G : xs = sx,\ \forall\ s \in S\}$, which is a subgroup of G. If $S \trianglelefteq G$, then $C_G(S) \trianglelefteq G$.*

Definition 2.6. *Let G be a nontrivial group. Then G is simple if and only if the only normal subgroups of G are the trivial group and itself.*

Definition 2.7. *A group G is said to be almost simple, if and only if it verifies one of the following equivalent conditions:*

i) There exists a simple nonabelian subgroup S such that $S \leq G \leq Aut(S)$.

ii) G has a nonabelian normal simple subgroup S such that $C_G(S)$ is trivial.

In this case, we say that G is an almost simple group with socle S.

Definition 2.8. *A maximal subgroup H of a group G is a proper subgroup, such that if there exists $H \leq K \leq G$, then $K = H$ or $K = G$.*

Definition 2.9. *Let G be a nontrivial group and $\{1\} < H \leq G$. We say that H is a minimal normal subgroup of G, if H is normal in G and for every $K \trianglelefteq G$ such that $\{1\} \leq K \leq H$, we have $K = \{1\}$ or $K = H$.*

Definition 2.10. *Let G be a nontrivial group. We say that G is characteristically simple, if and only if G has no proper nontrivial characteristic subgroup.*

Lemma 2.11. *Let G be a finite group. If M is a minimal normal subgroup of G, then M is characteristically simple.*

Proof. If K *char* M, then by the minimality of M we have either $K = \{1\}$ or $K = M$. Thus M is characteristically simple. □

Theorem 2.12. *If G is a characteristically simple finite group, then it is the direct product of a finite number of isomorphic simple groups.*

Proof. Let S be a minimal normal subgroup of G. Let

$$T = \{H \trianglelefteq G : H = S_1 \times S_2 \times ...S_k;\ S_i \cong S;\ 1 \leq i \leq k\}. \tag{2.1}$$

Notice that $T \neq \emptyset$ as $S \in T$. Let N be a maximal normal subgroup in T. Since $N \in T$, we can write $N = S_1 \times ... \times S_r$ where $S_i \cong S$ for every $i \in \{1,..,r\}$. We claim that $N = G$. If $N < G$, then N is not a characteristic subgroup of G as G is characteristically simple. Thus there exists $\phi \in Aut(G)$ such that $\phi(N) \nleq N$. Hence, there exists $1 \leq j \leq r$ such that $\phi(S_j) \nleq N$. Since ϕ is an automorphism of G and S_j is a minimal normal subgroup of G, we conclude that $\phi(S_j)$ is a minimal normal subgroup of G, which implies that $N \cap \phi(S_j) \trianglelefteq G$. Also $N \cap \phi(S_j) < \phi(S)$ $(\phi(S_j) \nleq N)$. Therefore, by the minimality of N we have $N \cap \phi(S_j) = \{1\}$. It follows then that $N\phi(S_j) = N \times \phi(S_j) = S_1 \times ... \times S_r \times \phi(S_j) \trianglelefteq G$. Thus $N\phi(S_j) \in T$, which contradicts the maximality of N.

Hence $G = N = S_1 \times .. \times S_r$. It remains to check that S is a simple group. Let $L \trianglelefteq S$. Since $S_i \cong S$ for every i, we conclude that $L \trianglelefteq S_i$ for every i. Thus $L \trianglelefteq G = S_1 \times ... \times S_r$. But since S is a minimal normal subgroup of G, we deduce that $L = \{1\}$ or $L = S$. □

Corollary 2.13. *A minimal normal subgroup of a finite group is the direct product of isomorphic simple groups.*

Proof. Combine the results of Lemma 2.11 and Theorem 2.12. □

Definition 2.14. *A Hall subgroup H of a group G is defined to be a subgroup whose order is relatively prime with its index.*

Definition 2.15. *Let G be a group, $N \leq G$ and consider a prime p. We say that N is a normal p-complement of G, if and only if:*

i) $N \trianglelefteq G$,

ii) $(|N|, p) = 1$,

iii) $|G : N| = p^{\alpha}$ *for some positive integer* α.

Definition 2.16. *Let G be a finite group, we define $\pi(G) = \{p : p$ is prime and $p \mid |G|\}$.*

Let γ be a set of primes. A group G is called a γ-group if and only if $\pi(G) \subseteq \gamma$. If $H \leq G$, then it is called a γ-subgroup of G if and only if H is a γ-group.

Definition 2.17. *Let G be a group and $\{x, y\} \subseteq G$. The commutator of x and y is defined to be $[x, y] = x^{-1}y^{-1}xy$.*

The subgroup generated by all commutators of G is called the commutator subgroup and it is denoted by G' or $[G, G]$.

Definition 2.18. *Let p be a prime. An abelian p-group G is called elementary abelian p-group if and only if $|x| = p$ for all $x \in G \setminus \{1\}$.*

Definition 2.19. *Let $(G, \diamond)$ and $(H, \circ)$ be two groups. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$.*

i) The direct product of G and H is defined to be the group $G \times H$ with the following operation: $(g_1, h_1) * (g_2, h_2) = (g_1 \diamond g_2, h_1 \circ h_2)$.

ii) Let $\phi : G \longrightarrow Aut(H)$ be a group homomorphism such that $g\phi = \phi(g) := \phi_g$, *where $g \in G$. Recall that $(Aut(H), \circ)$ is also a group. Consider $H \times K$ and define the following operation: $(g_1, h_1) * (g_2, h_2) = (g_1 \diamond g_2, (h_1(g_2\phi)) \circ h_2)$. Then $H \times K$ is a group with respect to this operation. It is called the semidirect product of H and K with respect to ϕ, and it is denoted by $H \rtimes_\phi K$ or simply by $H \rtimes K$.*

Theorem 2.20. *(Recognition Theorem for Direct Products, Theorem 7.6 in [Rose, 2009]):*

Suppose G is a group and $H, K \leq G$ such that:

i) $H, K \trianglelefteq G$ or each element of H commutes with each element of K,

ii) $H \cap K = \{1\}$, and

iii) $G = HK = \{hk : h \in H \text{ and } k \in K\}$.

Then $G = H \times K$.

Theorem 2.21. *(Recognition Theorem for Semidirect Products, Theorem 7.17 in [Rose, 2009]):*

Suppose G is a group and $H, K \leq G$ such that:

i) $H \trianglelefteq G$,

ii) $H \cap K = \{1\}$, and

iii) $G = HK$.

Let ϕ be a homomorphism from K into $Aut(H)$ such that $\phi_k(h) = k^{-1}hk$. Then $G = H \rtimes K$.

Definition 2.22. *Let G be a group,*

i) A subnormal series of G is a finite chain of subgroups of G,

$$\{1\} = G_0 \leq G_1 \leq ... \leq G_r = G \tag{2.2}$$

where $G_{i-1} \trianglelefteq G_i$ for every $1 \leq i \leq r$.

ii) A normal series of G is a finite chain of subgroups of G,

$$\{1\} = G_0 \leq G_1 \leq ... \leq G_r = G \tag{2.3}$$

where $G_i \trianglelefteq G$ for every $1 \leq i \leq r$.

iii) *A factor of a subnormal or a normal series is any quotient of the form :* G_i/G_{i-1} *for some* $i \geq 1$.

iv) *A normal series of G is called a central series if*

$$G_i/G_{i-1} \leq Z(G/G_{i-1}) \tag{2.4}$$

for all $1 \leq i \leq r$.

v) *The derived series of G is a sequence of subgroups of G which is defined as follows:* $G^{(0)} = G$ *and* $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ *for all* $n \geq 1$.

vi) *G is a nilpotent group if and only if G has a central series.*

If G is nilpotent, then the length of the shortest central series of G is called the nilpotency class of G. It is denoted by $n(G)$.

vii) *G is a solvable group if and only if it verifies one of the following equivalent conditions:*

- *G has a subnormal series such that* G_i/G_{i-1} *is abelian for all* $1 \leq i \leq r$.
- *There exists a non-negative integer k such that the* k^{th} *term of the derived series,* $G^{(k)}$, *is trivial.*

If G is solvable, then the length of the shortest subnormal series which verifies the above property or the smallest integer k such that $G^{(k)} = \{1\}$ *(in the derived series) is called the derived length of G. It is denoted by* $dl(G)$.

Remarks 2.23.

- *If G is a nilpotent group then it is solvable, however the converse is not necessarily true.*
- *The direct product of two nilpotent groups is nilpotent.*
- *If G is a finite group, then it is nilpotent if and only if it is the direct product of its Sylow subgroups.*
- *The semidirect product of two solvable groups is solvable.*
- *If G is solvable, then for every* $N \trianglelefteq G$, *N and* G/N *are solvable.*
- *A subgroup H of G is called perfect if and only if* $H' = H$.
- *Let* $H \trianglelefteq G$. *Then* G/H *is abelian if and only if* $G' \leq H$.

Theorem 2.24. *(Burnside's Theorem in [Alperin and Bell, 1991])*

Let p and q be two primes. Let G be a group of order $p^{\alpha}q^{\beta}$. Then G is solvable.

Definition 2.25. *The Fitting subgroup of a group G, $F(G)$, is the subgroup generated by all normal nilpotent subgroups of G. If G is finite, then it is the largest normal nilpotent subgroup of G.*

Definition 2.26. *The upper Fitting series of a finite group G is a subnormal series of G:*

$$F_0(G) = \{1\} \leq F_1(G) = F(G) \leq ... \leq F_i(G) \leq .... \tag{2.5}$$

where for every i, $F_i(G)$ is defined as follows: $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$.

Remark that, there exists r such that $F_r(G) = G$ if and only if G is solvable. In this case, the smallest integer h such that $F_h(G) = G$ is called the Fitting height of G and it is denoted by $h(G)$.

Theorem 2.27. *(Satz 10 in [Huppert, 1957]) If G is a solvable group, then $h(G) \leq dl(G)$.*

Definition 2.28. *Let H be a proper subgroup of a finite group G. Assume that $H \cap H^g = \{1\}$ for every $g \in G \setminus H$. Then H is called a Frobenius complement and the group G which contains H is called a Frobenius group. Recall that $H^g = \{g^{-1}hg : h \in H\}$.*

Theorem 2.29. *(Frobenius Theorem: Theorem 7.2 in [Isaacs, 1976]) Let G be a Frobenius group with complement H. Then there exists $N \trianglelefteq G$ such that $H \cap N = \{1\}$ and $G = HN$.*

The normal subgroup N whose existence is guaranteed by the previous theorem is called the Frobenius kernel of G. Furthermore, by Theorem 7.3 in [Isaacs, 1976], we can see that the kernel N is uniquely determined by the complement H. Also by Theorem 2.21, we can write G as the semidirect product of H and N, that is $G = H \rtimes N$.

Definitions and Properties 2.30. *([Alperin and Bell, 1991]) Let G be any group and X be any set. Let $x \in X$.*

- *The left action of G on X is defined to be the following map:*

$$. : G \times X \longrightarrow X \tag{2.6}$$

where $.(g, x) = g.x$, *such that:*

i) $1.x = x$, *for all* $x \in X$,

ii) $(g.h).x = g.(h.x)$, *for all* $x \in X$ *and* $g, h \in G$.

- *The stabilizer of x in G is a subgroup of G, which is defined as follows:*

$$St_G(x) = \{g \in G : g.x = x\}. \tag{2.7}$$

- *The orbit of x in G is the following subset of X:*

$$Orb_G(x) = \{g.x : g \in G\}. \tag{2.8}$$

- *For any group action we have:*

$$\bigsqcup_{x \in G} Orb_G(x) = G. \tag{2.9}$$

- *Every group G acts on itself via conjugation by the following action:* $g.x = x^g = g^{-1}xg$ *where* $g, x \in G$. *In this case, we have* $St_G(x) = C_G(x)$ *and we substitute the notion of orbits by conjugacy classes which is denoted by* $cl_G(x)$.
- *If* $X = V$ *is a finite dimensional vector space over a field* F. *Then the action of G on V is linear if and only if:*

i) $g.(v + w) = g.v + g.w$ *for all* $g \in G$ *and* $v, w \in V$,

ii) $g.(c.v) = c.(g.v)$ *for all* $g \in G$, $v \in V$ *and* $c \in F$.

Theorem 2.31. *(The Orbit-Stabilizer Theorem: Corollary 5 in Section 3 of [Alperin and Bell, 1991]) Let G acts on X and* $x \in X$. *Then there exists a 1-1 correspondence between* $Orb_G(x)$ *and the set of right cosets of* $St_G(x)$ *in G. Furthermore, if* $|Orb_G(x)| < \infty$, *then* $|G : St_G(x)| = |Orb_G(x)|$.

In the following we define characters of finite groups and some of its properties.

For more information, you can see “Character Theory of Finite Groups” [Isaacs, 1976].

Definition 2.32. *Let G be a finite group and V be a vector space of dimension n over a field F. We define a representation of G in V of degree n to be a homomorphism of groups:*

$$\rho : G \longrightarrow GL(V), \tag{2.10}$$

where $GL(V)$ is the set of all invertible linear transformations of V.

Recall that $GL(V) \cong GL(n, |F|) = \{A \in M_n(F) : |A| \neq 0\}$.

Proposition 2.33. *(Propositions 1 and 6 of Section 12 in [Alperin and Bell, 1991]) There is a 1-1 correspondence between the set of all linear actions of a group G on a finite dimensional F-vector space V and the set of homomorphisms from G to $GL(V)$.*

Definition 2.34. *Let G be a finite group and V be an F-vector space.*

A representation of G in V, ρ, is called irreducible if one of the following equivalent conditions holds:

- *$Orb_G(v)$ is a spanning set for all $v \in V$.*
- *V has no nontrivial invariant subspaces.*

Example 2.35. *Any finite group G can be represented trivially in $\mathbb{C}$ by considering the following homomorphism:*

$$\rho : G \longrightarrow GL(\mathbb{C}) \tag{2.11}$$

such that $\rho(g).z = z$. Or, by considering the linear action of G on $\mathbb{C}$ which is defined as follows: $g.z = 1.z = z$.

Definition 2.36. *Let ρ be a representation of G in $\mathbb{C}$ of degree n. We define the character of G associated with ρ to be the following map:*

$$\chi : G \longrightarrow \mathbb{C} \tag{2.12}$$

where $\chi(g) = tr(\rho(g))$.

Example 2.37.

i) *The principal character of a finite group G is the character associated with the trivial representation of G. In this case we have* $\chi(g) = tr((1)_{1\times 1}) = 1$ *for all* $g \in G$.

ii) *For every representation of G in V,* ρ*, we have* $\chi(1) = \chi(\rho(1)) = tr(I_n) = n = dim_F(V)$.

Definition 2.38.

i) *A character which is associated with an irreducible representation is called an irreducible character.*

ii) *The kernel of* χ*,* $ker\chi$*, is a normal subgroup of G which is defined as follows:*

$$ker\chi = \{g \in G : \chi(g) = \chi(1)\}. \tag{2.13}$$

iii) $Irr(G) = \{\chi : \chi \text{ is an irreducible character of } G\}$.

iv) *If* $\chi(1) = 1$*, then* χ *is called a linear character of G.*

Note that the number of conjugacy classes of G is equal to the number of irreducible characters of G (see Theorem 3 in Section 14 of [Alperin and Bell, 1991]).

Proposition 2.39. *(Lemmas 6 and 13 in Section 15 in [Alperin and Bell, 1991]) If* $N \trianglelefteq G$*, then there is a 1-1 correspondence between the set of irreducible characters of* G/N *and the set of irreducible characters of G whose kernel contain N.*

Definition 2.40. *Let* χ*,* $\psi \in Irr(G)$*. The inner product of* χ *and* ψ *is defined as:*

$$[\chi, \psi] = \frac{1}{|G|} \sum_{g\in G} \chi(g)\overline{\psi(g)}. \tag{2.14}$$

Definition 2.41. *Let* $H \leq G$*. Let* $\psi \in Irr(H)$*. Then* ψ^G *which is defined by:*

$$\psi^G(g) = \frac{1}{|H|} \sum_{x\in G} \psi^{\circ}(xgx^{-1}) \tag{2.15}$$

is called the induced character of ψ on G, where ψ° is defined by: $\psi^\circ(h) = \psi(h)$ if $h \in H$ and $\psi^\circ(y) = 0$ if $y \notin H$.

Definitions and Properties 2.42. *Let $H \leq G$.*

- *If $\chi \in Irr(G)$, then $\chi_H : H \longrightarrow \mathbb{C}$ is a character of H where $\chi_H(h) = \chi(h)$ for all $h \in H$. It is called the restriction character of χ on H.*
- *If $H \trianglelefteq G$, $\theta \in Irr(H)$ and $g \in G$, we define $\theta^g : H \longrightarrow \mathbb{C}$ such that $\theta^g(h) = \theta(ghg^{-1})$. We have $\theta^g \in Irr(H)$ and we say that θ^g is a conjugate of θ in G.*
- *If $H \trianglelefteq G$ and $\theta \in Irr(H)$. Then*

$$I_G(\theta) = \{g \in G : \theta^g = \theta\} \tag{2.16}$$

  is the inertia group of θ in G.
- *If $H \trianglelefteq G$ and $\theta \in Irr(H)$ such that $\theta^g = \theta$ for all $g \in G$. Then θ is G-invariant or we say that θ is invariant under G.*
- *Let $H \trianglelefteq G$, $\theta \in Irr(H)$ and $\chi \in Irr(G)$. If $\chi_H = \theta$, then θ is extendible to G.*

Theorem 2.43. *(Clifford's Theorem: Theorem 6.2 in [Isaacs, 1976]) Let $H \trianglelefteq G$, $\chi \in Irr(G)$ and θ be an irreducible constituent of χ_H. Let $\theta = \theta_1, \theta_2, \ldots, \theta_t$ be the distinct conjugates of θ in G. Then*

$$\chi_H = e \sum_{i=1}^{t} \theta_i, \tag{2.17}$$

where $e = [\chi_H, \theta]$.

It follows from Lemma 6.1 [Isaacs, 1976] that G acts on $Irr(H)$ via $\star : G \times Irr(H) \longrightarrow Irr(H)$ where $\star(g, \theta) = \theta^g$. Thus $I_G(\theta)$ is the stabilizer of θ via this action. As $|G : I_G(\theta)| = |Orb_G(\theta)|$ (Theorem 2.31), we can see that the index t in Clifford's Theorem is exactly $|G : I_G(\theta)|$.

Definition 2.44. *Let ϕ be a character of a finite group G. Let $\chi \in Irr(G)$. Then χ is an irreducible constituent of ϕ if and only if $[\chi, \phi] \neq 0$.*

Theorem 2.45. *(Theorem 2.8 in [Isaacs, 1976]) Every character of a finite group G can* *be expressed uniquely as:*

$$\phi = \sum_{\chi \in Irr(G)} a_\chi \chi, \tag{2.18}$$

where $a_\chi \in \mathbb{N}$ for every χ.

Proposition 2.46. *(Corollaries 2.14 and 2.17 in [Isaacs, 1976]) If χ and ψ are two characters of G. Then $[\chi, \psi] = [\psi, \chi]$. Also, ψ is irreducible if and only if $[\psi, \psi] = 1$. Furthermore, if $\theta \neq \psi$ is an irreducible character of G, then $[\psi, \theta] = 0$.*

Definition 2.47. *Let G be a finite group,*

i) The character degree set, $cd(G)$, is defined to be the set of all irreducible character degrees of G, that is,

$$cd(G) = \{\chi(1) : \chi \in Irr(G)\}. \tag{2.19}$$

ii) We define $\rho(G)$ to be the set of all primes which divide some character degree of G.

Notice that Proposition 2.39 implies that if $N \trianglelefteq G$, then $cd(G/N) \subseteq cd(G)$.

Theorem 2.48. *(Chapter 11 and Corollary 1.17 in [Isaacs, 1976]) Let G be a finite group and $\chi \in Irr(G)$. Then*

i) $\chi(1) \mid |G|$,

ii) $|G| = \sum_{\chi \in Irr(G)} \chi(1)^2$.

Definition 2.49. *The character table of a finite group G is the array $(\chi_i(g_i))_{1 \leq i, j \leq r}$, where $g_1, ..., g_r$ are representatives of the r conjugacy classes of G.*

2.2. Preliminary Definitions and Results in Graph Theory

In this section, we give some definitions and results in graph theory that are related to our study.

We follow in this section definitions and Theorems of [West, 2001].

Definition 2.50. *A graph* Φ *is an ordered pair* $(V(\Phi), E(\Phi))$*, where* $V(\Phi)$ *is a vertex set,* $E(\Phi)$ *is an edge set and each edge is associated with two vertices.*

Definition 2.51. *A subgraph of a graph* Φ *is a graph* Ω*, where* $V(\Omega) \subseteq V(\Phi)$ *and* $E(\Omega) \subseteq E(\Phi)$ *such that the assignment of endpoints to edges in* Ω *is the same as in* Φ*. We write in this case* $\Omega \subseteq \Phi$*.*

If $S \subseteq V(\Phi)$*, then we define the subgraph induced by S to be the subgraph of* Φ *whose vertex set is S and whose edge set contains all edges of* Φ *that join any two vertices in S.*

Definition 2.52. *Consider a graph* $\Phi = (V(\Phi), E(\Phi))$*.*

i) A walk on Φ *is an alternating sequence of vertices and edges:* v_0, e_1, v_1, ..., e_k, v_k, *such that* v_{i-1} *and* v_i *are the endpoints of the edge* e_i*, for all* $i \in \{1, ..., k\}$*.*

A walk is said to be closed if $v_0 = v_k$*. The length of a walk is defined to be the number of its edges. We denote an edge between two vertices u and v by* $u - v$*. In this case we say that u and v are adjacent in* Φ*.*

ii) A path of length n on Φ *is a walk of length n where all vertices are distinct. In this study, we use the notion* P_n *to denote a path of length n.*

iii) A cycle of length $n \geq 3$ *on* Φ *is a closed walk of length n where all its vertices are distinct except the endpoints. Usually, it is denoted by* C_n*.*

iv) If there exists a path between any two vertices a and b, we say that they are connected in Φ*. Furthermore, the distance between a and b in* Φ *is defined to be the length of a shortest path between them, and it is denoted by* $d_{\Phi}(a, b)$*.*

We call a triangle (resp. a square) any cycle of length three (resp. four).

Definition 2.53.

i) A loop is an edge connecting a vertex to itself.

ii) Multiple edges are two or more edges that are incident to the same two vertices.

iii) A simple graph is a graph that has no loops and multiple edges.

Definition 2.54. *If* Φ_1 *and* Φ_2 *are two graphs, then their union* $\Phi_1 \cup \Phi_2$ *is also a graph whose vertex set is* $V(\Phi_1) \cup V(\Phi_2)$ *and edge set is* $E(\Phi_1) \cup E(\Phi_2)$*.*

Definition 2.55. *A graph Φ is said to be connected if and only if for every $u \neq v \in V(\Phi)$ there exists a u, v-path (a path whose endpoints are u and v). Otherwise, we say that Φ is disconnected. In this case, we define the connected components of Φ to be its maximal connected subgraphs, and we denote by $n(\Phi)$ the number of such components. If $x \in V(\Phi)$, then by $[x]_\Phi$ we mean the connected component of Φ that contains x.*

It is clear that every disconnected graph can be written as the union of its connected components.

Definition 2.56. *Let Φ be a connected graph. Then,*

$$diam(\Phi) = Max\{d_\Phi(u,v) : u,\, v \in V(\Phi)\}. \tag{2.20}$$

If Φ is disconnected where $\Phi_1, \ldots, \Phi_r$ are its connected components, then

$$diam(\Phi) = Max\{diam\{\Phi_i\} : 1 \leq i \leq r\}. \tag{2.21}$$

Definition 2.57. *Let Φ_1 and Φ_2 be two graphs, then Φ_1 is isomorphic to Φ_2 if and only if:*

i) $V(\Phi_1) = V(\Phi_2)$.

ii) There exists an isomorphism f between $V(\Phi_1)$ and $V(\Phi_2)$ such that if u is adjacent to v in Φ_1, then $f(u)$ is adjacent to $f(v)$ in Φ_2.

Definition 2.58. *A complete graph of order n, K_n, is a simple graph on n vertices where there is an edge between every pair of vertices.*

Definition 2.59. *Let Φ be a graph and $v \in V(\Phi)$. The degree of v, $deg_\Phi(v)$, is defined to be the number of edges connected to v. Note that if v has some loops, then each loop will be counted as two edges.*

If $deg_\Phi(v) = 0$, then v is an isolated vertex.

Definition 2.60. *A graph Φ is called bipartite if its vertex set can be written as the union of two disjoint sets X and Y, such that each edge of Φ connects a vertex of X to a vertex of Y.*

Theorem 2.61. *(Theorem 1.2.18 in [West, 2001]) A graph Φ is a bipartite graph if and only if it contains no odd cycles.*

Definition 2.62. *A tree is a connected graph with no cycles.*

It should be mentioned that all graphs in this study are simple undirected graphs. Remark that in an undirected graph there is no difference between the edge $u-v$ and $v-u$. However, in a directed graph the order matters.

2.3. Basic Definitions and Results

Definition 2.63. *Let G be a finite group,*

i) The prime degree (or vertex) graph or simply the prime graph, $\Delta(G)$, is an undirected graph whose vertex set is $V(\Delta) = \rho(G)$, and there is an edge between two primes p and q in $\rho(G)$ if pq divides some degree in $cd(G)$.

ii) The common divisor degree graph, $\Gamma(G)$, is an undirected graph whose vertex set is $V(\Gamma) = cd(G)^* = cd(G) \setminus \{1\}$, and there is an edge between two distinct vertices x and y if $(x,y) \neq 1$.

iii) The bipartite divisor graph, $B(G)$, is an undirected graph whose vertex set is $V(B) = \rho(G) \cup cd(G)^*$, and there is an edge between $p \in \rho(G)$ and $x \in cd(G)^*$ if $p \mid x$.

Remark that if there exists $x \in cd(G)^*$ such that x is prime, then we count x as two distinct vertices in $V(B(G))$.

Definition 2.64. *If $N \trianglelefteq G$. Then, $cd(G|N) = \{\chi(1) : \chi \in Irr(G|N)\}$, where*

$$Irr(G|N) = \bigcup_{1_N \neq \theta \in Irr(N)} Irr(G|\theta), \tag{2.22}$$

and $Irr(G|\theta)$ is the set of all irreducible constituents of θ^G. It is well-known that

$$cd(G) = cd(G/N) \bigsqcup cd(G|N). \tag{2.23}$$

Lemma 2.65. *([Iranmanesh and Praeger, 2010]) Let $B = B(G)$ be a bipartite graph* *with $V(B) = \rho(G) \cup cd(G)^*$. Let $p, q \in \rho(G)$ and $x, y \in cd(G)^*$ such that $[p]_B = [q]_B$ and $[x]_B = [y]_B$.*

Then:

i) $d_B(p,q) = 2d_\Delta(p,q)$, $d_B(x,y) = 2d_\Gamma(x,y)$;

ii) if $p \mid x$ and $q \mid y$, then $[p]_B = [x]_B = [p]_\Delta \cup [x]_\Gamma$ and $d_B(p,q) - d_B(x,y) \in \{-2,0,2\}$;

iii) $n(B) = n(\Delta) = n(\Gamma)$;

iv) we have either:

- $diam(B) = 2max\{diam(\Delta), diam(\Gamma)\}$, *and* $|diam(\Delta) - diam(\Gamma)| \leq 1$;*or;*
- $diam(\Gamma) = diam(\Delta) = \frac{1}{2}(diam(B) - 1)$.

Proof. i) If $d_B(p,q) = k$, then there exists a shortest path in Δ of length k between p and q, say $(p_0 = p,\ p_1, ..., \ p_k = q)$. Remark that p_i and p_{i+1} are adjacent in Δ if and only if $d_B(p_i,\ p_{i+1}) = 2$. Hence if so, then there exists a path of length $2k$ in B connecting p and q, say $(p_0,\ x_1,\ p_1, ..., \ x_k,\ p_k)$. This implies that $d_B(p,q) \leq 2k$. But since p and q belong to the same bipartition part of B, we deduce that $d_B(p,q) = 2l \leq 2k$. If $(p'_0 = p, x'_1,\ p'_1, ..., \ x'_l,\ p'_l)$ is a shortest path in B between p and q then $(p'_0,\ p'_1, ..., \ p'_l)$ is a path in Δ of length l, hence $k \leq l$. Therefore $d_B(p,q) = 2k = 2d_\Delta(p,q)$. A similar proof shows that $d_B(x,y) = 2d_\Gamma(x,y)$.

ii) Suppose $p \mid x$ and $q \mid y$. If the above path $(p'_0 = p, x'_1, p'_1, ..., x'_l, p'_l = q)$ can be chosen such that $x'_1 = x$ and $x'_l = y$, then by a similar argument to above we can see that $d_B(p,q) - d_B(x,y) = 2$. Now, if only one of the following equalities: $x'_1 = x$ or $x'_l = y$ holds, then $d_B(p,q) = d_B(x,y)$. Finally if neither $x'_1 = x$ nor $x'_l = y$, then $d_B(p,q) - d_B(x,y) = -2$. Moreover, by the structure of a shortest path in B between p and q and the two induced paths in Δ and Γ which are respectively $(p'_0, p'_1 ...., p'_l)$ and $(x'_1, x'_2, ..., x_{l-1})$, we can deduce that $[p]_B$ contains $\{p,q,x,y\}$ and $[p]_B = [p]_\Delta \cup [x]_\Gamma$.

iii) Follows directly from part ii).

iv) Let $m = max\{diam(\Delta), diam(\Gamma)\}$. From part i) we can see that $diam(B) \geq 2m$. Let $diam(B) = M$ where $M \geq 2m$, and choose $a,\ b \in V(B)$ such that $d_B(a,b) = M$. If a and b are both in $\rho(G)$ (respectively in $cd(G)^*$), then $M = d_B(a,b) = 2d_\Delta(a,b) \leq 2diam(\Delta) \leq 2m$ (respectively, $M \leq 2diam(\Gamma) \leq 2m$). In each case we can see that $M = 2m$. Without loss of generality suppose that $a \in \rho(G)$ and $b \in cd(G)^*$. From the structure of $B(G)$ we can conclude that $M \geq 2m + 1$. Let $p \in \rho(G)$ be the vertex adjacent to b on a path on $B(G)$ of length M with endpoints a and b. Then by the definition of M, we can see that the p,a-subpath of this path must be the shortest path between these two vertices. Thus $M-1 = d_B(a,p) = 2d_\Delta(a,p)$ by part i) and so $M-1 \leq 2diam(\Delta)$. By a similar discussion, we can deduce that $M \leq 2diam(\Gamma)+1$. Hence $diam(\Delta) = diam(\Gamma) = \frac{M-1}{2}$.

It remains to check that $|diam(\Delta) - diam(\Gamma)| \leq 1$. Assume that $diam(B) = 2m$ and $f = diam(\Delta)$. Let p_0, $p_f \in \rho(G)$ such that $d_\Delta(p_0, p_f) = f$. It follows from part i) that there exists a path on B of length $2f$ connecting p_0 and p_f. Let x_0 and x_f be the vertices on this path adjacent to p_0 and p_f respectively. Consider now that subpath connecting x_0 to x_f. It is clear that this is the shortest path on B between x_0 and x_f. Hence, $diam_B(x_0,x_f) = 2f-2$. On the other hand, we have $diam_\Gamma(x_0,x_f) = f-1$ by part i). Thus $diam(\Gamma) \geq diam(\Delta) - 1$. Similarly we can show that $diam(\Delta) \geq diam(\Gamma) - 1$. Hence $|diam(\Delta) - diam(\Gamma)| \leq 1$. □

Theorem 2.66. *(Frobenius Reciprocity: Lemma 5.2 in [Isaacs, 1976]) Let $H \leq G$ and suppose that ψ is a character on H and θ is a character on G. Then $[\psi, \theta_H] = [\psi^G, \theta]$.*

Theorem 2.67. *(Gallagher's Theorem: Theorem 6.17 in [Isaacs, 1976]) Let N be a normal subgroup of G. Let $\chi \in Irr(G)$ such that $\chi_N = \theta \in Irr(N)$. Then for every $\beta \in Irr(G/N)$, the characters $\beta\chi$ are irreducible, distinct for distinct β and are all of the irreducible constituents of θ^G.*

Theorem 2.68. *(Ito-Michler's Theorem: Corollary 12.34 in [Isaacs, 1976]) Let G be a solvable group and p be a prime. Then every element of $cd(G)$ is relatively prime with p if and only if the Sylow p-subgroup of G is normal abelian.*

Theorem 2.69. *(Pálfy's Condition: Theorem 18.7 in [Manz and Wolf, 1993]) Let G be a solvable group and $\pi \subseteq \rho(G)$ such that $|\pi| \geq 3$. Then there exists u, $v \in \pi$ such that $uv \mid \chi(1)$ for some $\chi \in Irr(G)$ (equivalently, u and v are adjacent in $\Delta(G)$).*

Proposition 2.70. *Let G be a finite group and H be an abelian subgroup of G. Then for all $\chi \in Irr(G)$ we have $\chi(1) \leq |G:H|$.*

Proof. Let $\chi \in Irr(G)$. We know that χ_H is a character of H. If $\chi_H \in Irr(H)$, then χ_H is a linear character and we have $\chi_H(1) = 1 \leq |G:H|$. Otherwise, let $\theta \in Irr(H)$ such that $[\chi_H, \theta] \neq 0$. Then by Frobenius Reciprocity, we have $[\chi, \theta^G] = [\chi_H, \theta] \neq 0$. This implies that χ appears in the decomposition of θ^G as the sum of irreducible characters. Thus, $\chi(1) \leq \theta^G(1) = |G:H|\theta(1) = |G:H|$ (see Definition 2.41). □

Theorem 2.71. *(Fermat's Little Theorem [Andreescu et al., 2007]) If p is a prime and n is any integer not divisible by p, then $n^p - 1$ is divisible by p, which is equivalent to say that $n^p \equiv n$ (mod p).*

Theorem 2.72. *(Horosevskii Theorem: Corollary 3.3 in [Isaacs, 2008]) Let G be a nontrivial finite group and $\sigma \in Aut(G)$, then the order of σ in $Aut(G)$ is less than $|G|$.*

3. NO NONSOLVABLE GROUP G WHOSE $\Delta(G)$ IS A P_3 OR A C_4

In this chapter we see that there is no nonsolvable group G such that its $\Delta(G)$ is a P_3 or a C_4. We start by classifying the almost simple groups G where $S \leq G \leq Aut(S)$ for some nonabelian simple group S and such that $\Delta(G)$ is a subgraph of C_4. Then, we claim that every nonsolvable group G with $\Delta(G) = P_3$ or C_4 has a normal solvable subgroup N such that G/N is almost simple, and $\Delta(G/N)$ is a subgraph of C_4. Then, we use the above classifications for proving that no such G exists.

Note that if G is an almost simple group with socle S, then S is normal in G and hence $\rho(S) \subseteq \rho(G)$ by Clifford's Theorem. By Ito-Michler's Theorem and the simplicity of S we can deduce that $\rho(S) = \pi(S)$. Therefore we consider the case where S is a nonabelian simple group whose order is divisible by at most four primes. As a group with at most two prime divisors is solvable (Burnside's Theorem), we may consider those nonabelian simple groups S such that $\pi(S) = 3$ or 4.

It should be mentioned that if $\theta \in Irr(S)$ and $\chi \in Irr(G|\theta)$ then by Lemma 6.8 in [Isaacs, 1976] we can see that $\theta(1)|\chi(1)$, thus $\Delta(S) \subseteq \Delta(G)$. Hence if $\Delta(S)$ is not a subgraph of C_4 then so is $\Delta(G)$.

To reach the desired classification we use the list of simple groups whose prime degree graphs are incomplete [White, 2009] and the list of simple groups whose orders are divisible by three or four primes [Huppert and Lempken, 2000]. We conclude then that the only simple groups S whose $|\pi(S)| = 3$ or 4 are either $PSL_2(q)$ for a prime power q or one of the groups listed in Table 2.1.

Remark that we did not consider those simple groups whose prime degree graphs are complete graphs with three or four vertices since such graphs cannot be a subset of C_4.

We follow in this chapter lemmas and theorems of paper [Lewis and White, 2013].

Lemma 3.1. *If G is a finite group with socle S such that S is a nonabelian simple group, not isomorphic to $PSL_2(q)$ for any prime power q, then $\Delta(G)$ is not a subgraph of C_4.*

Proof. By the above notes, the only simple groups S whose $|\pi(S)| = 3$ or 4 and not isomorphic to $PSL_2(q)$ for any prime power q are the groups listed in Table 3.1. From

Table 3.1: $S \ncong PSL_2(q)$, $|\rho(S)| \leq 4$ and $\Delta(S)$ incomplete.

Group	Order	Degrees	Triangle
M_{11}	$2^4.3^2.5.11$	$10, 44, 55$	$2, 5, 11$
A_8	$2^6.3^2.5.7$	$14, 20, 35$	$2, 5, 7$
$PSL_3(4)$	$2^6.3^2.5.7$	$35, 45, 63$	$3, 5, 7$
$PSL_3(8)$	$2^9.3^2.7^2.73$	$72, 584, 657$	$2, 3, 73$
$PSU_3(4^2)$	$2^6.3.5^2.13$	$12, 39, 52$	$2, 3, 13$
$PSU_3(9^2)$	$2^5.3^6.5^2.73$	$72, 584, 657$	$2, 3, 73$
${}^2B_2(8)$	$2^6.5.7.13$	$35, 65, 91$	$5, 7, 13$
${}^2B_2(32)$	$2^{10}.5^2.31.41$	$775, 1025, 1271$	$5, 31, 41$

the table, it is clear that for each S, $\Delta(S)$ contains a triangle. Hence $\Delta(S)$ cannot be a subgraph of C_4 and so is $\Delta(G)$ (which contains $\Delta(S)$). $\square$

It is well known that for any $q = p^f$ for some positive integer f and a prime p,

- $cd(PSL_2(q)) = \{1, q, q-1, q+1\}$ if q is even,
- $cd(PSL_2(q) = \{1, q, (q+\varepsilon)/2, q-1, q+1\}$ if q is odd where $\varepsilon = (-1)^{\frac{q-1}{2}}$.

Therefore $\rho(PSL_2(q)) = \{p\} \cup \pi(q-1) \cup \pi(q+1)$.

Lemma 3.2. *(Lemma 1.3 in [Huppert and Lempken, 2000]) Let p be a prime and f a positive integer. Then,* $p^{2f} - 1$ *has at most two distinct prime divisors, if and only if,* $p^f \in \{2, 3, 4, 5, 7, 8, 9, 17\}$.

By Theorem 3.1, we can see that if G is an almost simple group such that $S \leq G \leq Aut(S)$ where S is a nonabelian simple group. Then, $\Delta(G)$ is a subgraph of a square if $S \cong PSL_2(q)$ for a prime power q. Therefore, till the end of this chapter we will assume that $S \cong PSL_2(q)$, for a prime power q. Since $PSL_2(2) \cong S_3$ and $PSL_2(3) \cong A_4$ where both S_3 and A_4 are not simple, we may assume that $q > 3$.

We consider first the case where $|\pi(S)| = 3$.

Lemma 3.3. *Let* $S = PSL_2(q)$ *where* $q = p^f > 3$ *for some positive integer f and a prime p. Then* $|\rho(\Delta(S))| = 3$ *if and only if* $p^f \in \{2^2, 2^3, 3^2, 5, 7, 17\}$. *In this case* $\Delta(S)$ *has at most one edge. And for every almost simple group G with socle S,* $\Delta(G)$ *has exactly one edge and is a subgraph of* C_4.

Before representing the proof of Lemma 3.3, we must mention some important remarks about the possible groups G such that $PSL_2(q) < G \leq Aut(PSL_2(q))$.

Definition 3.4. *The outer automorphism group of a group G, $Out(G)$, is defined to be the quotient group $Aut(G)/Inn(G)$, where $Inn(G) = \{\sigma_a : a \in G\}$ and $\sigma_a(g) = aga^{-1}$ for all $g \in G$.*

Remark that for any finite group G, $Inn(G) \cong G/Z(G)$. Thus $Inn(PSL_2(q)) \cong PSL_2(q)$ as $Z(PSL_2(q))$ is trivial for all prime power q.

Note 3.5.

i) *For $G \cong PSL_2(p^f)$, we have: $Out(G) \cong \mathbb{Z}_2$ if $p > 3$ is odd and $f = 1$, and $Out(G) \cong \mathbb{Z}_f$ if $p = 2$ and $f > 1$. Furthermore, for $q = p^f > 6$,*

$$Out(PSL_2(p^f)) = \langle\delta\rangle \times \langle\varphi\rangle \tag{3.1}$$

where δ is a diagonal automorphism of order $(2, p^f - 1)$, and φ is a field automorphism of order f.

ii) *Subgroups of $Aut(PSL_2(q))$ are discussed in [White, 2013] and can be summarized as follows:*

- *If q is even:*
  $PGL_2(q) = PSL_2(q)$, $Aut(PSL_2(q)) = PSL_2(q)\langle\varphi\rangle$, and any subgroup of $Aut(PSL_2(q))$ which contains $PSL_2(q)$ strictly, is of the form: $PSL_2(q)\langle\varphi^k\rangle$ for some $1 \leq k < f$ with $k \mid f$ and $G/PSL_2(q)$ is cyclic.

- *If q is odd:*
  $PGL_2(q) = PSL_2(q)\langle\delta\rangle$, $Aut(PSL_2(q)) = PSL_2(q)\langle\delta,\varphi\rangle$, and if $PSL_2(q) < G \leqslant Aut(PSL_2(q))$, then G has one of the following forms:

  i) *$\delta \in G$ so that $PGL_2(q) \leqslant G$ and $G = PGL_2(q)\langle\varphi^k\rangle$ for some $k \mid f$ with $1 \leq k \leq f$;*

  ii) *$G = PSL_2(q)\langle\varphi^k\rangle$ for some $k \mid f$ with $1 \leq k < f$;*

  iii) *$G = PSL_2(q)\langle\delta\varphi^k\rangle$ for some $k \mid f$ with $1 \leq k < f$ and $f \mid k$ is even.*

Proof of Lemma 3.3. By using Lemma 3.2, it is clear that $\Delta(S)$ has exactly three vertices if and only if $p^{2f} - 1$ is divisible by exactly two primes, if and only if $q =$

$p^f \in \{2^2, 2^3, 3^2, 5, 7, 17\}$. Remark that $PSL_2(4)$ and $PSL_2(5)$ are isomorphic groups, this group will be viewed as $PSL_2(4)$ in our discussion. By using character tables in [Conway et al., 1984], it is clear that $\Delta(S)$ has no edges for $q \in \{4, 8\}$, and has exactly one edge for $q \in \{7, 9, 17\}$. By note 3.5, it is clear that for every $p^f \in \{4, 8, 7, 17\}$, the outer automorphism group of S is of order 2 or 3, hence $[Aut(S) : G] = 1$ and $G = Aut(S)$. For $q \in \{4, 8, 7, 17\}$ the character tables of $Aut(S)$ are given in [Conway et al., 1984] where it is shown that that for each case, $\rho(G) = \rho(S)$ and $\Delta(G)$ has exactly one edge. For the case $q = 9$, the outer automorphism group of S is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the proper subgroups of $Aut(S)$ are as mentioned above. For each case the character table is known by [Conway et al., 1984] and shows that $\Delta(G)$ has exactly one edge. To finish the proof, consider $Aut(PSL_2(9)) = PSL_2(9)\langle\delta, \varphi\rangle$ where δ and φ are automorphisms of order 2. Since $q = 9$ is odd and $\delta \in Aut(PSL_2(9))$, we know by [White, 2013] that $|Aut(PSL_2(9)) : PGL_2(q)| = d = 2^a m$, where m is odd. Furthermore, $cd(Aut(PSL_2(9))) = \{1, 9, (9+\varepsilon)/2 : \varepsilon = (-1)^{(9-1)/2}\} \cup \{(9-1)2^a l : l \mid m\} \cup \{(q+1)j : j \mid d\}$. But since $|Aut(PSL_2(9)) : PGL_2(9)| = f = 2$ (see Note 3.5) and $Aut(PSL_2(9))$ contains strictly $PSL_2(9)\langle\delta\rangle$, we conclude by Theorem A of [White, 2013] that $(9+\varepsilon)/2 = 5 \notin cd(Aut(PSL_2(9)))$, $m = 1$ and $a = 1$. Thus $l = 1$, $j \in \{1, 2\}$ and $cd(Aut(PSL_2(9))) = \{1, 9, 10, 16, 20\}$. Therefore $2 - 5$ is the unique edge in this case and the proof is completed. □

Now consider the case were $S = PSL_2(q)$ such that $q^2 - 1$ has exactly three prime divisors, equivalently the case where $\Delta(S)$ has exactly four vertices. The cases where q is even or odd will be treated separately.

For the even case, i.e., when $q = 2^f$, it is clear that S is not simple for $f = 1$, and $\Delta(S)$ has three vertices for $f \in \{2, 3\}$, hence we may consider $f \geq 4$. Since 2, $2^f - 1$ and $2^f + 1$ are pairwise relatively prime and $2^{2f} - 1$ has three prime divisors, then one of $2^f - 1$ or $2^f + 1$ is a prime power and the other is a product of two prime powers. Precisely, by Lemma 1.2 of [Huppert and Lempken, 2000] one of $2^f - 1$ or $2^f + 1$ is a prime.

Lemma 3.6. *If $S = PSL_2(2^f)$ such that $|\rho(S)| = 4$, then either:*

i) $f = 4$, $2^f - 1 = 3.5$ and $2^f + 1 = 17$; or;

ii) $f \geq 5$ is a prime, $2^f - 1 = r$ is a prime and $2^f + 1 = 3.t^\beta$ where t is an odd prime *and $\beta \geq 1$ is odd.*

Furthermore, $\rho(S) = \{2,3,r,t\}$, $3-t$ is the only edge in $\Delta(S)$ and if G is almost simple with socle S, then $\Delta(G)$ is not a subgraph of C_4.

Proof. Parts i) and ii) follow directly from Lemma 3.5 in [Huppert and Lempken, 2000]. Since the character degrees of S are relatively prime, then $3-t$ is the only edge in $\Delta(S)$. Notice that for the even case, the outer automorphism group of S is a cyclic group generated by a field automorphism of order f. Hence if G is almost simple group with socle S, then $|G:S| \mid f$.

If $f = 4$, then $|G:S| = 2$ or 4 and $\rho(G) = \rho(S) = \{2,3,5,17\}$. For each case, 3.5.$|G:S| \in cd(G)$ [Conway et al., 1984], therefore the primes 2, 3 and 5 form a triangle in $\Delta(G)$ and it cannot be a subgraph of C_4.

If $f \geq 5$ is prime, then $|G:S| = f$. By Fermat's Little Theorem we know that $2^f \equiv 2 \pmod f$. Therefore $2^f + 1 \equiv 3 \pmod f$ and $2^f - 1 \equiv 1 \pmod f$. Since $f \geq 5$, then f is neither r nor t. Hence $\rho(G) = \{2,3,r,t,f\}$. Thus $\Delta(G)$ has five vertices and it cannot be a subgraph of C_4. □

Suppose now that the prime vertex graph of $S = PSL_2(p^f)$ has four vertices for some odd prime p and positive integer f, or equivalently q^2-1 has three distinct prime divisors. As $q-1$ and $q+1$ are both divisible by 2, then q^2-1 is divisible by exactly two odd primes r and t. So $\rho(S) = \{p,2,r,t\}$.

Lemma 3.7. *(Theorem 3.2 and Lemma 3.4 in [Huppert and Lempken, 2000]) If $q = p^f$ is an odd prime power such that q^2-1 is divisible by exactly three primes, then one of the following holds:*

i) $q \in \{3^4, 5^2, 7^2\}$;

ii) $p = 3$ *and* f *is an odd prime integer;*

iii) $11 \leq p$ *and* $f = 1$.

Note 3.8. *It follows from Lemma 3.4 in [Huppert and Lempken, 2000] that if $q = 3^f$ and $|q^2-1| = 3$, then $q-1 = 2r^\alpha$ and $q+1 = 2t^\beta$ for some odd primes $r \neq t$ and positive integers α, β.*

Lemma 3.9. *Let $S = PSL_2(q)$ where $q = p^f$ for an odd prime p and a positive integer f*

such that q^2-1 is divisible by exactly two odd primes r, t and $\rho(S)=\{p,2,r,t\}$. Let G be an almost simple group with socle S. Then the graph $\Delta(G)$ is a subgraph of C_4 if and only if one of the following holds:

i) $p=3$, *f is an odd prime and $G=S$ or $G=PGL_2(q)$;*

ii) $q\in\{3^4,5^2,7^2\}$ *or* $11\leq p$ *is neither a Mersenne prime nor a Fermat prime and* $f=1$. *Recall that a Mersenne prime (resp. a Fermat prime) is a prime number that can be written as* 2^n-1 *(resp.* $2^{2^n}+1$*) for some positive integer n.*

In all cases where $\Delta(G)$ is a subgraph of C_4, $\Delta(G)=\Delta(S)$.

Proof. By Lemma 3.7, $q\in\{3^4,5^2,7^2\}$, $q=p\geq 11$, or $q=3^f$ for an odd prime f. If either $q+1$ or $q-1$ is a power of 2, then the other is divisible by 2, r and t. Thus $\Delta(S)$ contains a triangle, i.e., $\Delta(G)$ cannot be a subgraph of C_4. Therefore, each of $q+1$ and $q-1$ is the product of a power of 2 and a power of an odd prime. By Note 3.8, this holds for $q=3^f$ with an odd prime f and clearly holds for $q\in\{3^4,5^2,7^2\}$. If $q=p\geq 11$, this holds if and only if q is neither a Mersenne prime nor a Fermat prime by definition.

Under the conditions of *i*) or *ii*), $\rho(S)=\{p,2,r,t\}$ and $\Delta(S)$ is a subgraph of C_4 which has exactly the following edges: $2-r$, $2-t$. Suppose that G is an almost simple group with socle S, then by Theorem A of [White, 2013], each character of G is either 1, q or a divisor of either $|G:S|(q+1)$ or $|G:S|(q-1)$. Note that, if $|G:S|$ is a power of 2, then $\Delta(S)=\Delta(G)$ is a subgraph of C_4. Since q is odd, $|Aut(S):S|=2f$. For each of the cases $q\in\{3^4,5^2,7^2\}$ or $11\leq q$ is a prime, f is a power of 2, hence so is $|G:S|$ (being a divisor of $|Out(S)|=2f$). Consider now the case where $q=3^f$ for an odd prime f and let G be an almost simple group with socle S. If $G=PGL_2(q)$, then $|G:S|$ is a power of 2 and $\Delta(G)$ is a subgraph of C_4 (as above). Assume now that $G\neq PGL_2(q)$ and $|G:S|\neq 2$, then f must divide $|G:S|$, precisely $f\in\rho(G)$ by [White, 2013]. Since f is an odd prime, then $3^f\equiv 3 \pmod f$ and neither 3^f-1 nor 3^f+1 is divisible by 3. Hence if $f\neq 3$, then $\rho(G)=\{2,3,r,t,f\}$, i.e., $\Delta(G)$ has five vertices and it cannot be a subgraph of C_4. Finally if $f=3$, then $S=PSL_2(3^3)$. If $S<G$ and G is not $PGL_2(3^3)$, then $|Out(S)|=2$ and $G=Aut(S)$. It follows from the character degree table of G in [Conway et al., 1984] that $3(q-1)$, $3(q+1)\in cd(G)$. Hence 3 is adjacent to 2, $r=7$ and $t=13$ in $\Delta(G)$, a contradiction. □

Theorem 3.10. *Let G be a finite group such that $S<G\leqslant Aut(S)$ where S is a nonabelian*

finite simple group. If $\Delta(G)$ is a subgraph of C_4, then $S \cong PSL_2(q)$ for some prime power $q = p^f$, such that $q^2 - 1$ is divisible by at most three distinct prime divisors. Furthermore, the values of q and the almost simple groups G with socle $PSL_2(q)$ such that $\Delta(G)$ is a subgraph of C_4 are as follows:

i) $q \in \{2^2, 2^3, 3^2, 3^4, 5, 5^2, 7, 7^2, 17\}$*, all groups G such that* $S < G \leqslant Aut(S)$.

ii) $q = 2^4$ *and* $G = S$.

iii) $q = 2^f$ *for a prime integer* $f \geq 5$ *and such that* $2^f - 1 = r$ *is a prime and* $2^f + 1 = 3t^\beta$ *for a prime t distinct from r, in this case* $G = S$.

iv) $q = 3^f$ *for a prime integer* $f \geq 3$ *and such that* $3^f - 1 = 2r^\alpha$ *and* $3^f + 1 = 2^2 t^\beta$ *where* $r \neq t$ *are two primes, and* $G = S$ *or* $G = PGL_2(q)$.

v) $q = p \geq 11$ *and each of* $q+1$, $q-1$ *is divisible by 2 and an odd prime, in particular q is neither a Mersenne prime nor a Fermat prime, all G.*

Proof. The proof follows directly from Lemmas 3.3, 3.6 and 3.9. □

In the following, we claim that if G is a nonsolvable group whose $\Delta(G) = P_3$ or C_4, then it has a solvable normal subgroup N such that G/N is one of the groups listed in the previous theorem. We conclude then that no such G exists. This requires the usage of Dickson's classification of subgroups of $PSL_2(q)$ (Hauptstaz II.8.27 in [Huppert, 1967]), the fact that the Schur multiplier for $PSL_2(q)$ has order 2 when $q \neq 9$ is odd and order 1 when $q \geq 4$ is even, and that $SL_2(q)$ is the Schur representation group of $PSL_2(q)$ when $q \notin \{4, 9\}$.

Lemma 3.11. *Let G be a nonsolvable group such that* $\Delta(G) = P_3$ *or* C_4. *Then there exist two normal subgroups* $N < M$ *of G such that N is solvable,* M/N *is a nonabelian simple group and* G/N *is one of the groups listed in Theorem 3.10.*

Proof. Let N be maximal such that N is a normal solvable subgroup of G. Take M to be a normal subgroup in G such that $N < M$ and M/N is a chief factor of G. By the maximality of N, M is nonsolvable and M/N is a nonabelian factor. Thus M/N is a minimal normal subgroup of G/N. Hence $M/N = S \times S \times ... \times S$ for a nonabelian simple group S (see Corollary 2.13). As mentioned previously $|\rho(S)| \geq 3$. If M/N is not simple, then M/N is isomorphic to the direct product of at least two copies of S. So by considering the product of the different characters of S divisible by the distinct primes of $\rho(S)$, we conclude that $\Delta(M/N)$ is a complete subgraph of $\Delta(G)$, a contradiction. Thus $M/N = S$ is simple. Let $C/N = C_{G/N}(M/N)$. Since M/N is simple, $M \cap C = N$. If $N < C$, then there exists a normal subgroup of G, say L, such that $N < L \leq C$ and L/N is a chief factor for G. By applying the previous argument of M/N to L/N, we can similarly conclude that L/N is a nonabelian simple group.

We claim now that any vertex in $\rho(M/N) \cap \rho(L/N)$ is adjacent to all vertices of $\rho(M/N) \cup \rho(L/N)$. Indeed, since $L/N \leq C_{G/N}(M/N)$, then every element in $M/N \cap L/N$ commutes with all elements of M/N, i.e., all elements in the intersection belongs to $Z(M/N)$. But $Z(M/N) = 1_{G/N}$, being the center of a nonabelian simple group. Hence $L/N \cap M/N = 1_{G/N}$. Thus $L/N.M/N = L/N \times M/N$ is a normal subgroup of G/N (note that M/N is normal in G/N by the definition of an almost simple group, and L/N is normal in G/N being the centralizer of a normal subgroup). Let $p \in \rho(L/N) \cap \rho(M/N)$, then there exist $\chi \in Irr(M/N)$ and $\psi \in Irr(L/N)$ such that $p \mid \chi(1)$, $\psi(1)$. Let $\theta \in Irr(L/N)$ and $\nu \in Irr(M/N)$. Then $\chi(1)\theta(1)$, $\psi(1)\nu(1) \in cd(M/N \times L/N)$ (see Example 2 of section 15 in [Alperin and Bell, 1991]). Hence, p is adjacent to t for all $t \in \rho(M/N)$ and p is adjacent to r for all $r \in \rho(L/N)$. Therefore p is adjacent to q for all $q \in \rho(M/N) \cup \rho(L/N)$.

Particularly, $\rho(M/N) \cap \rho(L/N)$ induces a complete subgraph of $\Delta(G)$. Therefore $|\rho(M/N) \cap \rho(L/N)| \leq 2$ and so $|\rho(M/N) \cup \rho(L/N)| \geq 4$. By Theorem 3.10, we know that both M/N and L/N are projective linear groups of order 2, hence $2 \in \rho(M/N) \cap \rho(L/N)$. It follows then that 2 has degree at least three in $\Delta(G)$ which implies a contradiction. □

By the last result (Lemma 3.11) and since our purpose is to prove that there does not exist any nonsolvable group G whose $\Delta(G)$ is a P_3 or a C_4 , we may assume the following hypothesis.

Hypothesis 3.12.

- *G is a nonsolvable group whose $\Delta(G)$ is a P_3 or a C_4.*
- *$N < M$ are two normal subgroups of G such that N is solvable, and $M/N = S = PSL_2(q)$ where q is one of the values described in Theorem* 3.10.
- *G/N is an almost simple group with socle S, precisely, it is one of the groups*

described in Theorem 3.10.

If Hypothesis 3.12 is true, we discuss the existence of G in separate theorems depending on the parity of p and whether $|\rho(S)| = 3$ or 4.

Theorem 3.13. *There is no G that satisfies Hypothesis 3.12 if p is odd and* $|\rho(S)| = 4$.

Proof. Let $|\rho(S)| = 4$ such that p is odd. Suppose that such G exists. It follows from Lemma 3.9 that $\rho(G) = \rho(S) = \{2, p, r, s\}$, $q - 1 = 2^{\alpha} r^{a}$ and $q + 1 = 2^{\beta} s^{b}$ for some positive integers α, β, a and b. Indeed, q is one of the following:

i) $q = p \geq 11$.

ii) $q = 25$, with $r = 3$ and $s = 13$.

iii) $q = 49$, with $r = 3$ and $s = 5$.

iv) $q = 81$, with $r = 5$ and $s = 41$.

v) $q = 3^{f}$, such that $f \geq 3$ is a prime.

Notice that none of the character degrees of G is divisible by three primes, and that 2 is adjacent to both r and s, hence no degree in $\Delta(G)$ is divisible by either rs or $2p$. Let $\theta \in Irr(N)$ and $T = I_M(\theta)$. Since $[\chi_N, \theta] = [(\chi_M)_N, \theta] = [\chi_M, \theta^M]$ for all $\chi \in Irr(G|\theta)$, then each irreducible constituent of χ_M belongs to $Irr(M|\theta)$. Thus $|M : T|\theta(1) \mid \chi(1)$ for all $\chi \in Irr(G|\theta)$. This implies that $|M : T|$ is divisible by at most two primes, and if $|\pi(|M : T|)| = 2$ then any prime divisor of $\theta(1) \in \pi(M/T)$.

We claim that $T = M$. Suppose that $T < M$. We consider the different possibilities for the subgroup T/N of $S = PSL_2(q)$ described in Hauptstaz II.8.27 of [Huppert, 1967].

If T/N is abelian, then it is either an elementary abelian p-group or a cyclic group of order z such that $z \mid \dfrac{p^f \pm 1}{k}$ and $k = (p^f - 1, 2)$. This implies that $|M : T|$ must be divisible by three primes (as $|T : N|$ is divisible by exactly one prime and $|S| = |M/N| = |T/N|.|M/N : T/N| = |T/N||M : T|$ has four prime divisors), a contradiction. So T/N is nonabelian.

If T/N is a dihedral group of order $2z$ where z is as described above, then $p \mid |M : T|$. Let C/N be the cyclic normal subgroup of index 2 in T/N. Since θ is invariant in C, then by Corollary 11.22 in [Isaacs, 1976] θ extends to $\varphi \in Irr(C)$. If φ is T-invariant, then θ extends to T by Corollary 6.20 in [Isaacs, 1976] and since 2 is a character degree in any dihedral group, then $2\theta(1) \in cd(T|\theta)$ by Gallagher's Theorem.

Otherwise, apply Clifford's Theorem to any $\eta \in Irr(T|\varphi)$, we get: $\eta_C(1) = [\eta_C, \varphi].|T : I_T(\varphi)|.\varphi(1)$ such that $1 < |T : I_T(\varphi)|$ divides 2. Hence $\varphi(1) = [\eta_C, \varphi]2\varphi(1)$ with $[\eta_C, \varphi] \mid |T : C|$. Thus $[\eta_C, \varphi] = 1$ or 2. If $[\eta_C, \varphi] = 2$, then $2, p \mid |T : C| \mid |M : T| \mid \chi(1)$ for all $\chi \in Irr(G|\theta)$, i.e., 2 is adjacent to p in $\Delta(G)$, a contradiction. Therefore $\eta_C(1) = \eta(1) = 2\varphi(1) = 2\theta(1) \in cd(T|\theta)$. So in both cases we get, $2\theta(1) \in cd(T|\theta)$. Applying again Clifford's Theorem implies that $|M : T|2\theta(1)$ divides $\phi(1) \in Irr(M)$ which divides a character degree in G as $M \trianglelefteq G$. Thus 2 and p are adjacent in $\Delta(G)$, a contradiction.

If T/N is a Frobenius group, then by the structure of T/N given in Hauptstaz II.8.27 in [Huppert, 1967], we know that T/N is a direct product of a cyclic group of order t and a Frobenius kernel F/N, where F/N is an elementary abelian p-group and $t \mid 2^{\alpha} r^{a}$. Thus, $p \mid |T : N|$ and $s \nmid |T : N|$. As a consequence s must divide $|M : T|$. Assume now that $r \nmid |T : N|$, then $r, s \mid |M : T|$. But $|M : T| \mid \chi(1)$ for all $\chi(1) \in cd(G|\theta)$ where $\theta \in Irr(N)$. This generates a character degree in G/N and consequently in G which is divisible by both r and s, a contradiction. Hence $p, r \mid |T : N|$ and $2, s \mid |M : T|$. If θ does not extend to F, then $[F : I_F(\theta)] > 1$, but it divides the p-group F/N, hence p divides all character degrees in $cd(G|\theta)$. It follows from the last result, Clifford's Theorem and the fact that each irreducible constituent of $\chi \in Irr(G|\theta)$ in T belongs to $Irr(T|\theta)$, that each character degree in $cd(T|\theta)$ and $cd(G|\theta)$ is divisible by p. And since the character degrees in $cd(G|\theta)$ are divisible by $|M : T|$, then $cd(G)$ has a degree divisible by 2 and p, a contradiction. Thus θ extends to F. If R/N is either a Sylow r-subgroup or a Sylow 2-subgroup of T/N, then it is cyclic and extends to R by Corollary 11.22 in [Isaacs, 1976]. Moreover it extends to T by Theorem 11.31 in [Isaacs, 1976]. Applying Gallagher's Theorem and the fact that $|T : F| \in cd(T/N)$ gives that $|T : F|\theta(1) \in cd(T|\theta)$. Then by applying Clifford's Theorem to any $\psi \in Irr(M|\upsilon.\theta(1))$ where $\upsilon(1) = |T : F|$, we can conclude that $\psi(1)$, which divides a character degree in G, is divisible by $|M : T||T : F|\theta(1)$. Hence r is adjacent to s in $\Delta(G)$, a contradiction.

If $T/N \cong A_4$ or S_4. We have $\pi(A_4) = \pi(S_4) = \{2, 3\}$, hence $3 \in \{p, r, s\}$ and $|M : T|$ is divisible by the other two primes. If θ extends to T, then it follows from the character degrees of A_4 and S_4, Gallagher's Theorem and Clifford's Theorem, that $|M : T|2\theta(1)$ or $|M : T|3\theta(1)$ divides a character degree in M and consequently in G. I.e., there exists a character degree in G divisible by three distinct primes, a contradiction.

Otherwise $|T : I_T(\theta)| > 1$, and since it divides $|A_4|$ or $|S_4|$, then $|T : I_T(\theta)|$ is divisible by either 2 or 3. Hence, for any $\gamma \in Irr(T|\theta)$, we have $\gamma_N(1) = [\gamma_N, \theta]|T : I_T|\theta(1)$. Again by Clifford's Theorem we get: $\varphi_T(1) = |M : T|[\varphi_T, \gamma]\gamma(1)$ for some $\varphi \in cd(M|\gamma)$, which means that, $\varphi(1)$ that divides a character degree in G, is divisible by 2 (or 3) and two other distinct primes, a contradiction.

If $T/N \cong A_5$, then $\pi(A_5) = \{2,3,5\}$. So if p is neither 3 nor 5 , then $p \mid |M : T|$. Whether θ extends to T or not, there exists some degree in $cd(T|\theta)$ divisible by 2 (see character table of A_5 in [Conway et al., 1984]). Hence there exists a character degree of G divisible by both 2 and p, a contradiction. If $p = 5$, then $q = 25$ and $|PSL_2(25)|$ is divisible by $\{2,3,5,13\}$, hence $13 \mid |M : T|$ and as above we obtain from [Conway et al., 1984] a character in $cd(T|\theta)$ divisible by $r = 3$. This gives a character degree of G divisible by both r and $s = 13$, a contradiction. Similarly, if $p = 3$, then $q = 81$ and $\pi(PSL_2(81)) = \{2,3,5,41\}$. Hence, 41 must divide $|M : T|$ and we obtain a character degree of G divisible by both $r = 5$ and $s = 41$, a contradiction.

Finally, assume that $T/N \cong PSL_2(p^m)$ or $PGL_2(p^m)$. As the cases $PGL_2(3) \cong S_4$ and $PSL_2(5) \cong A_5$ have been already discussed, the remaining possibilities are $q = 25$ with $T/N \cong PGL_2(5)$, $q = 81$ with $p^m = 9$ and $q = 49$ with $p^m = 7$. By a similar discussion to the previous paragraph and by using the character tables found in [Conway et al., 1984], we can obtain for each case a character degree of G divisible by two primes which are not adjacent in $\Delta(G)$, a contradiction.

Therefore, $T = M$. If θ extends to M, then by Gallagher's Theorem $\theta(1)(q-1)$, $\theta(1)(q+1)$ and $\theta(1)q$ are all character degrees of M. So, if $\theta(1) \neq 1$, then any prime divisor of $\theta(1)$ is adjacent to all other primes in $\rho(G)$, a contradiction. Hence, $\theta(1) = 1$. On the other hand, if θ does not extend to M, then we obtain the degrees $\theta(1)(q+1)$ and $\theta(1)(q-1)$ in $cd(M)$, as the representation group is $SL_2(q)$. Now, if $t \mid \theta(1)$, then it is adjacent to the other two primes among $\{2,r,s\}$. And since p cannot be adjacent to three distinct primes, then $p \nmid \theta(1)$. Also as r and s are not adjacent, $\theta(1)$ is not divisible by either r or s. Thus $\theta(1)$ is a power of 2. Notice that since q is odd, then $|M : N| = |PSL_2(q)| = q.(q^2-1)/2$. Consider now $a \in Irr(M|\theta)$. As $T = M$, we can write $a(1) = e.\theta(1)$ where $e = [a_N, \theta] \mid |M : N| = q.(q^2-1)/2$. If $e \nmid q^2-1$, then e must be a power of p. This implies that 2, $p \mid a(1)$, a contradiction as the vertex 2 will be adjacent to three distinct vertices in $\Delta(G)$. Hence $e = a(1)/\theta(1)$ divides q^2-1 for all

$a \in Irr(M|\theta)$. Since $p \nmid q^2-1$ or $\theta(1)$, we conclude that p does not divide any degree in $cd(M|\theta)$.

As $p \in \rho(G)$, there exists $\chi \in Irr(G)$ such that $p \mid \chi(1)$. Let $\mu \in Irr(M)$ be a constituent of χ_M and $\theta \in Irr(N)$ be a constituent of μ_N. By the possibilities of G for each q found in Theorem 3.10, we can see that $|G:M|$ is a power of 2. For example, if $q = 3^f$, then G/N is either S or $PGL_2(3^f)$, i.e., $|G:M| = 1$ or 2 in this case. It follows from Clifford's Theorem and the fact that $|G:M|$ is a power of 2, that $p \mid \mu(1)$. But as mentioned before, the only possible case to have a degree in $cd(M|\theta)$ divisible by p, is when θ is linear and extends to M. If follows then, that $\mu(1) = q$. Hence, 2 and p are the only possible prime divisors of $\chi(1)$. But since p is not adjacent to 2 in $\Delta(G)$, then p is an isolated vertex, which implies a contradiction as $\Delta(G)$ is a connected graph. Therefore, no such G exists. □

Theorem 3.14. *There is no G that satisfies Hypothesis 3.12 if* $p \neq 5$ *is odd and* $|\rho(S)| = 3$.

Proof. By Theorem 3.10, we have $q \in \{7, 9, 17\}$. Let $\rho(S) = \{2, p, r\}$ and s be a prime such that $\rho(G) = \rho(S) \cup \{s\}$. Since 2 is adjacent to r in $\Delta(M/N)$, and consequently in $\Delta(G)$, then 2 and r have no common neighbors in $\Delta(G)$. Let $\chi \in Irr(G)$ such that $s \mid \chi(1)$, and θ be an irreducible constituent of χ_N. Notice that, S is a nontrivial finite group, thus $|\sigma| \leq |S|$ for all $\sigma \in Aut(S)$ (Horosevskii Theorem). Hence, as $s \nmid |S|$, for all the possibilities of G/N mentioned in Theorem 3.10, $s \nmid |G:N|$. It follows from Corollary 11.29 in [Isaacs, 1976] that $s \mid \theta(1)$. Suppose that θ is M-invariant. If θ extends to M or does not extends to M such that either $q \neq 9$, so that $SL_2(q)$ is the Schur representation group of $PSL_2(q)$, or $q = 9$ and θ corresponds to the character of the Schur multiplier of order 2, then $\theta(1)(q-1), \theta(1)(q+1) \in cd(M)$. But $2 \mid (q-1)$ and r divides one of $q \pm 1$, hence s is adjacent to r and 2, a contradiction. If $q = 9$ and θ corresponds to a character of the Schur multiplier of order 3 or 6, then by [Conway et al., 1984], $6\theta(1) \in cd(M)$. Again a contradiction, as 2, p and s are adjacent in $\Delta(G)$. Therefore, θ is not invariant in M. Let $T = I_M(\theta)$. If $|M:T|$ is divisible by two primes among $\{2, p, r\}$, then by Clifford's Theorem, we obtain a character degree of G divisible by $|M:T|\theta(1)$, i.e., it is divisible by s and the two divisors of $|M:T|$, a contradiction. So $|M:T|$ is a prime power. By the lists of maximal subgroups given in [Conway et al., 1984], we have $q = 7$ and either $|M:T| = 7$ and $T/N \cong S_4$, or $|M:T| = 8$ and T/N is a Frobenius group of order 21. In the first case and as we discussed in Theorem 3.13, there exists a character degree in $cd(T|\theta)$ which is divisible by either 2 or 3. This gives a character degree of G divisible by 7, s and either 2 or 3, a contradiction. In the latter case and as mentioned in Theorem 3.13, the Sylow subgroups of T/N are cyclic, thus θ extends to T by Corollary 11.22 in [Isaacs, 1976]. Since $3 \in cd(T/N)$, we conclude by Gallagher's Theorem that $3\theta(1) \in cd(T|\theta)$, which gives a character degree in M divisible by $|M:T|3\theta(1)$. Hence G has a character which is divisible by 2, 3 and s, a contradiction. Thus no such G exists. □

Theorem 3.15. *There is no G that satisfies Hypothesis 3.12 if $p = 2$ and $|\rho(S)| = 4$.*

Proof. We have $q = 2^f$, then by Theorem 3.10, either:

i) f is odd, such that $2^f - 1$ is a Mersenne prime and $2^f + 1 = s^{\alpha} t^{\beta}$ for some distinct primes s, t and positive integers α, β; or;

ii) $f = 4$, such that $q + 1 = r = 17$ and $q - 1 = s \times t = 3 \times 5 = 15$.

As a consequence, s is adjacent to t in $\Delta(G)$. Let $\theta \in Irr(N)$. If θ is M-invariant, then θ extends to M, as the Schur multiplier of $PSL_2(q)$ is trivial. It follows then from Gallagher's Theorem that $\theta(1)(q-1)$, $\theta(1)(q+1) \in cd(G)$. Thus each prime divisor of $\theta(1)$ is adjacent to all other primes in $\rho(G)$, which leads to a contradiction. Hence, $\theta(1) = 1$. Let $T = I_M(\theta)$ and suppose $T < M$. We claim that r and a power of 2 divide $|M:T|$ and $a/\theta(1)$ is a power of 2 for all $a \in cd(T|\theta)$. Moreover, either $2 \mid |M:T|$, or there is a degree $a \in cd(T|\theta)$ which is divisible by 2. The claim will be proved by showing that the other possibilities of the Dickson's classification cannot occur (see Hauptsatz II.8.27 in [Huppert, 1967]). If T/N is abelian, then it is either an elementary abelian 2-subgroup, or a cyclic subgroup of order $q-1$ or $q+1$. In other words, T/N is an abelian 2-group, a cyclic group of order r, or a cyclic group whose order is the product of powers of the primes s and t. In the first case, since $|\rho(s)| = 4$ and $\pi(T/N) = \{2\}$, then $|M:T|$,which divides all characters in $cd(T|\theta)$, is divisible by the three primes in $\rho(S)$ which are different from 2, a contradiction. If T/N is cyclic of order r, then $|M:T|$ is divisible by 2, s and t, a contradiction. Thus, if T/N is abelian, then $|M:T|$ is divisible by both 2 and r. Remark that $q \mid |S|$ and $2 \nmid |M:T|$, so $q \mid |M:T|$. Thus, neither s nor t divides $|M:T|$ and $|M:T| = qr$. Since T/N is cyclic, it follows from Corollary 11.22 in [Isaacs, 1976] that θ extends to T, hence $cd(T|\theta) = \{\theta\}$.

Let T/N be nonabelian. Suppose first it is a dihedral group. Then $p = 2 \mid |M : T|$. It follows from the order of the dihedral group in Hauptsatz II.8.27 of [Huppert, 1967] , that if $r \mid |T : N|$, then $|T : N| = 2r$ and s, t will divide $|M : T|$ in this case, a contradiction as 2 also divides $|M : T|$. Hence $r \mid |M : T|$. It follows that neither s nor t divides $|M : T|$, and since $|T : N|_2 = 2$ (i.e. 2 is the largest 2-power that divides $|T : N|$), then $|M : T| = \frac{qr}{2}$. In this case all the Sylow subgroups of T/N are cyclic and thus θ extends to T by Corollary 11.22 and Corollary 11.31 of [Isaacs, 1976]. Notice that $cd(T/N) = \{1,2\}$, and by Gallagher's Theorem we conclude that $cd(T|\theta) = \{\theta(1), 2\theta(1)\}$.

If T/N is a Frobenius group with a Fitting subgroup F/N, then $(q+1) \mid |M : T|$, and F/N is a 2-group (discussed previously). Thus $q^2 - 1 \mid |M : F|$. If θ extends to T, then by Gallagher's Theorem we have, $|T : F|\theta(1) \in cd(T|\theta)$. It follows then by Clifford's Theorem that G has a character degree divisible by r, s and t, a contradiction. Therefore, θ does not extend to T, particularly, θ does not extend to F. Hence, by Clifford's Theorem, $2 \mid a/\theta(1)$ for all $a \in cd(T|\theta)$. Again by Clifford's Theorem, we know that there is a character degree in G divisible by $a|M : T|$. So, if $q+1$ is the product of a power of s and a power of t, then s, t and 2 are adjacent in $\Delta(G)$, a contradiction. Thus $q+1 = r$ and $r \mid |M : T|$. If a or $|M : T|$ is divisible by either s or t, then there is a degree in $cd(G)$ divisible by r, s and t, again a contradiction. Therefore, $|M : T|$ is a power of 2 times r and $a/\theta(1)$ is a power of 2 for each $a \in cd(T|\theta)$.

If $T/N \cong A_4$, and since $2^{2f} - 1 \equiv 0 \pmod 5$, then f is even and so $f = 4$. It follows that $|M : T|$ is divisible by 2, 5 and 17, a contradiction.

If $T/N \cong A_5$, then similar to previous we have $f = 4$. In this case 2, $17 \mid |M : T|$, and it follows from [Conway et al., 1984] that either $3\theta(1)$ or $6\theta(1)$ belongs to $cd(T|\theta)$, which implies that G has a character degree divisible by 2, 3 and 17, a contradiction.

Finally, if $M/T \cong PGL_2(4)$, again by Hauptsatz II.8.27 in [Huppert, 1967] we can see that $f = 4$. Also we have 2, $7 \mid |M : T|$, and by [Conway et al., 1984] we have $6\theta(1) \in cd(T|\theta)$. This implies that $cd(G)$ has a character divisible by 2, 3 and 17, a contradiction which proves our claim.

Now, if $\theta \in Irr(N)$, we know by Clifford's Theorem that G has some character degrees divisible by $a|M : T|$ for all $a \in cd(T|\theta)$. By previous, we have $a|M : T|$ is the product of a power of 2 and r. Thus $\theta(1)$ which divides $a|M : T|$ belongs to $\{2, r\}$. If θ is G-invariant, then $cd(G|\theta) \subseteq \{2, r\}$. If not, then $cd(G|\theta) = cd(G)$. In both cases we get a contradiction, since neither 2 nor r is adjacent to either s or t in $\Delta(G)$. $\square$

To complete the discussion, we need to consider the last case where $p=2$ and $|\rho(S)|=3$. The only possibilities here are $q=4$ or $q=8$. Notice that the case $q=5$ is included in the following theorem as $PSL_2(4)=PSL_2(5)$.

Theorem 3.16. *There is no G that satisfies Hypothesis* 3.12 *if $p=2$ and $|\rho(S)|=3$.*

Proof. As noted before, $S\cong PSL_2(4)$ or $PSL_2(8)$. In the first case, $\rho(S)=\{2,3,4\}$. In the second one, $\rho(S)=\{2,3,7\}$. Let r be a prime such that $\rho(G)=\rho(S)\cup\{r\}$. Consider $\chi\in Irr(G)$ such that $r\mid\chi(1)$, then by Corollary 11.29 of [Isaacs, 1976] we have $r\mid\theta(1)$. If θ extends to M, then by Gallagher's Theorem r is adjacent to the other three primes in $\rho(S)$, a contradiction. If θ is M-invariant but does not extend to M, then $S\cong PSL_2(4)$. By [Conway et al., 1984] we have $6\theta(1)\in cd(M)$, which gives a character degree in G divisible by 2, 3 and r, a contradiction. Hence $T=I_M(\theta)<M$. If $|M:T|$ is divisible by two or more primes, then $cd(G)$ contains a character divisible by three primes, a contradiction. Therefore, $|M:T|$ is a prime power. By the list of maximal subgroups of $PSL_2(q)$ found in [Conway et al., 1984], we can conclude that T/N is a Frobenius group with Frobenius kernel F/N. If $q=4$, we have $|M:T|=5$ and $|T:N|=3\times 2$, and when $q=8$, $|M:T|=3^2$ and $|T:N|=2^3\times 7=56$. So if θ extends to T, it follows from Gallagher's Theorem that $|T:F|\theta(1)\in cd(T|\theta)$. Otherwise, θ does not extend to F particularly. Thus each character degree in $cd(T|\theta)$ is divisible by all prime divisors of $|F:N|$. Therefore, if $q=4$ then $cd(T|\theta)$ has a degree divisible by 2 or 3. This gives a character degree of G divisible by 2 (or 3), 5 and r, a contradiction. Similarly, if $q=8$, we obtain some degrees in $cd(T|\theta)$ divisible by 2 or 7, which yields to a character degree of G divisible by 2 (or 7), 3 and r, a contradiction. Thus, no such G exists. $\square$

Theorem 3.17. *If G is a nonsolvable group, then $\Delta(G)$ is neither a P_3 nor a C_4.*

Proof. Combine results of Theorems 3.13, 3.14, 3.15 and 3.16. $\square$

4. GRAPHS OF FINITE GROUPS WITH THREE OR FOUR DEGREE-VERTICES

In this part, we discuss classifications of all graphs with three vertices that can occur as $\Gamma(G)$ for any finite group G. In addition, we classify all graphs with four vertices that can occur as $\Gamma(G)$ when G is nonsolvable.

Remark that if $\Gamma(G)$ has at most two vertices, then G is solvable by Theorems 12.5 and 12.15 of [Isaacs, 1976]. Also, $n(\Gamma(G)) = n(\Delta(G)) = n(B(G)) \leq 3$ for any finite group G by Theorem 6.4 of [Lewis, 2008], and $n(\Gamma(G)) \leq 2$ if G is solvable (see Corollary 4.2 of [Lewis, 2008]). Thus graphs composed of four isolated vertices cannot occur as $\Gamma(G)$. Moreover, if $\Gamma(G)$ is a complete graph, then G is solvable by Theorem 7.3 in [Lewis, 2008]. On the other hand, if G is nonsolvable and $\Gamma(G)$ is disconnected, then $n(\Gamma(G)) \leq 3$ and either one connected component is an isolated vertex and the other has diameter at most two if $n(\Gamma(G)) = 2$, or each component is an isolated vertex if $n(\Gamma(G)) = 3$ (Theorem 7.1 in [Lewis, 2008]) . Observe that if $\Gamma(G)$ has exactly $n \geq 1$ vertices, then $|cd(G)| = n+1$, or equivalently, $|cd(G/G')| = n$ (see Example 5 in Section 15 of [Alperin and Bell, 1991]).

We follow in this chapter lemmas and theorems of paper [LiGuo and GuoHua, 2015].

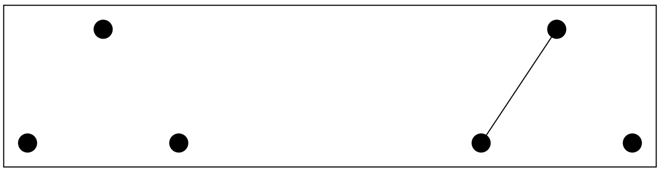

Figure 4.1: $\Gamma(G)$ when G is nonsolvable and $|cd(G)^*| = 3$.

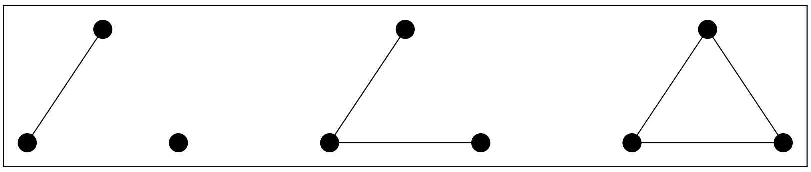

Figure 4.2: $\Gamma(G)$ when G is solvable and $|cd(G)^*| = 3$.

Lemma 4.1. *(Corollary B of [Malle and Moretó, 2005]) Let G be a nonsolvable group whose* $\Gamma(G)$ *has three vertices. Then either* $cd(G) = \{1, 9, 10, 16\}$ *or* $cd(G) = \{1, q, q-1, q+1\}$ *where q is a prime power strictly greater than* 3.

Theorem 4.2. *If G is a finite group whose $\Gamma(G)$ has three vertices, then $\Gamma(G)$ is one of the graphs found in Figures 4.1 or 4.2 depending on whether G is solvable or not.*

Proof. If $\Gamma(G)$ has three vertices with no edges, then $n(\Gamma(G)) = 3$, which implies by Corollary 4.2 in [Lewis, 2008] that G is nonsolvable. By Lemma 4.1, we know that for any nonsolvable group G with precisely three degree-vertices ($|cd(G)^*| = 3$), $cd(G)^*$ is either $\{9,10,16\}$ or $\{q-1,q,q+1\}$ for some prime power $q > 3$. Thus, an angle cannot occur in $\Gamma(G)$ if G is nonsolvable. As mentioned previously, if $\Gamma(G)$ is a complete graph, then G is solvable by Theorem 7.3 in [Lewis, 2008]. Based on these results, we can see easily that graphs in Figures 4.1 and 4.2 represent all possibilities of $\Gamma(G)$ when it has exactly three vertices. □

Examples 4.3.

i) *The alternating group A_5 is a nonsolvable group which is isomorphic to $PSL_2(4)$. So $cd(A_5) = \{1,3,4,5\}$ and $\Gamma(A_5)$ is the first graph in Figure 4.1.*

ii) *The symmetric group S_5 is a nonsolvable group which is isomorphic to $PGL_2(5)$. Thus $cd(S_5) = \{1,4,5,6\}$ and $\Gamma(S_5)$ is the second graph in Figure 4.1. Remark that, if q is an odd prime power, then $cd(PGL_2(q)) = \{1,q-1,q,q+1\}$.*

iii) *The character degree set of the general linear group $GL(2,3)$ is known as $\{1,2,3,4\}$. Thus the first graph in Figure 4.2 represents $\Gamma(GL(2,3))$. Recall that $GL(2,3)$ is the set of all 2×2 matrices of nonzero determinant defined over a field of order three. This group is solvable. Actually, it is the smallest solvable group whose derived length is 4.*

iv) *The product group $S_3 \times A_4$ is a solvable group, whose character degree set is $\{1,2\}\times\{1,3\} = \{1,2,3,6\}$ (see Example 6.3). Hence the second graph in Figure 4.2 represents its common divisor degree graph.*

v) *The extra special 3-group P of order 27, is a 3-group whose center $Z(P)$ is cyclic of order 3 and whose quotient $P/Z(P)$ is nontrivial elementary abelian 3-group. This group has exactly 3^2 linear representations and 2 nonlinear irreducible representations of order 3. Thus $cd(P) = \{1,3\}$. Notice that since P is a 3-group then it is solvable. Consider now $G = P\times P\times P$. It is clear that G is solvable and $cd(G) = \{1,3\}\times\{1,3\}\times\{1,3\} = \{1,3,3^2,3^3\}$ (see Example 2 p.153 in*

[Alperin and Bell, 1991]). Thus, $\Gamma(G)$ is the third graph in Figure 4.2.

The following three theorems will be useful in classifying common divisor degree graphs $\Gamma(G)$ for an almost simple group G.

Theorem 4.4. *(Theorem A in [White, 2013]) Let $S = PSL_2(q)$ for a prime power $q = p^f \geq 4$. Let G be an almost simple group with socle S. Set $N = S\langle\delta\rangle$ if $\delta \in G$ and $N = S$ otherwise. Let $\varepsilon = (-1)^{(q-1)/2}$ and $k = 2^a m = |G : N|$ where m is an odd integer. Then,*

$$cd(G) = \{1, q, (q+\varepsilon)/2\} \cup \{(q+1)j : j \mid k\} \cup \{(q-1)2^a l : l \mid m\}, \tag{4.1}$$

with the following exceptions:

i) If $p = 2$, or p is odd with $G \not\leq S\langle\phi\rangle$, then $(q+\varepsilon)/2 \notin cd(G)$.

ii) If $p = 3$, f is odd and $G = S\langle\phi\rangle$, then $l \neq 1$.

iii) If $p = 3$, f is odd and $G = Aut(S)$, then $j \neq 1$.

iv) If $p \in \{2,3,5\}$,f is odd and $G = S\langle\phi\rangle$, then $j \neq 1$.

v) If $p = 2$ or 3, $f \equiv 2$ (mod 4) and $G = S\langle\phi\rangle$ or $G = S\langle\delta\phi\rangle$, then $j \neq 2$.
Remark that δ and ϕ are both automorphisms described in Note 3.5.

By applying the result of Theorem 4.4, we can obtain the following consequence:

Theorem 4.5. *(Theorem 2.6 in paper [He and Zhu, 2012]) Assume the hypothesis of Theorem 4.4 is true. Then:*

i) If $\pi(G/N) \geq 4$, then $|cd(G|S)| \geq 6$.

ii) If $\pi(G/N) = 3$, then $|cd(G|S)| \geq 6$ unless $p = 2$, or p odd with $G \not\leq S\langle\phi\rangle$, in which case $|cd(G|S)| = 5$.

iii) If $\pi(G/N) = 2$, then $|cd(G|S)| \geq 6$ except possibly when k is prime.

iv) If $\pi(G/N) = 1$, then $|cd(G|S)| = 3$ except when $q > 5$ is odd, where $|cd(G|S)| = 4$.

Remark that the principal character of G, 1_G, does not belong to $Irr(G|S)$. Indeed, if $1_G \in Irr(G|S)$, then $1_G \in Irr(G|\theta)$ for some $\theta \neq 1_S$. Thus $[1_G, \theta^G] = [1_S, \theta] \neq 0$ (see Theorem 2.66). But since both 1_S and θ are irreducible characters of S, we conclude by Proposition 2.46 that $1_S = \theta$, a contradiction. Therefore, $1_G \notin Irr(G|S)$. So, if $|cd(G|S)| \geq m$, then $|cd(G)| \geq m+1$.

Theorem 4.6. *(Theorem 2.22 in [He and Zhu, 2012]) Let S be a nonabelian simple group such that $S \leq G \leq Aut(S)$. Then $|cd(G|S)| > 5$ except possibly when S is one of the following:*

i) $S \cong PSL_2(q)$, $q \geq 4$, *where* $|cd(G|S)| \geq 3$.

ii) $S \cong PSL_3(4)$ *and* G/S *is abelian, where* $|cd(G|S)| \geq 5$.

iii) $S \cong {}^2B_2(q^2)$, $q^2 \neq 2$, *where* $|cd(G|S)| \geq 5$.

Theorem 4.7. *Let S be a nonabelian simple group such that $S \leq G \leq Aut(S)$. Assume that $\Gamma(G)$ has four vertices, or equivalently $|cd(G)| = 5$. Then, $S \cong PSL_2(q)$ for a prime power $q = p^f > 5$. Furthermore, $\pi(G/N) \leq 2$ and we have the following:*

- *If $\pi(G/N) = 1$ and $G = PSL_2(q)$ for an odd $q > 5$, then*

$$cd(G) = \{1, q, (q+\varepsilon)/2, q-1, q+1\}. \tag{4.2}$$

- *If $\pi(G/N) = 2$,*

  i) $k = 2$, $p = 2$ *and* $G < S\langle\phi\rangle$. *In this case,*

$$cd(G) = \{1, q, 2(q-1), q+1, 2(q+1)\}. \tag{4.3}$$

  ii) $k = 2$, $p = 2$, $f = 2$ *and* $G = S\langle\phi\rangle$. *In this case,*

$$cd(G) = \{1, q, 2(q-1), q+1, 2(q+1)\}. \tag{4.4}$$

  iii) $k = 2$, $p = 3$, $f = 2$ *and* $G = S\langle\phi\rangle$. *In this case,*

$$cd(G) = \{1, q, (q+1)/2, 2(q-1), q+1\}. \tag{4.5}$$

  iv) $k = 2$, $p = 3$, $G \not\leq S\langle\phi\rangle$ *and* $f \equiv 0$ *(mod 4) or* $G \neq S\langle\delta\phi\rangle$. *In this case,*

$$cd(G) = \{1, q, 2(q-1), q+1, 2(q+1)\}. \tag{4.6}$$

v) $k = 2$, $p > 3$ and $G \not\leq S\langle\phi\rangle$. In this case,

$$cd(G) = \{1, q, 2(q-1), q+1, 2(q+1)\}. \tag{4.7}$$

vi) k is an odd prime, $p = 2$, $f = k$ and $G = S\langle\phi\rangle$. In this case,

$$cd(G) = \{1, q, q-1, k(q-1), k(q+1)\}. \tag{4.8}$$

vii) k is an odd prime, $p = 3$, $f = k$ and $G = S\langle\phi\rangle$. In this case,

$$cd(G) = \{1, q, (q+\varepsilon)/2, k(q-1), k(q+1)\}. \tag{4.9}$$

viii) k is an odd prime, $p = 3$, $f = k$ and $G = Aut(S)$. In this case,

$$cd(G) = \{1, q, q-1, k(q-1), k(q+1)\}. \tag{4.10}$$

Proof. Since $|cd(G)^*| = 4$, we can conclude by Theorem 4.6 that $S \cong PSL_2(q)$ where $q \geq 4$ and $|cd(G|S)| \geq 3$. By applying Theorem 4.5, we deduce that $\Gamma(G)$ can have exactly four vertices if either $\pi(G/N) = 2$ with k prime, or $\pi(G/N) = 1$ with $q > 5$ is odd. Finally, we check on a case-by-case basis via Theorem 4.4 to obtain the above results. □

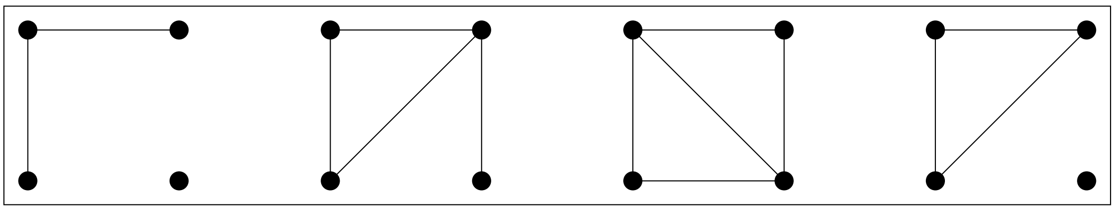

Figure 4.3: $\Gamma(G)$ when G is nonsolvable and $|cd(G)^*| = 4$.

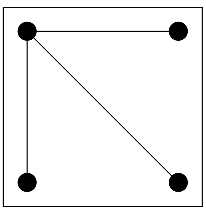

Figure 4.4: Unknown graph.

Corollary 4.8. *Let G be an almost simple group with a nonabelian socle S. Assume that $|cd(G)^*| = 4$. Then $\Gamma(G)$ is precisely one of the graphs listed in Figure 4.3.*

Proof. By checking the arithmetic properties of elements of $cd(G)^*$ when q is even or odd, we can see that in each case of Theorem 4.7, $\Gamma(G)$ is exactly one of the graphs found in Figure 4.3. $\square$

Examples 4.9.

i) *Let $S = PSL_2(9)$ and $G = S$. By case one of Theorem 4.7, we have $cd(G) = \{1,5,8,9,10\}$. Thus $\Gamma(G)$ is the first graph in Figure 4.3.*

ii) *Let $S = PSL_2(3^3)$ and $G = S\langle\phi\rangle$. Since $\delta \notin G$, we have $k = |G : N| = |\langle\phi\rangle| = f = 3$, which implies by case two vii) in Theorem 4.7 that $cd(G) = \{1,13,27,3\times 26,3\times 28\}$. Thus $\Gamma(G)$ is the second graph in Figure 4.3.*

iii) *Let $S = PSL_2(3^3)$ and $G = Aut(S)$. Since $\delta \in G$, then $N = S\langle\delta\rangle = PGL_2(3^3)$ and $k = |G : N| = |Aut(S) : PGL_2(3^3)| = f = 3$. Thus $cd(G) = \{1,26,27,3\times 26,3\times 28\}$ by case two viii) in Theorem 4.7 and so $\Gamma(G)$ is the third graph in Figure 4.3.*

iv) *Let $S = PSL_2(5^2)$ and $G = Aut(S)$. As $\delta \in G$, we have $N = PGL_2(25)$ and $k = |G : N| = |\langle\phi\rangle| = f = 3$. Thus $cd(G) = \{1,25,26,2\times 24,2\times 26\}$ by case two v) in Theorem 4.7 and $\Gamma(G)$ is the fourth graph in Figure 4.3.*

Thus for every graph Ω in Figure 4.3, there exists a nonsolvable group G whose $\Gamma(G) = \Omega$.

Table 4.1: Character degrees of almost simple groups with no prime dividing three character degrees ($\varepsilon = (-1)^{(q-1)/2}$).

Group G	$cd(G)$
$PSL_2(q), q > 5$ odd	$\{1,q-1,q,q+1,(q+\varepsilon)/2\}$
$PSL_2(q), q \geq 4$ even	$\{1,q-1,q,q+1\}$
$PGL_2(q), q$ odd	$\{1,q-1,q,q+1\}$
$PSL_2(3^2) \rtimes \mathbb{Z}_2 \cong PGL_2(9)$	$\{1,8,9,10\}$
$PSL_2(3^2) \rtimes \mathbb{Z}_2 \cong M_{10}$	$\{1,9,10,16\}$
$PSL_2(3^2) \rtimes \mathbb{Z}_2 \cong S_6$	$\{1,5,9,10,16\}$
$PSL_2(3^f) \rtimes \mathbb{Z}_f, f > 3$ prime	$\{1,3^f,(3^f-1)f,(3^f+1)f,(3^f-1)/2\}$
$PSL_2(2^2) \rtimes \mathbb{Z}_2 \cong S_5$	$\{1,4,5,6\}$
$PSL_2(2^f) \rtimes \mathbb{Z}_f, f > 2$ prime	$\{1,2^f-1,2^f,(2^f-1)f,(2^f+1)f\}$
$PSL_2(2^f) \rtimes \mathbb{Z}_r$, r odd prime, $r \mid f$, $r < f$	$\{1,2^f-1,2^f,2^f+1,(2^f-1)r,(2^f+1)r\}$

Theorem 4.10. *Let G be a nonsolvable group such that $\Gamma(G)$ has four vertices. Then $\Gamma(G)$ is one of the graphs shown in Figure 4.3 or 4.4.*

Proof. Since G is nonsolvable, $n(\Gamma(G)) \leq 3$ (see Theorem 6.4 in [Lewis, 2008]). If $n(\Gamma(G)) = 3$, then by Theorem 7.1 of [Lewis, 2008] we know that each connected component is an isolated vertex, but this contradicts the structure of $\Gamma(G)$ which has four vertices. Thus $n(\Gamma(G)) \leq 2$. Assume next that $n(\Gamma(G)) = 2$. By Theorem 7.1 in [Lewis, 2008], we can see that one connected component is an isolated vertex and the other has diameter at most 2. Therefore, $\Gamma(G)$ is either the first or the last graph in Figure 4.3. Suppose now that $\Gamma(G)$ is connected or equivalently $n(\Gamma(G)) = 1$. We discuss the rest of the proof according to whether $\Gamma(G)$ contains a triangle or not.

Assume first that $\Gamma(G)$ has no triangles. By Theorem 4 of [Lewis and White, 2011] we can deduce that no prime in $\rho(G)$ divides three distinct character degrees of G. This implies by Section 5 of [Lewis and White, 2011] that there exists $L \trianglelefteq G$ such that G/L is almost simple and no prime in $\rho(G)$ divides three degrees in $cd(G/L) \subseteq cd(G)$. Let $K/L \leq G/L \leq Aut(K/L)$ for some nonabelian simple group K/L. By Theorem 1 of [Lewis and White, 2011], we can see that $K/L \cong PSL_2(q)$ for a prime power $q \geq 4$. Furthermore, G/L is one of the following:

i) $S = PSL_2(q)$,

ii) $PGL_2(q)$ where q odd,

iii) $PSL_2(3^f) \rtimes \mathbb{Z}_f$ for some prime $f \neq 3$,

iv) $PSL_2(2^f) \rtimes \mathbb{Z}_f$ for a prime f,

v) $PSL_2(2^f) \rtimes \mathbb{Z}_r$ for an odd prime $r < f$ such that $r \mid f$, $r \nmid 2^f - 1$ and $r \nmid 2^f + 1$.

If $q \geq 7$ is odd, we conclude via Theorem 5.10 of [Lewis and White, 2011] that $L = Z(G)$ and $cd(G) = cd(G/L)$. Now by checking character degrees of the possible groups G/L shown in Table 4.1, we can see that all graphs $\Gamma(G) = \Gamma(G/L)$ that have four vertices are disconnected (see the first, sixth, seventh, ninth and tenth rows of Table 4.1). So this case is not possible. If $q = 2^f \geq 4$ and $K < G$, we can obtain via Theorem 5.12 of [Lewis and White, 2011] that $L = Z(G)$ and $cd(G) = cd(G/L)$. Similarly by checking Table 4.1, we can observe that all $\Gamma(G) = \Gamma(G/L)$ with four vertices are disconnected. Thus, this case in not possible too. If $q = 2^f \geq 4$ and $K = G$, we can conclude by Theorem 5.15 of [Lewis and White, 2011] that $cd(G) = \{1, q-1, q, q+1, a\}$, where

$f \geq 2$ and $a > q+1$. Indeed, a has common prime divisors with at least two numbers among $\{q-1, q, q+1\}$. Now as $\{q-1, q, q+1\}$ is a pairwise relatively prime set when q is even, we deduce that $\Gamma(G)$ is either the first graph in Figure 4.3 or the graph shown in Figure 4.4, according to whether a has common divisors with two or three integers among $\{q-1, q, q+1\}$.

Finally, assume that $\Gamma(G)$ contains at least one triangle. Since $\Gamma(G)$ is connected but not complete (see Theorem 7.3 of [Lewis, 2008]), it has at least one triangle and $|cd(G)^*| = 4$, we conclude that $\Gamma(G)$ is either the second or the third graph in Figure 4.3. $\square$

5. GROUPS WHOSE PRIME GRAPHS HAVE NO TRIANGLES

We discuss in this chapter, finite groups whose prime graphs have no triangles. We start by claiming that prime graphs of such groups have at most five vertices. Then we obtain a classification of finite graphs with five vertices and no triangles that can occur as the prime graph of a finite group G. Finally, we claim that the prime graph of any finite group cannot be a cycle or a tree with at least five vertices. We remark that a finite group G is said to be almost simple with socle S, if there exists a nonabelian simple group S, such that $S \trianglelefteq G \leq Aut(S)$.

For $\varepsilon = \pm$, the convention $PSL_n^{\varepsilon}(q)$ means $PSL_n(q)$ if $\varepsilon = +$ and $PSU_n(q)$ if $\varepsilon = -$.

We follow in this chapter lemmas and theorems of paper [Tong-Viet, 2013].

Theorem 5.1. *Let G be a finite group such that $\Delta(G)$ has no triangles, then $|\rho(G)| \leq 5$.*

Theorem 5.2. *Let G be a finite group such that $|\rho(G)| = 5$ and $\Delta(G)$ has no triangles, then one of the following holds:*

i) If $\Delta(G)$ is disconnected, then $G \cong PSL_2(2^f) \times A$, where A is an abelian group, $|\pi(2^f \pm 1)| = 2$ and $\Delta(G)$ is the second graph in Figure 5.1;

ii) If $\Delta(G)$ is connected, then $G = H \times K$ such that $H \cong A_5$ or $PSL_2(8)$, K is a solvable group whose prime graph has exactly two vertices and two connected components, and $\rho(H) \cap \rho(K) = \phi$. Furthermore $\Delta(G)$ is the first graph in Figure 5.1.

Theorem 5.3. *If G is a finite group where $\Delta(G)$ is a cycle or a tree, then $|\rho(G)| \leq 4$.*

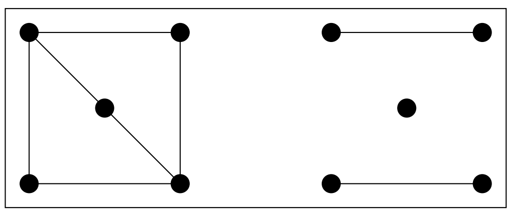

Figure 5.1: Prime graphs of finite groups having five vertices and no triangles.

In the following we present a series of lemmas and results that will be needed in the proofs of our main theorems.

Lemma 5.4. *Let G be solvable such that $\Delta(G)$ has no triangles, then $|\rho(G)| \leq 4$.*

Proof. If $|\rho(G)| \leq 3$, we are done. Suppose G is solvable, $\Delta(G)$ has no triangles and at least four vertices. We claim that $\Delta(G)$ is a square (C_4). If $\Delta(G)$ is disconnected then by Corollary 4.2 of [Lewis, 2008] we know that $\Delta(G)$ has two connected components which are complete graphs. Since K_m contains a triangle for all $m \geq 3$, we deduce that each connected component can have at most two vertices. On the other hand, Pálfy proved that if G is a solvable group whose $\Delta(G)$ has two connected components C_1 and C_2 such that $|C_1| = n$ and $|C_2| = m$ with $n \leq m$, then $m \geq 2^n - 1$ (see Theorem 3 in [Pálfy, 2001]). This implies that there does not exist any disconnected prime graph such that each connected component has two vertices. Therefore $\Delta(G)$ must be connected. If $\Delta(G)$ has a vertex of degree at least 3, then Pálfy's condition implies that there must be an edge between at least two of its neighbors, a contradiction as $\Delta(G)$ contains a triangle in this case. Hence, any vertex in $\Delta(G)$ has at most degree 2, which means that $\Delta(G)$ is either a path or a cycle. Again, Pálfy's condition excludes paths with five or more vertices and cycles with six or more vertices. Remark that, the prime graph of a solvable group cannot be a P_3 by Theorem 4.5 in [Lewis, 2008]. Finally, by using Lewis's result which says that there does not exist any finite solvable group G whose $\Delta(G)$ is a C_5 (see [Lewis, 2004]), we conclude that $\Delta(G)$ is a square. Thus if G is solvable and its prime graph has no triangles, then $|\rho(G)| \leq 4$. $\square$

Theorem 5.5. *(Zsigmondy's Theorem [Zsigmondy, 1892]) Let $a, n \geq 2$ be integers. Then there exists a prime l such that $l \mid a^n - 1$ and $l \nmid a^m - 1$ for all $1 \leq m < n$ unless either $n = 6$ and $a = 2$, or $n = 2$ and $a = 2^r - 1$ is a Mersenne prime , for some prime integer r.*

We call such a prime the *primitive prime divisor*, and for any integers $a, n \geq 2$ the smallest prime divisor of $a^n - 1$; if exists; is denoted by $l_n(a)$.

Lemma 5.6. *Let $f = nb \geq 6$ be an integer such that $n \geq 3$ is a prime and $b \geq 2$ is an* *integer. Then* $(2^{2f}-1)/(2^{2b}-1)$ *is not a prime power.*

Proof. Assume on the contrary that $(2^{2f}-1)/(2^{2b}-1) = r^m$ for a prime r and a positive integer m. If $f = 6$, then $b = 2$ and $n = 3$. Thus $(2^{2\times 6}-1)/(2^{2\times 2}-1) = 3.7.13$ is not a prime power. Hence we may assume that $f > 6$, which implies that $2f > f > 6$. It follows then from Zsigmondy's theorem that both $l_f(2)$ and $l_{2f}(2)$ exist, they are distinct, and do not divide $2^{2b}-1$ ($f > 2b$, $n \geq 3$). Therefore $l_f(2)$ and $L_{2f}(2)$ must divide r^m, i.e. $l_{2f}(2) = l_f(2) = r$, a contradiction. □

Lemma 5.7. *(Theorem 11.7 of [Isaacs, 1976]) Let G be a finite group and* $N \trianglelefteq G$*. Assume that* $\theta \in Irr(N)$ *is G-invariant and the Schur multiplier of* G/N *is trivial. Then* θ *extends to G.*

The next lemma is a consequence of Guralnick's classification of subgroups with prime power indices in abelian simple groups [Guralnick, 1983].

Lemma 5.8. *Let H be a proper subgroup of a nonabelian simple group G. If* $|G:H| = r^a$ *for some prime r and a positive integer a, then H is either nonsolvable or a nonabelian Hall subgroup of G.*

Proof. Guralnick's classification of prime power index subgroups of finite simple nonabelian groups implies that one of the following holds:

i) $(G,H) = (A_n, A_{n-1})$ where $n = r^a$;

ii) $G \cong PSL_n(q)$, H is the stabilizer of a line or a hyperplane, and

$$r^a = (q^n-1)/(q-1) \tag{5.1}$$

for a prime n.

iii) $(G,H,r^a) = (PSL_2(11), A_5, 11)$ or $(PSL_4(2), 2^4 : A_5, 3^3)$;

iv) $(G,H,r^a) = (M_{23}, M_{22}, 23)$ or $(M_{11}, M_{10}, 11)$.

Suppose the first case holds, then $G \cong A_n$, $H \cong A_{n-1}$ and $r^a = n$. If $n = 5$, then $(|A_4|, |A_5 : A_4|) = (24,5) = 1$, i.e., $H \cong A_4$ is a nonabelian Hall subgroup of $G \cong A_5$. So we may assume that $n \geq 7$. Hence $H \cong A_{n-1}$ is nonsolvable as $n-1 \geq 6$.

Suppose next that case ii) occurs. If $n=2$, then $G \cong PSL_2(q)$ with $q \geq 4$ and H is a nonabelian group such that $|H| = q(q-1)/(2,q-1)$ and $r^a = q+1$. We claim that $q(q-1)$ and $q+1$ are relatively prime, which implies that H is a Hall subgroup of G. If q is even, we know from number theory that $\{q, q-1, q+1\}$ is a pairwise relatively prime set, hence $(|H| = q(q-1), |G:H| = q+1) = 1$. Otherwise, q is odd, which means that $q+1 = r^a$ is even, but r is a prime, thus $r=2$. Since $q \geq 4$, it follows that $a \geq 3$, which implies that $(q-1)/2 = 2^{a-1}-1$ is odd. Hence, $|H| = q(q+1)/2$ is odd and it is relatively prime with $q+1 = 2^a$. I.e., H is a Hall subgroup. If $n=3$, then $G \cong PSL_3(q)$, H is the stabilizer of a line or a hyperplane and $|G:H| = (q^3-1)/(q-1) = r^a$. If $q=2$, then $G \cong PSL_3(2)$ and $H \cong S_4$ such that $|G:H| = r^a = 7$, thus H is a nonabelian Hall subgroup of G. Similarly for the case $q=3$ where $G \cong PSL_3(3)$ and $H \cong 3^2 : 2S_4$ with $|G:H| = 13$. Assume now that $q \geq 4$. In this case, it is well-known that H possesses a section isomorphic to the nonabelian simple group $PSL_2(q)$, thus H is not solvable.

Finally suppose that $G \cong PSL_n(q)$ such that $n \geq 5$ and H is the stabilizer of a line or a hyperplane. In this case, it is well-known that H possesses a section which is isomorphic to $PSL_{n-1}(q)$, which is nonsolvable as $n-1 \geq 4$. Hence, H is nonsolvable.

For the last two cases, we remark that none of A_5, M_{22}, M_{10} and $2^4 : A_5$ is solvable. Thus, H is nonsolvable in these cases. □

Notice that, if G is an almost simple group such that $S \trianglelefteq G \leq Aut(S)$, then any Sylow p-subgroup of G is not normal. Indeed, if there exists a normal Sylow p-subgroup, say P, we can see that $S \cap P \trianglelefteq S$ as it is normal in G. Thus, either $S \cap P = 1$ or $S \cap P = S$ (S is simple). The latter case cannot occur since if $S < P$, then S is a nontrivial p-group, hence $Z(S) \neq 1$, a contradiction as S is simple. So, $P \cap S = 1$, which implies that P centralizes S, a contradiction as $C_G(S)$ is trivial by the definition of an almost simple group. Therefore, for every $p \in \pi(G)$, G has no nontrivial normal abelian Sylow p-subgroup, which implies by Ito-Michler's Theorem that $\rho(G) = \pi(G)$.

In the following two lemmas, we classify the almost simple groups G such that $S \leq G \leq Aut(S)$ and $\Delta(G)$ has no triangles. We start by considering the nonabelian simple groups, then we take a general almost simple group G with socle S. It should be mentioned that proofs of these lemmas are based on the following classifications of prime graphs of simple groups due to D. White:

- Degree graphs of simple groups of exceptional Lie type [White, 2004].

- Degree graphs of simple linear and unitary groups [White, 2006].
- Degree graphs of simple orthogonal and symplectic groups [White, 2008].

Table 5.1: Simple groups with $|\pi(S)| = 3$ (See [Huppert and Lempken, 2000]).

G	$\|G\|$
$A_5 \cong PSL_2(4) \cong PSL_2(5)$	$2^2.3.5$
$A_6 \cong PSL_2(9) \cong S_{P4}(2)'$	$2^3.3^2.5$
$PS_{P4}(3) \cong PSU_4(2)$	$2^6.3^4.5$
$PSL_2(7) \cong PSL_3(2)$	$2^3.3.7$
$PSL_2(8) \cong {}^2G_2(3)'$	$2^3.3^2.7$
$PSU_3(3) \cong G_2(2)'$	$2^5.3^3.7$
$PSL_3(3)$	$2^4.3^3.13$
$PSL_2(17)$	$2^4.3^2.17$

Lemma 5.9. *If S is a nonabelian simple group such that $\Delta(S)$ has no triangles, then either:*

- *$S \cong PSL_2(2^f)$ such that $|\pi(2^f \pm 1)| \leq 2$ and so $|\pi(S)| \leq 5$; or;*
- *$S \cong PSL_2(q)$ such that $q = p^f$ is an odd prime power and $|\pi(q \pm 1)| \leq 2$ and so $|\pi(S)| \leq 4$.*

Proof. As S is a nonabelian simple group, we have $\rho(S) = \pi(S)$. And since any group with at most two prime divisors is solvable (Burnside's Theorem), we can assume that $|\pi(S)| \geq 3$. Suppose first that $S \cong PSL_2(q)$, where $q \geq 4$ is a prime power. As mentioned previously $PSL_2(4) \cong PSL_2(5)$, so we may assume that $q > 5$ when q is odd. Remark that $|S| = q(q-1)(q+1)/(2, q-1)$. Since $\Delta(S)$ has no triangles, we conclude that $|\pi(a)| \leq 2$ for all $a \in cd(S)$. If $q > 5$ is odd, then $q^2 - 1$ can have at most two odd prime divisors distinct from p, thus $|\pi(PSL_2(q))| \leq 4$. If q is even, then $|\pi(q \pm 1)| \leq 2$, which implies that $|\pi(S)| \leq 5$. Suppose now that $S \ncong PSL_2(q)$. It follows from Corollary 1.2 in [White, 2008] that $\Delta(G)$ is connected and either $\Delta(S)$ is complete or one of the following cases holds:

i) $S \in \{M_{11}, M_{23}, J_1, A_8\}$;

ii) $S \cong {}^2B_2(q^2)$, with $q^2 = 2^{2m+1}$ and $m \geq 1$;

iii) $S \cong PSL_3^{\varepsilon}(q)$, where $q > 2$ is a prime power and $\varepsilon = \pm$.

If $\Delta(S)$ is complete and since $|\pi(S)| \geq 3$, then it has a triangle. So we may assume that $\Delta(S)$ is not complete. The character tables of groups in case *i*) are found in [Conway et al., 1984] and show that the prime graph contains a triangle in each case, which implies a contradiction. Suppose case *ii*) occurs, then $S \cong {}^2B_2(q^2)$ with $q^2 = 2^{2m+1}$ and $m \geq 1$. It follows from Theorem 3.3 in [White, 2004] that $\pi(S) = \{2\} \cup \pi(q^2-1) \cup \pi(q^4+1)$ where the subgraph of $\Delta(S)$ on $\pi(S) \setminus \{2\}$ is complete. In this case $|\pi(S)| \geq 4$ and so $|\pi(S) \setminus \{2\}| \geq 3$. Therefore, $\Delta(S)$ has a triangle since $\pi(S) \setminus \{2\}$ is a complete graph with at least three vertices. Thus this case cannot occur too. Finally, suppose that $S \cong PSL_3^{\varepsilon}(q)$ where $q = p^f > 2$ and $\varepsilon = \pm$. If $q = 4$, then it follows from the character tables of $PSL_3^{\varepsilon}(4)$ in [Conway et al., 1984] that $\Delta(PSL_3^{\varepsilon}(4))$ has at least a triangle, which is not possible. So $q \neq 4$. By Theorems 3.2 and 3.4 of [White, 2006], we can see that $\pi(S) = \{p\} \cup \pi(q^2-1) \cup \pi(q^2+\varepsilon q+1)$ and the subgraph of $\Delta(S)$ on $\pi(S) \setminus \{p\}$ is complete. Hence, if $|\pi(S)| \geq 4$ and since $\pi(S) \setminus \{p\}$ is a complete subgraph with at least three vertices, $\Delta(S)$ possesses a triangle, a contradiction. Thus $|\pi(S)| = 3$. It follows then from Table 5.1 that $q = 3$. By the character tables of $PSL_3(3)$ and $PSU_3(3)$ given in [Conway et al., 1984], we can see that in each case $\Delta(S)$ is a triangle, a contradiction. Hence, all other cases different from $PSL_2(q)$ cannot occur. $\square$

Lemma 5.10. *Let G be an almost simple finite group such that $S \leq G \leq Aut(S)$, where S is a nonabelian simple group. Assume that $\Delta(G)$ has no triangles. Then $S \cong PSL_2(q)$ where $q = p^f \geq 4$ is a prime power, $\pi(S) = \pi(G)$ and $|\pi(G)| \leq 5$. Furthermore, if $|\pi(G)| = 5$, then $G = S \cong PSL_2(2^f)$ with $f \geq 6$ and $|\pi(2^f \pm 1)| = 2$.*

Proof. As $S \trianglelefteq G$ and $\Delta(G)$ has no triangles, then $\Delta(S)$ is a subgraph of $\Delta(G)$ with no triangles too. Thus every degree of S and G has at most two prime divisors. By Lemma 5.9, we obtain that $S \cong PSL_2(q)$ for a prime power $q = p^f \geq 4$ such that $|\pi(q \pm 1)| \leq 2$, and we may assume that $q \neq 5$. We claim that $\pi(G) = \pi(S)$. Suppose on the contrary that $\pi(G) \neq \pi(S)$ and let $r \in \pi(G) \setminus \pi(S)$. It follows that $r \in \pi(|G:S|)$. If q is even, then $PGL_2(q) = PSL_2(q)$ and $|G : G \cap PGL_2(q)| = |G:S|$. If q is odd, then $|G \cap PGL_2(q) : S| = 1$ or 2 according to whether the diagonal automorphism belongs to G or not (see Note 3.5). If it belongs to G, then $G \cap PGL_2(q) = PGL_2(q)$, which implies that $|PGL_2(q) : S| = (q-1, 2) = 2$. Since $2 \in \pi(S)$, we conclude in this case that if $r \in \pi(|G:S|)$, then $r \in \pi(|G : PGL_2(q)|) = \pi(|G : G \cap PGL_2(q)|)$. Otherwise, $G \cap PGL_2(q) = S$, which implies that $|G:S| = |G : G \cap PGL_2(q)|$. In all cases we deduce that $r \mid m := |G : G \cap PGL_2(q)|$. But $m \mid |Aut(S) : S| = |Out(S)| = (q-1,2)f$.

Therefore r must divide f. Remark that $\{2,3\} \subseteq \pi(S) = \rho(S)$ for all prime power q, and as $r \notin \pi(S)$, we deduce that $f \geq 5$. Hence $q > 9$. If follows from Lemma 4.5 in [Lewis and White, 2011] that $m(q \pm 1) \in cd(G)$. If either $|\pi(q+1)| = 2$ or $|\pi(q-1)| = 2$, then $m(q+1)$ or $m(q-1)$ is divisible by three primes, which gives a triangle in $\Delta(S)$, a contradiction. Thus each of $q \pm 1$ is a prime power. But this can occur if and only if q is even. Precisely, q must be 4 or 8, a contradiction since $q > 9$. Therefore, $\pi(G) = \pi(S)$. Now by applying previous lemma we obtain that $|\pi(G)| = |\pi(S)| \leq 4$ for an odd q and $|\pi(G)| = |\pi(S)| \leq 5$ for an even q. Thus, $|\pi(G)| = |\pi(S)| \leq 5$ for all q. Finally, assume that $|\pi(S)| = |\pi(G)| = 5$. By the above argument, we must have q even and $|\pi(q \pm 1)| = 2$. So $f \geq 6$, because if $f = 5$ then $2^5 - 1 = 31$ is divisible by only one prime which is not the case. We claim now that $G = S$. Suppose that $S < G$. Then there exists some prime r such that $r \mid |G : S|$. By Lemma 4.5 of [Lewis and White, 2011] we know that both $|G : S|(q \pm 1)$ are degrees in $cd(G)$. If $r \nmid (q^2 - 1)$, then $|G : S|(q-1)$ is divisible by three distinct primes, a contradiction. Thus $r \mid (q^2 - 1)$. As $(q-1, q+1) = 1$, we conclude that $r \in \pi(q - \alpha)$ where $\alpha = 1$ or -1. And since $r \notin \pi(q + \alpha)$, we deduce that $|G : S|(q + \alpha)$ is divisible by three distinct primes, again a contradiction. Thus $G = S$ as required. □

We consider now G to be a nonsolvable or a general group. And we discuss several auxiliary lemmas that will be needed in the proof of our main results. Some of these lemmas are special cases of the main theorems. Recall that the *solvable radical* of G is the largest solvable normal subgroup in G.

Lemma 5.11. *Let N be the solvable radical of a finite nonsolvable group G whose $\Delta(G)$ has no triangles. Then there exists $M \trianglelefteq G$ such that $M/N \cong PSL_2(q)$ where $q \geq 4$ is a prime power, G/N is an almost simple group with socle M/N and $\rho(M) = \rho(G)$.*

Proof. Let N be the solvable radical of G and take M to be a normal subgroup such that M/N is a chief factor of G. By applying the same discussion in Lemma 3.11, we can conclude that M/N is a nonabelian simple group. Let $C/N = C_{G/N}(M/N)$. Then $C \trianglelefteq G$ as $C/N \trianglelefteq G/N$ (being the centralizer of a normal subgroup). Also $M \cap C = N$ as M/N is a nonabelian simple group. Assume that $N \neq C$. Then C is nonsolvable and we can find a normal subgroup L such that $N \leq L \leq C$ and L/N is a nonabelian chief factor of G. By a similar argument to that found in Lemma 3.11, we can deduce that L/N is a nonabelian simple group and every vertex in $\pi(L/N) \cap \pi(M/N)$ is adjacent to all vertices in $\pi(L/N) \cup \pi(M/N)$. Thus the subgraph of $\Delta(G)$ induced by $\pi(L/N) \cap \pi(M/N)$ is complete and so $|\pi(L/N) \cap \pi(M/N)| \leq 2$. By Lemma 5.10 we know that $L/N \cong PSL_2(q')$ and $M/N \cong PSL_2(q)$ for some prime powers q and q'. In particular $2, 3 \in \pi(L/N) \cap \pi(M/N)$, thus $\rho(L/N) \cap \rho(M/N) = \{2,3\}$. As $|\pi(M/N)| \geq 3$, M/N has a divisor $r > 3$. This implies that 2, 3 and r form a triangle in $\Delta(G)$, a contradiction. Therefore $C = N$. Thus G/N is an almost simple group with socle $M/N \cong PSL_2(q)$ where $q \geq 4$ is a prime power. We claim next that $\rho(M) = \rho(G)$. As mentioned previously, since $M \trianglelefteq G$, $\rho(M) \subseteq \rho(G)$. Let $r \in \rho(G)$, then there exists $\chi \in Irr(G)$ such that $r \mid \chi(1)$. Consider $\theta \in Irr(N)$ such that $[\theta, \chi_N] \neq 0$. By Clifford's Theorem we know that r divides either $\theta(1)$ or $\chi(1)/\theta(1)$. Suppose first that $r \mid \chi(1)/\theta(1)$, then $r \in \pi(|G:N|)$ as $\chi(1)/\theta(1)$ divides $|G:N|$ (see Corollary 11.29 of [Isaacs, 1976]). If $r \mid \theta(1)$, then $r \in \rho(N)$. Therefore, $r \in \pi(G/N) \cup \rho(N)$. But since $\pi(G/N) = \pi(M/N)$, we conclude that $r \in \pi(M/N) \cup \rho(N)$ (see Lemma 5.10). Notice that $\pi(M/N) = \rho(M/N) \subseteq \rho(M)$ and $\rho(N) \subseteq \rho(M)$ as $N \trianglelefteq M$. Thus, $r \in \rho(M)$ and we obtain the second inclusion. Hence, $\rho(M) = \rho(G)$ as required. □

The next result is frequently used. The proof of this lemma is based on Guralnick's classification of subgroups of prime power index in nonabelian simple groups (see Lemma 5.8), and the following result which is due to Higgs: If N is a normal subgroup in a finite group G and $\theta \in Irr(N)$ is G-invariant such that $\chi(1)/\theta(1)$ is a power of a fixed prime p for all $\chi \in Irr(G|\theta)$, then G/N is solvable (see Theorem 2.3 of [Moretó, 2006]).

Lemma 5.12. *Let G be a finite group and $N \trianglelefteq G$ such that $G/N \cong S$ where S is a nonabelian simple group. Let $\theta \in Irr(N)$. Then either there exists $\chi \in Irr(G|\theta)$ such that $\chi(1)/\theta(1)$ is divisible by two distinct primes in $\pi(G/N)$ or θ extends to G and $G/N \cong A_5$ or $PSL_2(8)$.*

Proof. Let $\theta \in Irr(N)$. It follows from Corollary 11.29 of [Isaacs, 1976], that $\chi(1)/\theta(1)$ divides $|G:N|$ for all $\chi \in Irr(G|\theta)$. Thus if $\chi(1)/\theta(1)$ is divisible by two distinct primes, then these primes belong to $\pi(G/N) = \pi(|G:N|)$. So we can assume that

$\chi(1)/\theta(1)$ is not divisible by two distinct primes in $\pi(G/N)$, or equivalently $\chi(1)/\theta(1)$ is a prime power for all $\chi \in Irr(G|\theta)$. Then, we claim that θ extends to G with $G/N \cong A_5$ or $PSL_2(8)$.

First, we claim that θ is G-invariant. Let $I = I_G(\theta)$ and suppose on the contrary that $I < G$. Since $N \trianglelefteq G$, then I/N is a proper subgroup of G/N. By Clifford's Theorem we can write: $\theta^I = \sum_{i=1}^{m} e_i\phi_i$ where $\phi_i \in Irr(I|\theta)$ and $m \geq 1$. Again, Clifford's Theorem and Theorem 6.11 of [Isaacs, 1976] imply that for every i, we have $\phi_i^G \in Irr(G|\phi)$ and $\phi_i^G(1) = |G:I|e_i\theta(1) \in cd(G)$. Hence, $|G:I|e_i$ is a prime power for all $1 \leq i \leq m$. In particular, we have $|G:I| = r^a$ for a prime r and a positive integer a. By Lemma 5.8, we know that I/N is either nonsolvable or a nonabelian Hall subgroup of G/N. Suppose the latter case holds. If for every i, $e_i = 1$, we conclude by Clifford's Theorem that $(\phi_i)_N \in Irr(N)$. Then by Gallagher's Theorem we have $\phi_i\beta \in Irr(I)$ for all $\beta \in Irr(I/N)$. But we supposed $e_i = 1$ for all i, thus $\phi_i\beta$ must be ϕ_i for all $1 \leq i \leq m$, which implies that $\beta(1) = 1$ for all $\beta \in Irr(I/N)$. Hence I/N is abelian, a contradiction. Therefore, there exists $1 \leq j \leq m$ such that $e_j > 1$. As $e_j \mid |I:N|$ and $(|I:N|, |G:I|) = (|I:N|, |G/N:I/N|) = 1$, we conclude that $(e_j, r) = 1$. So $\phi_j^G(1)/\theta(1) = r^a e_j$ is divisible by at least two prime divisors, a contradiction. Suppose now the first case holds, i.e., I/N is nonsolvable. It follows from Theorem 2.3 of [Moretó, 2006] that there exists some $1 \leq k \leq m$ such that $\phi_k(1)/\theta(1)$ is not a power of r. This implies that $\phi_k(1)/\theta(1)$ has another prime divisor s distinct from r. So $|\pi(\phi_k^G(1)/\theta(1))| \geq 2$, a contradiction. Therefore θ is G-invariant.

Assume now that θ does not extend to G. In the sense of Chapter 11 in [Isaacs, 1976], we have (G,N,θ) is a character triple isomorphic to the triple (L,A,λ) where L is perfect, $A \leq Z(L)$, $L/A \cong G/N$ and $\lambda \in Irr(A)$ is nontrivial. Since λ is not trivial, then $ker(\theta) \neq G$, which implies that $o(\lambda) = |G:ker(\lambda)|$ is divisible by a prime p. By applying Lemma 2.1 of [Moretó, 2006] to the perfect group L and to the L-invariant linear character λ, we obtain that $o(\lambda) \mid \chi(1)$ for all $\chi \in Irr(L|\lambda)$. Thus $p \mid \chi(1)$ for all $\chi \in Irr(L|\lambda)$, which implies that $\chi(1) = \chi(1)/\lambda(1)$ is a nontrivial p-power for all $\chi \in Irr(L|\lambda)$. Hence $L/A \cong G/N$ is solvable by Theorem 2.3 in [Moretó, 2006]. But this implies a contradiction as G/N is a nonabelian simple group. Therefore θ is extendible to $\theta_0 \in Irr(G)$. It follows then by Gallagher's Theorem that $\theta_0\psi \in Irr(G|\theta)$ for all $\psi \in Irr(G/N)$. Hence, $\theta_0(1)\psi(1)/\theta(1) = \psi(1)$ is a prime power for all

$\psi \in Irr(G/N)$. Finally, by applying Corollary of [Manz et al., 1988] to the nonsolvable group G/N we obtain that $G/N \cong B \times S$, where B is abelian and $S \in \{A_5, PL_2(8)\}$. But since G/N is a simple group, we conclude that $G/N \cong PSL_2(8)$ or A_5. $\square$

Lemma 5.13. *Let G be a finite group and $N \trianglelefteq G$ such that $G/N \cong PSL_2(2^f)$, $|\rho(G)| = |\pi(G/N)| = 5$ and $\Delta(G)$ has at most two connected components. Then $\Delta(G)$ contains a triangle.*

Proof. We proceed by contradiction. Suppose that $\Delta(G)$ has at most two connected components with no triangles. Since $|\pi(G/N)| = 5$ and $\Delta(G/N)$ must have no triangles too, we conclude by Lemma 5.10 that $|\pi(2^f \pm 1)| = 2$ and $f \geq 6$. Recall that $cd(PSL_2(2^f))^* = \{2^f - 1, 2^f, 2^f + 1\}$ is a pairwise relatively prime set. Hence $\{2\}$, $\pi(2^f - 1)$ and $\pi(2^f + 1)$ are the three connected components of $\Delta(G/N)$.

We split now our discussion according to whether $\Delta(G)$ is connected or not.

If $\Delta(G)$ is connected. Then the vertex 2 is adjacent to a vertex r, where $r \in \pi(2^f - 1)$ or $r \in \pi(2^f + 1)$, and r is odd as both $2^f \pm 1$ are so. Hence G has an irreducible character χ such that $\pi(\chi(1)) = \{2, r\}$. Let $\theta \in Irr(N)$ be an irreducible constituent of χ_N. If θ is the principal character of N, then $[\chi_N, 1_N] \neq 0$, which implies by Corollary 6.7 of [Isaacs, 1976] that $N \subseteq ker\chi$. So χ corresponds to an irreducible character of G/N, a contradiction as 2 is an isolated vertex in $\Delta(G/N)$. Thus θ is a nontrivial character.

Suppose that θ is G-invariant. By a result of Steinberg we know that the Schur multiplier of $PSL_2(q)$ is isomorphic to $\mathbb{Z}/(q-1,2)\mathbb{Z}$ for all prime powers q except 4 and 9. Since $f \geq 6$, we conclude that the Schur multiplier of $G/N \cong PSL_2(2^f)$ is trivial. Thus θ extends to $\theta_0 \in Irr(G)$ by Lemma 5.7. It follows then from Gallagher's Theorem that $Irr(G|\theta) = \{\theta_0\lambda \mid \lambda \in Irr(G/N)\}$. In particular, as $\chi \in Irr(G|\theta)$ we have $\chi = \theta_0\mu$ where $\mu \in Irr(G/N)$. Since $\mu \in Irr(G/N)$, then $\mu(1) \in \{1, 2^f - 1, 2^f, 2^f + 1\}$. Assume first that $\mu(1) = 1$ or 2^f, then $r \mid \theta_0(1)$ as $\pi(\chi(1)) = \pi(\theta(1)) \cup \pi(\mu(1)) = \pi(\theta_0(1)) \cup \pi(\mu(1))$. Thus $r \in \pi(2^f + 1)$ or $\pi(2^f - 1)$. If $r \in \pi(2^f + 1)$, then by taking $\eta \in Irr(G/N)$ with $\eta(1) = 2^f - 1$, we obtain a character of G, $\theta_0(1)\eta(1)$, which is divisible by r and the two divisors of $2^f - 1$, a contradiction. Similarly we can argue the case $r \in \pi(2^f - 1)$ and obtain a contradiction. Therefore $\mu(1) \in \{2^f \pm 1\}$. Since each of $2^f \pm 1$ is divisible by two odd primes and $\mu(1) \mid \chi(1)$, we conclude that

$\chi(1)$ is divisible by two odd primes, which is impossible as $\pi(\chi(1)) = \{2, r\}$. So we conclude that θ is not invariant under G and $N \leq I = I_G(\theta) < G$. Let $K \leq G$ such that $I/N \leq K/N$ and K/N is a maximal subgroup of $G/N \cong PSL_2(2^f)$ with $f \geq 6$. By writing $\theta^I = \sum_{i=1}^{m} e_i\phi(1)$ where $\phi \in Irr(I|\theta)$ and $e_i \geq 1$ for all $1 \leq i \leq m$, we can conclude through Theorem 6.11 of [Isaacs, 1976], Definition 2.41 and Clifford's Theorem that $\phi_i^G(1) = |G : I|e_i\theta(1) \in cd(G)$. Now, since $\chi \in Irr(G|\theta)$, we have $\chi = \phi_j^G$ where $1 \leq j \leq m$. Thus, $\chi(1) = |G : I|e_j\theta(1) = 2^c r^d$ for some positive integers c and d. Notice that, the index of the maximal subgroup of G/N, $|G/N : K/N| = |G : K|$, divides $|G : I|$. Hence $\pi(G/K) \subseteq \{2, r\}$, which means that $|G : K|$ can be divisible by at most one odd prime. The list of maximal subgroups of $PSL_2(2^f)$ with $2^f \geq 8$ is given in Theorem 2.4 of [Lewis et al., 2017] and shows that K/N is one of the following:

i) $C_2^f \rtimes C_{2^f-1}$: The stabilizer of a point of the projective line, where $|C_2^f \rtimes C_{2^f-1}| = 2^f(2^f-1)$,

ii) $D_{2(2^f-1)}$, where $|D_{2(2^f-1)}| = 2(2^f-1)$,

iii) $D_{2(2^f+1)}$, where $|D_{2(2^f+1)}| = 2(2^f+1)$,

iv) $PGL_2(q_0)$, where $2^f = (q_0)^n$, n is prime, $q_0 = 2^a$ with $a > 1$ and $|PGL_2(q_0)| = (2^a-1)2^a(2^a+1)$.

Hence, the index of K/N, $|G : K|$, is one of the following:

$$2^{f-1}(2^f+1), \qquad 2^{f-1}(2^f-1), \qquad 2^f+1, \qquad \frac{2^f(2^{2f}-1)}{2^a(2^{2a}-1)}, \tag{5.2}$$

where $2^f = (q_0)^n$, n is prime and $q_0 = 2^a$ with $a > 1$. Since both $2^f \pm 1$ have exactly two odd prime divisors, $f \geq 6$ and $|G : K|$ is divisible by at most one odd prime, we can conclude that $|G : K| = \frac{2^f(2^{2f}-1)}{2^a(2^{2a}-1)}$. As $\pi(|G : K|) = \{2, r\}$ and $(2^{2f}-1)/(2^{2a}-1) > 1$ is odd, we deduce that $\frac{2^{2f}-1}{2^{2a}-1} = r^k$, for some integer $k \geq 1$. If $\frac{f}{a} = n = 2$, then $(2^{2f}-1)/(2^{2a}-1) = 2^f+1 = r^k$, a contradiction as $|\pi(2^f+1)| = 2$. Hence $n \geq 3$ is a prime. Again, we obtain a contradiction as $\frac{2^{2f}-1}{2^{2a}-1}$ cannot be a prime power by Lemma 5.6. Thus this case cannot happen.

Suppose now that $\Delta(G)$ is disconnected, then $n(\Delta(G)) = 2$ (by hypothesis). By Theorem 6.4 (3) of [Lewis, 2008], we know that the smaller connected component of $\Delta(G)$ has exactly one vertex. Since $\rho(G) = \pi(G/N)$, we conclude from the structure of $\Delta(G/N)$ that this vertex must be 2. Thus $\pi(2^f+1)$ and $\pi(2^f-1)$ lie in the same connected component. Hence, there exists $\chi \in Irr(G)$ such that $\pi(\chi(1)) = \{u,v\}$, where $u \mid (2^f - 1)$ and $v \mid (2^f + 1)$. As $|\pi(2^f \pm 1)| = 2$ and $(2^f - 1, 2^f + 1) = 1$, we have $\pi(2^f - 1) = \{u,r\}$ and $\pi(2^f + 1) = \{v,s\}$, where $\{u,r\} \cap \{v,s\} = \emptyset$. It follows then that r, $s \nmid \chi(1)$. So either $2^f - 1$ or $2^f + 1$ does not divide $\chi(1)$. Let $\theta \in Irr(N)$ be a constituent of χ_N. If θ is trivial, then $\chi(1) \in cd(G/N)$, which cannot happen as $\pi(\chi) = \{u,v\}$. So θ is not the principal character of N. Suppose first that θ is not invariant under G and let $I = I_G(\theta)$. Then $I/N < G/N \cong PSL_2(2^f)$, and thus $|G : I|$ is divisible by the index of a maximal subgroup of G/N (see case $n(\Delta(G)) = 1$). Furthermore, we conclude that this index is odd as $|G : I| \mid \chi(1)$ by Clifford's Theorem. The possible list of indicies that a maximal subgroup of G/N can have are listed above and show that the only odd index is $2^f + 1$. Hence $2^f + 1 \mid |G : I| \mid \chi(1)$, a contradiction as $\chi(1)$ is divisible by three distinct primes. Therefore, θ is G-invariant. Since the Schur multiplier of $G/N \cong PSL_2(2^f)$ is trivial where $f \geq 6$, we conclude from Lemma 5.7 that θ is extendible to $\theta_0 \in Irr(G)$. It follows then from Gallagher's Theorem that $Irr(G|\theta) = \{\theta_0\psi : \psi \in Irr(G/N)\}$. Since $\chi \in Irr(G|\theta)$, we have $\chi = \theta_0\mu$, where $\mu \in Irr(G/N)$. Thus $\mu(1) \in \{1, 2^f, 2^f \pm 1\}$ and $\mu(1)$ divides $\chi(1)$. As $\chi(1)$ is odd and neither $2^f + 1$ nor $2^f - 1$ divides $\chi(1)$, we conclude that $\mu(1) = 1$. Hence, $\chi(1) = \theta_0(1)$. It follows again from Gallagher's Theorem, that $\chi(1)(2^f - 1) \in cd(G)$, a contradiction as this degree is divisible by the three primes u, v and r. Therefore, $\Delta(G)$ contains always a triangle. □

Lemma 5.14. *Let N be a solvable normal subgroup of G. Assume that G/N is a nonabelian simple group and $\Delta(G)$ has no triangles. Let $\tau = \rho(G) \setminus \pi(G/N)$. Then the following hold:*

i) $\tau \subseteq \rho(N)$, there is no edges between the primes of τ and $|\tau| \leq 2$.

ii) If $\tau \neq \emptyset$, then for each $r \in \tau$ and $\theta \in Irr(N)$ such that $r \mid \theta(1)$, θ extends to G and $G/N \cong A_5$ or $PSL_2(8)$.

Proof. As $\Delta(G)$ has no triangles, we have $|\pi(\psi(1))| \leq 2$ for all $\psi \in Irr(G)$. If τ is empty, then $i)$ holds trivially. So assume that $\tau \neq \emptyset$. Let $r \in \tau$. Then there exists $\chi \in Irr(G)$ such that $r \mid \chi(1)$ and $r \nmid |G : N|$. Let $\theta \in Irr(N)$ be a constituent of χ_N. It follows from Corollary 11.29 in [Isaacs, 1976] that $\chi(1)/\theta(1)$ divides $|G : N|$. Since $r \nmid |G : N|$, we conclude that $(r, \chi(1)/\theta(1)) = 1$, thus $r \mid \theta(1)$, which implies that

$r \in \rho(N)$. Notice that r is an arbitrary element of τ, hence $\tau \subseteq \rho(N)$. We claim next that there is no edges between primes in τ. If $|\tau| \leq 1$, the result is clear. Suppose now that there exist two distinct primes in τ, say r and s, such that $r, s \mid \chi(1)$. Since $|\pi(\chi(1))| \leq 2$, we have $\pi(\chi(1)) = \{r,s\}$. As $r, s \notin \pi(G/N)$, we can see that $(r, |G:N|) = (s, |G:N|) = 1$. Therefore $(\chi(1), |G:N|) = 1$, so $r, s > 2$ as $2 \mid |G:N|$. By Corollary 11.29 in [Isaacs, 1976], we have $\chi_N \in Irr(N)$. It follows then by Gallagher's Theorem that $\chi(1)\lambda(1) \in cd(G)$ for each $\lambda \in Irr(G/N)$. So by taking any $\nu \in Irr(G/N)$ with $2 \mid \nu(1)$, we can see that $\chi(1)\nu(1)$ is divisible by r, s and 2, a contradiction as $\Delta(G)$ contains no triangles. Hence there is no edges joining the primes of τ. As $\tau \in \rho(N)$, where N is solvable and there is no edges among the vertices of τ, we deduce via Pálfy's condition that $|\tau| \leq 2$, which completes the proof of part *i*).

For part *ii*), let $r \in \tau$ and $\theta \in Irr(N)$ where $r \mid \theta(1)$. By Lemma 5.12 , we have either $\chi(1)/\theta(1)$ is divisible by two distinct prime divisors of $|G:N|$ where $\chi \in Irr(G|\theta)$, or θ extends to G and $G/N \cong A_5$ or $PSL_2(8)$. If the first case holds, then since $r \nmid |G:N|$, we obtain by Clifford's Theorem that $\chi(1)$ is divisible by r. Thus $\chi(1)$ is divisible by three distinct primes (r and the two prime divisors of $\chi(1)/\theta(1)$), a contradiction. Therefore, this case cannot occur and the second case must hold, which completes the proof of the lemma. $\square$

Proof of Lemma 5.1. Let G be a group whose prime graph has no triangles. If G is solvable, then $|\rho(G)| \leq 4$ by Lemma 5.4. Thus we may assume that G is nonsolvable. Let N be the solvable radical of G. It follows from Lemma 5.11 that there exists $M \trianglelefteq G$ such that G/N is an almost simple group with socle $M/N \cong PSL_2(q)$ for a prime power $q \geq 4$, and $|\rho(G)| = |\rho(M)|$. As $M \trianglelefteq G$, $\Delta(M) \subseteq \Delta(G)$. Thus $\Delta(M)$ must contain no triangles too. By Lemma 5.9, we can see that $|\pi(M/N)| \leq 5$. Let $\tau \in \rho(M) \setminus \pi(M/N)$. It follows from Lemma 5.14 i) that $|\tau| \leq 2$ and if $\tau \neq \emptyset$, then $G/N \cong A_5$ or $PSL_2(8)$. Suppose first that $\tau = \emptyset$. Then $\rho(G) = \rho(M) = \rho(M/N) \leq 5$ as required. Assume next that $\tau \neq \emptyset$, then G/N is either A_5 with $cd(A_5) = \{1,3,4,5\}$ or $PSL_2(8)$ with $cd(PSL_2(8)) = \{1,7,8,9\}$. In both cases, $|\pi(G/N)| = 3$. Thus, $|\rho(G)| = |\rho(M)| = |\tau| + |\pi(G/N)| \leq 2+3 = 5$, which completes the proof of the theorem. $\square$

Proof of Lemma 5.2. Let G be a finite group with minimal order such that $\Delta(G)$ has five vertices with no triangles and G does not satisfy the results described in Theorem 5.2. By Lemma 5.4, if G is solvable, then $|\rho(G)| \leq 4$. Thus G is a nonsolvable group. Furthermore, if $n(\Delta(G)) = 3$, we conclude by Theorem 6.4 (2) of [Lewis, 2008] that part *i*) holds. Thus we may assume that $n(\Delta(G)) \leq 2$. Let N be the solvable radical of G. By Lemma 5.11, we know that there exists a normal subgroup $N \lhd M \unlhd G$ such that G/N is an almost simple group with socle $M/N \cong PSL_2(q)$ for a prime power $q \geq 4$, and $\rho(G) = \rho(M)$. As $M \unlhd G$ and $\rho(G) = \rho(M)$, we conclude that $\Delta(M) \subseteq \Delta(G)$ with the same set of vertices. Notice that as N is the solvable radical of G , then M is not solvable and N is also the solvable radical of M.

Step 1. $M = G$.

Suppose on the contrary that $M < G$. As $\Delta(M)$ has five vertices with no triangles, we conclude by the minimality of $|G|$ that $\Delta(M)$ is either the second graph in Figure 5.1 and $M \cong PSL_2(2^f) \times A$ where A is abelian and $|\pi(2^f \pm 1)| = 2$, or $\Delta(M)$ is the first graph in Figure 5.1 and $M \cong H \times K$ where H and K satisfy conditions of part *ii*) in Theorem 5.2.

Assume the first case holds. As A is an abelian normal subgroup of M and $M/A \cong PSL_2(2^f)$ is a nonabelian simple group containing N/A strictly, we conclude that $N = A$. So $M/N \cong PSL_2(2^f)$ and $|\rho(M)| = |\pi(M/N)| = 5$, which implies that $|\pi(G/N)| = 5$ as $|\pi(M/N)| \leq |\pi(G/N)| \leq |\rho(G)| = |\rho(M)| = 5$. It follows then from Lemma 5.10 that $G/N = M/N$. Thus $G = M$, a contradiction as M is a proper subgroup of G.

Assume the latter case holds. Since $\Delta(M) \subseteq \Delta(G)$ and they have the same set of vertices, we conclude that $\Delta(G)$ is obtained from $\Delta(M)$ by adding some edges. By the structure of $\Delta(M)$ given in Figure 5.1, we can realize that adding any edge to the graph $\Delta(M)$ will release a triangle. But $\Delta(G)$ has no triangles. Thus $\Delta(G) = \Delta(M)$. Notice that K is a sovable normal subgroup of G, hence $K \leq N \lhd M$. As $M/K \cong H$ is a nonabelian simple group containing N/K strictly, we deduce that $N = K$ and so $M/N \cong H$, where $H \cong A_5$ or $PSL_2(8)$. We claim now that each vertex in $\rho(H)$ has degree 2 in $\Delta(M)$. Suppose that there exists a vertex $r \in \rho(H)$ which is adjacent to three other vertices in $\Delta(M)$. As $\rho(M) = \rho(H) \cup \rho(K)$ and $|\rho(K)| = 2$, there exists $s \neq r \in \rho(H)$ such that $r.s \mid \chi(1)$ for some $\chi \in Irr(M)$. Let $\theta \in Irr(H)$ be a constituent of χ_N. By applying Clifford's Theorem, we can write $\chi(1) = e.t.\theta(1)$ for some positive integers e and t which divide $|M : H|$ by Corollary 11.29 in [Isaacs, 1976]. Thus $r.s$ cannot divide $e.t$ as $r, s \mid H$ and $(|M : H| = |K|, |H|) = 1$. This implies that $r.s$, which divides $\chi(1)$, must divide $\theta(1)$. Hence r is adjacent to s in $\Delta(H)$, a contradiction as $\Delta(H)$ is the graph formed of three isolated vertices. Therefore, each prime in $\rho(H)$ is of degree 2, which implies by the structure of $\Delta(M)$ that each prime in $\rho(K)$ has degree 3 in $\Delta(M) = \Delta(G)$. Hence there is no edges among primes in $\rho(H)$. As G/N is an almost simple group with socle $M/N \cong H$ and $|G/N : M/N| \neq 1$, it follows from [Conway et al., 1984] that $G/N \cong A_5 \cdot 2$ or $PSL_2(8) \cdot 3$. Both cases lead to a contradiction, as the character tables of G/N are given in [Conway et al., 1984] and show that G/N possesses in each case a character degree divisible by two distinct primes in $\rho(H)$, which is impossible as showed above. Thus, this case cannot occur.

Therefore, $M = G$ as required.

Step 2. Let $\tau = \rho(G) \setminus \pi(G/N)$. Then $G/N \cong A_5$ or $PSL_2(8)$, $|\tau| = 2$ and $|\pi(G/N)| = 3$. Furthermore, if $r \in \tau$ and $\theta \in Irr(N)$ such that $r \mid \theta(1)$, then θ extends to G.

From step 1, we know that $G/N = M/N \cong PSL_2(q)$ for a prime power $q \geq 4$. Suppose that $\tau = \emptyset$, then $\rho(G) = \pi(G/N)$, so $|\pi(G/N)| = |\rho(G)| = 5$. It follows from Lemma 5.10 that $G/N \cong PSL_2(2^f)$ with $f \geq 6$ and $|\pi(2^f \pm 1)| = 2$. Since $\Delta(G)$ has at most two connected components, Lemma 5.13 implies that $\Delta(G)$ has a triangle, a contradiction. Thus $\tau \neq \emptyset$. This implies by Lemma 5.14 ii) that $G/N \cong A_5$ or $PSL_2(8)$. Therefore, $|\pi(G/N)| = 3$. Since $|\rho(G)| = |\tau| + |\pi(G/N)| = 5$, we deduce that $|\tau| = 2$. The remaining statement follows directly from Lemma 5.14 ii) as $\tau \neq \emptyset$.

Let $\pi(G/N) = \{p_1, p_2, p_3\}$ and $\tau = \{r_1, r_2\}$. Then $r_1 \neq r_2$, $p_i \neq p_j$ for all $1 \leq i \neq j \leq 3$ and $\{r_1, r_2\} \cap \{p_1, p_2, p_3\} = \emptyset$.

Step 3. For each $1 \leq j \leq 3$, p_j is of degree 2 and each of $\{r_1, r_2\}$ is of degree 3 in $\Delta(G)$. Hence, there is no edges among the primes $\{p_1, p_2, p_3\}$ in $\Delta(G)$.

It follows from Lemma 5.14 i) that $\emptyset \neq \tau \subseteq \rho(N)$. So for each $i \in \{1, 2\}$, there exists $\theta_i \in Irr(N)$ such that $r_i \mid \theta_i(1)$. We know by Step 2. that each θ_i extends to G, hence as $\pi(G/N) = \{p_1, p_2, p_3\}$, we can conclude by Gallagher's Theorem that each

r_i is adjacent to all vertices p_j. Thus r_i is of degree 3 in the graph $\Delta(G)$. Finally, if there is any edge between the primes $\{p_1, p_2, p_3\}$, then $\Delta(G)$ has a triangle, which is not possible.

Step 4. If H is the last term of the derived series of G, then $H \cong A_5$ or $PSL_2(8)$ and $G \cong H \times N$.

Since G is nonsolvable and H is the last term of its derived series, we conclude that H is perfect, i.e, $H' = H$. Let $U = H \cap N$, then $U \trianglelefteq G$ (being the intersection of two normal subgroups). Notice that, as N is solvable and H is not, then $H \not\leq N$, which implies that $HN \neq N$. Now, since $1 \neq HN/N \trianglelefteq G/N$ and G/N is a nonabelian simple group, we conclude that $G = HN$. Thus $G/N = HN/N \cong H/(H \cap N) = H/U$ (Second Isomorphism Theorem) and so $H/U \cong A_5$ or $PSL_2(8)$ by Step 2. It remains to show that U is trivial. Remark that $\Delta(H) \subseteq \Delta(G)$ and as there is no edges among the primes $\{p_1, p_2, p_3\}$ in $\Delta(G)$ (Step 3), then there is no edges among the vertices $\{p_1, p_2, p_3\}$ in $\Delta(H)$. Suppose on the contrary that U is nontrivial. Since U is solvable then $U' < U$, which implies that $|U/U'| > 1$. But $|U/U'|$ represents the number of linear characters of U, see Example 5 in Section 15 of [Alperin and Bell, 1991]. Hence there exists a nontrivial character $\lambda \in Irr(U)$ such that $\lambda(1) = 1$. By Lemma 5.12 we can see that λ extends to H as H/U is a nonabelian simple group and there is no edges among primes in $\pi(|H : U|)$. Thus H has a nontrivial linear character, which is not possible as $H' = H$ and so $|H/H'| = 1$. Therefore U is trivial and the proof is completed.

We have proved that $G = H \times N$, where $H \cong A_5$ or $PSL_2(8)$, $\rho(G) = \rho(H) \cup \rho(N)$ and $\rho(H) = \pi(G/N) = \{p_1, p_2, p_3\}$. If $\rho(H) \cap \rho(N) \neq \emptyset$, then $\rho(N) = \{r_1, r_2\} \cup L$ where $L \subseteq \{p_1, p_2, p_3\}$. Suppose without loss of generality that $p_1 \in \rho(N)$. As $\Delta(N)$ has at most two connected components, r_1 is not adjacent to r_2 and there is no edges among $\{p_1, p_2, p_3\}$, we conclude that p_1 must be adjacent to exactly one of $\{r_1, r_2\}$. Again, without loss of generality assume that p_1 is adjacent to r_1. Then there exists $\mu \in Irr(N)$ such that $p_1 r_1 \mid \mu(1)$. Consider now $\nu \in Irr(H)$ such that $p_2 \mid \nu(1)$. This gives a character degree of G, $\mu \times \nu(1)$, which is divisible by both p_1 and p_2, a contradiction. Hence $\rho(H) \cap \rho(N) = \emptyset$. As there is no edges among vertices in τ, we conclude via Lemma 5.14 i), that $\Delta(N)$ has two connected components. By Taking $K = N$, we can see that $G = H \times K$ satisfies part ii) of Theorem 5.2 which contradicts our assumption. Therefore, G must satisfy one of the conclusions of Theorem 5.2 and the proof is completed. □

Proof of Lemma 5.3. If there exists a group G whose $\Delta(G)$ is a cycle or a tree with n vertices such that $n \geq 5$, then $\Delta(G)$ has no triangles. This implies by Theorem 5.1 that $n \leq 5$. Thus $n = 5$. It follows from Theorem 5.2 that $\Delta(G)$ is one of the graphs in Figure 5.1, a contradiction as these graphs are neither cycles nor trees. Therefore, any finite group G whose $\Delta(G)$ is either a cycle or a path has at most four vertices in its prime graph. □

6. GROUPS WHOSE BIPARTITE DIVISOR GRAPHS ARE PATHS

In this chapter, we study finite groups whose bipartite divisor graphs are paths. In particular, we claim that any finite group G whose $B(G) = P_n$ for some positive integer n is solvable. Furthermore, $n \leq 6$ and $dl(G) \leq 5$. Moreover, we discuss some group theoretical properties of such groups.

We follow in this chapter lemmas, propositions and theorems of paper [Hafezieh, 2017].

Theorem 6.1. *Let G be a finite group. Then $diam(B(G)) \leq 7$ and this bound is sharp.*

Proof. By Corollary 4.2, Theorem 6.5 and Theorem 7.2 of [Lewis, 2008], we can see that $diam(\Delta(G))$ and $diam(\Gamma(G))$ are less or equal than 3. Also from Lemma 2.65 we can deduce that either:

i) $diam(B(G)) = 2.max\{diam(\Delta(G), diam(\Gamma(G))\} \leq 2 \times 3 = 6$; or;

ii) $diam(B(G)) = 2diam(\Delta(G)) + 1 = 2diam(\Gamma(G)) + 1 \leq (2 \times 3) + 1 = 7$.

So in all cases we have $diam(B(G)) \leq 7$.

Consider now Lewis's example [Lewis, 2001b] of a group of order $2^{45}.(2^{15} - 1).15$ whose character degree set is:
$cd(G) = \{1, 3, 5, 3 \times 5, 7 \times 31 \times 151, 2^7 \times 7 \times 31 \times 151, 2^{12} \times 31 \times 151,$
$2^{12} \times 3 \times 31 \times 151, 2^{12} \times 7 \times 31 \times 151, 2^{13} \times 7 \times 31 \times 151, 2^{15} \times 3 \times 31 \times 151\}$; and prime degree set is:

$$\rho(G) = \{2, 3, 5, 7, 31, 151\}. \tag{6.1}$$

By a simple construction of $B(G)$, we can see that a shortest path between $5 \in cd(G)^*$ and $7 \in \rho(G)$ is the following path:

$$5 - 5 - 3 \times 5 - 3 - 2^{12} \times 3 \times 31 \times 151 - 2 - 2^{12} \times 7 \times 31 \times 151 - 7 \tag{6.2}$$

Hence, $d_{B(G)}(7, 5) = 7$. Therefore, 7 is the best possible upper bound for $diam(B(G))$ and the bound is sharp. □

Proposition 6.2. *If G is a finite group whose $B(G)$ is a path of length n. Then:*

- $n \leq 6$*,*
- *G is solvable,*
- $dl(G) \leq 5$*.*

  Furthermore, if $B(G) \cong P_5$, then $h(G) \leq 3$. And if $B(G) \cong P_6$, then $h(G) \leq 4$.

Proof. By the definition of $B(G)$ and since $B(G)$, $\Delta(G)$ and $\Gamma(G)$ have the same number of connected components (Lemma 2.65), we can conclude that $\Delta(G)$ and $\Gamma(G)$ are paths as $B(G)$ is so. Also, we can deduce from Corollary 4.2 and Theorem 6.5 of [Lewis, 2008] that $\Delta(G)$ is a path of length m where $m \leq 3$. Furthermore, by Theorem 4.5 of [Lewis, 2008] and Theorem 3.17, we can see that if G is solvable or not, $\Delta(G)$ cannot be a P_3. Thus $m \leq 2$. In other words, $\Delta(G)$ is an isolated vertex, a path of length one or a path of length two. On the other hand, we know from Lemma 2.65 part iv), that $|diam(\Delta(G)) - diam(\Gamma(G))| \leq 1$. Hence $diam(\Gamma(G)) \leq 3$, $|\rho(G)| \leq 3$ and $|cd(G)^*| \leq 4$. Therefore $n \leq 6$.

We claim that G is solvable. If $|cd(G)| \leq 3$, then by Theorems 12.5 and 12.15 in [Isaacs, 1976] we can see that G is solvable. So we may assume that $|cd(G)^*| > 2$ and thus $n > 3$. As $3 < n \leq 6$, we conclude that $3 \leq |cd(G)^*| \leq 4$. Hence G is solvable by Theorem 4.2, Theorem 4.10 and the fact that $\Gamma(G)$ is a path. Since $|cd(G)| \leq 5$, we have $dl(G) \leq 5$ by [Lewis, 2001a]. Assume next that either $B(G) \cong P_5$ or $B(G) \cong P_6$. In the first case, we have $|cd(G)| = 4$, in the second, $|cd(G)| = 5$ (remark that if $B(G) \cong P_6$, then the case where $|\rho(G)| = 4$ and $|cd(G)^*| = 3$ cannot happen as there is no solvable group G such that $\Delta(G)$ is a P_3). Thus in both cases $|cd(G)| \geq 4$. It follows then from Theorem 1.2 in [Riedl, 2003] that $h(G) \leq |cd(G)| - 1$. In particular, $h(G) \leq 3$ if $B(G)$ is isomorphic to P_5 and $h(G) \leq 4$ if $B(G)$ is isomorphic to P_6. □

Example 6.3. *Let $G = S_3 \times A_4$.*

Since each irreducible character degree divides the order of the group and the order of any finite group is equal to the the sum of the square of its irreducible character degrees (see Theorem 2.48), we can conclude that $cd(S_3)$, $cd(A_4) \subseteq \{1,2,3\}$. Let $P \in Syl_3(S_3)$, then $|P| = 3$ and $|G : P| = 2$. Thus P is an abelian normal subgroup of G. It follows then by Ito-Michler's Theorem that $3 \notin \rho(S_3)$. Thus $3 \notin cd(S_3)$ and so $cd(S_3) = \{1,2\}$. If $Q \in Syl_2(A_4)$, then it is not hard to see that Q is the Klein group of

A_4, $\{1,(1\quad 2)(3\quad 4),(1\quad 3)(2\quad 4),(1\quad 4)(2\quad 3)\}$, which is the unique subgroup of A_4 of order 4. Similar to previous, we have $2 \notin \rho(A_4)$ and $cd(A_4) = \{1,3\}$. Recall that both S_3 and A_4 are solvable groups as their respective subnormal series: $\{1\} \trianglelefteq A_3 \trianglelefteq S_3$ and $\{1\} \trianglelefteq \{1,(1\quad 2)(3\quad 4)\} \trianglelefteq Q \trianglelefteq A_4$ have abelian factors. Hence $G = S_3 \times A_4$ is solvable. Finally, by using Example 2 of section 15 in [Alperin and Bell, 1991], we can deduce that $cd(G) = \{1,2\} \times \{1,3\} = \{1,2,3,6\}$. Therefore, $\rho(G) = \{2,3\}$ and $B(G)$ is the graph in Figure 6.1.

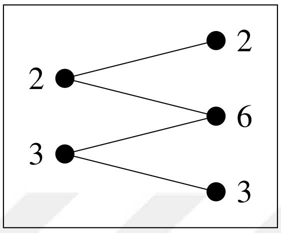

Figure 6.1: $B(S_3 \times A_4)$.

This gives an example of a finite group G whose $B(G) \cong P_4$.

Theorem 6.4. *Let G be a finite group such that $B(G) = P_n$ for some positive integer n. Then one of the following cases holds:*

i) There exists a normal abelian subgroup of G, say N, such that $cd(G) = \{1, |G : N|\}$ and G/N is abelian. In addition n is either 1 or 2 in this case.

ii) There exist two normal subgroups of G, N and K, and a prime number p such that:

- *G/N is abelian.*
- *$\pi(G/K) \subseteq \rho(G)$.*
- *Either p divides every element in $cd(N)^*$ and thus N has a normal p-complement by Corollary 12.2 of [Isaacs, 1976], or $cd(N) = \{1, l, , k, \frac{h}{m}\}$;*

where $n \in \{4,5,6\}$ in this case.

iii) $cd(G) = \{1, p^{\alpha}, q^{\beta}, p^{\alpha}q^{\beta}\}$, where p and q are two distinct primes and $n = 4$ in this case.

iv) There exists a prime number s such that G has a normal s-complement H, where H is either abelian and $n \in \{1,2\}$, or nonabelian and one of the following cases occurs:

- *$cd(G) = \{1, h, hl\}$, where h and l are two positive integers, and $n = 3$ in this case.*

- *$n = 4$, G/H is abelian, and as is explained in [Lewis, 2001c] either H is a group of type one and $cd(H) = |H : F(H)| \cup cd(F(H))$, or H is a group of type four with $cd(H) = \{1, |F_2(H) : F(H)|, |H : F_2(H)|\}$. Furthermore, $|G : F(G)| \in cd(G)$ and $cd(F(G)) = \{1, h_{s'}\}$ where $h \neq |G : F(G)| \in cd(G)$.*

- *$n = 3$, G/H is abelian, $|G : F(G)| \in cd(G)$, $F(G) = P \times A$, where P is a p-subgroup, $A \leq Z(G)$, $cd(G) = cd(G/A)$, and $cd(P) = \{1, m_{s'}\}$ where $m \in cd(G)$ such that $m \neq |G : F(G)|$.*

Proof. Since $B(G)$ is a P_n, then by Proposition 6.2, G is solvable and $n \leq 6$.

Suppose first that $n \geq 4$. This implies that $|cd(G)^*| \geq 2$. We claim that G has a normal subgroup $K > 1$ such that G/K is nonabelian. We discuss this according to whether G' is a minimal normal subgroup of G or not. If not, and since G is solvable, then it has a minimal normal subgroup K such that $G' \nleq K$, which means that G/K is nonabelian. Assume now that G' is a minimal normal subgroup of G. Since not all the nonlinear irreducible character degrees of G are equal, we conclude by Lemma 12.3 of [Isaacs, 1976] that G' is not unique. So G has a nontrivial minimal normal subgroup $H \neq G'$, thus $G' \nleq H$, i.e., G/H is nonabelian. Let K be maximal with respect to the property that G/K is nonabelian. We claim that $(G/K)'$ is the unique minimal normal subgroup of G/K. If $1 \neq H/K \lhd G/K$, then $K < H$ and by the maximality of K, we deduce that G/H is abelian. But $G/H \cong (G/K)/(H/K)$, thus $(G/K)' \leq H/K$, which implies that $(G/K)'$ is a minimal normal group. The uniqueness is direct by taking another minimal normal subgroup of G/K, say H/K, and deducing that $(G/K)' \leq (H/K)$, then by the minimality of H/K we obtain the equality.

Since $(G/K)'$ is the unique minimal normal subgroup of G/K, it follows that G/K satisfies the hypothesis of Lemma 12.3 in [Isaacs, 1976]. Consequently, all degrees in $cd(G/K)^*$ are equal to f and we have the following cases:

Case 1. $B(G) \cong P_6$.

Since $\Delta(G)$ cannot be a P_3 (Theorem 4.5 of [Lewis, 2008]), then $\Delta(G)$ is a P_2, and we have $B(G) : m - p - h - q - l - r - k$ where p, q and r are three distinct primes and $cd(G)^* = \{m, h, l, k\}$. If G/K is an s-group for some prime s, then by symmetry we can assume that $s = p$ and $f = m$. Let $\chi \in Irr(G)$ such that $\chi(1) = k$. Since $p \nmid \chi(1)$, it follows from Corollary 11.29 in [Isaacs, 1976] that $\chi_K \in Irr(K)$, then by Gallagher's Theorem we conclude that $\chi(1)f = km \in cd(G)$. Thus p is adjacent to r in $\Delta(G)$, a contradiction. It follows from Lemma 12.3 in [Isaacs, 1976] that G/K is a Frobenius group whose Frobenius complement is abelian of order f, and whose Frobenius kernel $N/K = (G/K)'$ is elementary abelian s-group for some prime s. So G/N is abelian, and we deduce by the structure of $B(G)$ and Ito's Theorem (Theorem 6.15 of [Isaacs, 1976]) that N is nonabelian.

Suppose first that $f = m$. Let $\chi(1)$, $\psi(1) \in cd(G)$ such that $\chi(1) = k$ and $\psi(1) = l$. By Corollary 11.29 of [Isaacs, 1976] we can conclude that $\chi_N(1)$, $\psi_N(1) \in cd(N)$, i.e., $l, k \in cd(N)$. If $s \notin \rho(G)$, then $(\chi(1), |G : K|) = 1$. It follows then by Corollary 11.29 in [Isaacs, 1976] and Gallagher's Theorem, that $km \in cd(G)$, which contradicts the structure of $B(G)$. Therefore, $s \in \rho(G)$. Let $\theta \in cd(N)^*$ such that $s \nmid \theta(1)$. By Theorem 12.4 of [Isaacs, 1976], we can deduce that $|G : N|\theta(1) \in cd(G)$, so $q \mid \theta(1)$, $|G : N|\theta(1) = h$ and $s = r$. Thus $\theta(1) = \frac{h}{m}$. Notice that if $\eta \in cd(N)$ such that $s \nmid \eta(1)$, then by Theorem 12.4 in [Isaacs, 1976] we have $|G : N|\eta(1) \in cd(G)$, hence $|G : N|\eta(1)$ is either m or h. This implies that θ is unique with respect to the property that $s \nmid \theta(1)$. Let $\gamma \in cd(N)^*$ such that $\gamma(1) \neq \theta(1)$. Let μ be an irreducible constituent of γ^G. By the uniqueness of θ, we have $s \mid \gamma(1)$ and since $s \nmid |G : N|$, it follows from Clifford's Theorem that $s \mid \mu(1)$ and $\mu(1)$ is either l or k. Thus $(\mu(1), |G : N|) = 1$, which implies that $\mu_N \in Irr(N)$ and $\gamma(1) = \mu(1)$. Therefore, according to whether s divides all the nonlinear irreducible characters of N or not, N has one of the following properties:

i) If yes, then N has a normal s-complement by Corollary 12.2 in [Isaacs, 1976].

ii) If not, then $cd(N) = \{1, k, l, \frac{h}{m}\}$.

In a similar way the case $f = k$ can be discussed. Suppose now that $f = h$. Let $\theta \in Irr(N)$ such that $\theta(1) \neq 1$. If $|G : N|\theta(1) \in cd(G)$, then by the structure of $B(G)$ we conclude that $|G : N|\theta(1) = h = |G : N|$, which implies that $\theta(1) = 1$, a contradiction. So $|G : N|\theta(1) \notin cd(G)$. By Theorem 12.4 in [Isaacs, 1976], we deduce that $s \mid \theta(1)$. I.e., $\pi(G/K) \subseteq \rho(G)$. Similarly we can discuss the case $f = l$. Hence for all the different possibilities of f, we obtain part *ii*).

Case 2. $B(G) \cong P_5$.

Assume that $B(G) : p - m - q - l - r - h$, where p, q and r are distinct primes and $cd(G)^* = \{m,l,h\}$. Similar to the previous case, as $mh \notin cd(G)$, then G/K is a Frobenius group whose Frobenius complement is abelian of order f and Frobenius kernel, $N/K = (G/k)'$, is elementary abelian s-group for some prime s. Thus G/N is abelian.

Suppose first that $f = h$. Since $mh \notin cd(G)$, it follows that $s \in \rho(G)$ (see the previous case). If there exists $\theta \in Irr(N)$ such that s does not divide $\theta(1)$, then by Theorem 12.4 of [Isaacs, 1976] we have $|G:N|\theta(1) \in cd(G)$, which implies that $q \mid \theta(1)$ and $s = p$. Let $\psi \in Irr(N)$ be a nonlinear character such that $\psi(1) \neq \theta(1)$. Then, $s \mid \psi(1)$. Let $\chi \in Irr(G)$ such that $[\psi^G, \chi] \neq 0$. By Clifford's Theorem we have: $\chi_N(1) = [\psi^G, \chi]|G : I_G(\psi)|\psi(1)$. Since $s \mid \psi(1)$, then $s \mid \chi(1) = \chi_N(1)$. By the structure of $B(G)$, we can see that $\chi(1) = m$. As $(\chi(1), |G:N|) = 1$, we know by Corollary 11.29 in [Isaacs, 1976] that $\chi_N \in Irr(N)$, thus $\chi(1) = \psi(1) = m$. This implies that q divides every nonlinear character degree in $cd(N)$. It follows that N has a normal q-complement (see Theorem 12.2 in [Isaacs, 1976]). Therefore case *ii*) occurs.

Suppose now that $f = m$. Similar to previous, as $hm \notin cd(G)$, we have $s \in \rho(G)$, precisely it must be r. Let θ be a nonlinear irreducible character of N. Since $|G : N|\theta(1) \notin cd(G)$, Theorem 12.4 in [Isaacs, 1976] implies that s divides $\theta(1)$. Thus N has a normal s-complement by Corollary 12.2 in [Isaacs, 1976], so case *ii*) occurs.

For the last case, $f = l$, let θ be a nonlinear irreducible character of N. Since $|G:N|\theta(1) \notin cd(G)$, we conclude by Theorem 12.4 in [Isaacs, 1976] that $s \mid \theta(1)$. This implies that N has a normal s-complement. Thus case *ii*) occurs.

Case 3. $B(G) \cong P_4$

Suppose first that $B(G) : p - m - q - h - r$ where $cd(G)^* = \{m,h\}$ and p, q and r are three distinct primes. As q divides every character in $cd(G)^*$, then G has a normal q-complement (Corollary 12.2 in [Isaacs, 1976]). Hence, $G = HQ = H \rtimes Q$, where H is a normal q-complement and Q is a Sylow q-subgroup of G. Notice that $cd(G)$ contains no powers of q, and if $Q \cong G/H$ is nonabelian then Q has a power of q character degree, which is impossible as $cd(Q) = cd(G/H) \subseteq cd(G)$. Thus Q is abelian. On the other hand, H is not abelian, because if so then by Corollary 6.15 in [Isaacs, 1976], we obtain that each character degree in G is a power of q, which is not the case. Since $q \nmid |H|$ and each character degree of H divides $|H|$ and a character in $cd(G)$, it follows from the structure of $B(G)$ that $cd(H) = \{1, a = p^\alpha, b = r^\beta\}$, for some positive integers α and

β. We claim now that H is not nilpotent. Indeed, if it is nilpotent, then it is the direct product of its normal Sylow subgroups, which implies that $ab \in cd(H)$, a contradiction. Thus $h(H) > 1$. Since $h(H) \leq dl(H) \leq cd(H) \leq 3$ (see Taketa inequality in [Garrison, 1973] and Theorem 1.27), we conclude that $2 \leq h(H) \leq 3$. It follows from Theorem 3.5 in [Noritzsch, 1995], that $dl(H) = 3$. Also by applying Lemma 3.1 in [Isaacs, 1976] to H, we have either $h(H) = 2$, $|H : F(H)| = a$ and $cd(F(H)) = \{1, b\}$, or $h(H) = 3$ and $cd(H) = \{1, |F_2(H) : F(H)|, |H, F_2(H)|\}$. As $\Delta(H)$ is the graph composed of the two isolated vertices p and r, we can see that H is either a group of type one or four in the classification of solvable groups whose prime graphs have two connected components which is found in [Lewis, 2001c]. Finally, by applying Lemma 5.1 in [Noritzsch, 1995] we deduce that $m = |G : F(G)|$ and $cd(F(G)) = \{1, h_{q'}\}$. Thus case *iv*) occurs, with $s = q$.

Assume now that $B(G) : m - q - l - p - h$, where p and q are two distinct primes and $cd(G)^* = \{m, l, h\}$. Suppose that G/K is a Frobenius group whose Frobenius complement is abelian of order f and Frobenius kernel, $N/K = (G/K)'$, is elementary abelian s-group for some prime s. Remark that G/N is abelian in this case. Let $m = q^{\alpha}$ and $h = p^{\beta}$ for some positive integers α and β. If $f = m$ and $s \notin \rho(G)$, then by Corollary 11.29 of [Isaacs, 1976] and Gallagher's Theorem we conclude that $p^{\alpha}q^{\beta} \in cd(G)$, i.e., $cd(G) = \{1, p^{\alpha}, q^{\beta}, p^{\alpha}q^{\beta}\}$, so case *iii*) occurs. If $f = m$ and $s \in \rho(G)$, then since $(f, s) = 1$, $s = p$. Let $\theta \in Irr(N)$ be a nonlinear character. If $s \nmid \theta(1)$, it follows from Theorem 12.4 in [Isaacs, 1976] that $|G:N|\theta(1) \in cd(G)$. Thus $m\theta(1) = l$. As $(s, f) = 1$ and $p \mid l$, then $p = s \mid \theta(1)$, a contradiction. Therefore, s divides each nonlinear character in $cd(N)$, which implies that N has a normal s-complement. So case *ii*) holds. The case $f = h$ can be done similarly. Consider now the case where $f = l$. As $(f, s) = 1$, then $s \notin \rho(G)$. Let $\psi \in Irr(N)$ such that $\psi(1) \neq 1$. If $|G : N|\theta(1) \in cd(G)$, then it must be l, which is impossible. Thus $|G:N|\theta(1) \notin cd(G)$. Theorem 12.4 in [Isaacs, 1976], implies that s divides $\psi(1)$ which divides a character degree of G and so $s \in \rho(G)$, a contradiction. Hence, $f \neq l$. Finally, assume that G/K is an s-group for some prime s. By symmetry we can suppose that $s = p$ and $f = p^{\alpha} = h$. Let $\chi \in Irr(G)$ such that $\chi(1) = m$. By Corollary 11.29 in [Isaacs, 1976] and Gallagher's Theorem, we have $mh \in cd(G)^*$. Thus $l = p^{\alpha}q^{\beta} = mh$, $cd(G) = \{1, p^{\alpha}, q^{\beta}, p^{\alpha}q^{\beta}\}$ and case *iii*) holds.

We consider now the cases where $B(G) = P_n$ with $n \leq 3$:

Case 4. $B(G) \cong P_3$

Assume $B(G): p - m - q - h$, where p and q are two distinct primes and $cd(G)^* = \{m, h\}$. As q divides every character in $cd(G)^*$, we conclude that $G = HQ = H \rtimes Q$, where H is a normal q-complement and Q is a q-subgroup of G (see Corollary 12.2 in [Isaacs, 1976]). If $h \mid m$, then $cd(G) = \{l, h, hl\}$ for some positive integer l. Thus part one of case *iv*) holds. Otherwise, we have $cd(H) = \{1, p^\alpha\}$ for some positive integer α (similar to Case 3). We claim that Q is abelian. If not, then $Q \cong G/H$ has a power of q character degree. But since $cd(Q) \subseteq cd(G)$, we conclude that $cd(Q) = cd(G/H) = \{1, h\}$. Let $\theta \in Irr(H)$ such that $\theta(1) = p^\alpha$. Remark that $\theta(1)$ and $o(\theta) = |H : ker\theta|$ divide $|H|$, thus $\theta(1)o(\theta) \mid |H|$. Since $(|H|, |G : H|) = 1$, we conclude by Corollary 6.28 in [Isaacs, 1976] that θ extends to $\nu \in Irr(I_G(\theta))$. It follows then from Theorem 6.11 in [Isaacs, 1976], that $\nu^G(1) = |G : I_G(\theta)|\nu(1) \in cd(G)$. Thus $\nu^G(1) = m$. It should be remarked that, since $p \mid |H|$, then $p \nmid |G : H|$, in particular, $p \nmid |G : I_G(\theta)|$. Hence $|G : I_G(\theta)| = m_q$. Also note that if $I_G(\theta)/H$ is abelian, then by Proposition 2.70 we have $h \leq |G/H : I_G(\theta)/H| = |G : I_G(\theta)| = m_q$, a contradiction as $h \nmid m$. So $I_G(\theta)/H$ is nonabelian, i.e., there exists $\psi \in Irr(I_G(\theta)/H)$ such that $\psi(1) > 1$. It follows then by Gallagher's Theorem that $\psi\nu \in Irr(I_G(\theta))$, which implies by Theorem 6.11 in [Isaacs, 1976] that $(\psi\nu)^G \in Irr(G)$. Hence $m < \psi(1)\nu(1)|G : I_G(\theta)| \in cd(G)$, a contradiction. Therefore Q is abelian. Now as $h \nmid m$ and $\pi(h) \subset \pi(m)$, Lemma 5.2 of [Noritzsch, 1995] implies that $|G : F(G)| = h$, $F(G) = P \times A$, where $A \leq Z(G)$, P is a p-group, $cd(G) = cd(G/A)$ and $cd(P) = \{1, m_{q'}\}$. Thus case *iv*) holds with $s = q$.

Case 5. $B(G) \cong P_2$

Suppose first that $cd(G) = \{1, m\}$ where m is not a prime power. Then $m = p^\alpha q^\beta$ where p and q are distinct primes and α, $\beta \subseteq \mathbb{N}^*$. It follows from Theorem 12.5 in [Isaacs, 1976], that G has a normal abelian subgroup N such that $|G : N| = m$. Notice that if G/N is not abelian, then $cd(G/N) = cd(G)$, which is not possible as $N > 1$. Thus G/N is abelian and case *i*) occurs.

Suppose now that $cd(G) = \{1, m, h\}$. Then both m and h are powers of a prime integer s, which implies that G has a normal s-complement H. If H is nonabelian, then it has a nonlinear character degree which divides a character in $cd(G)$, but since $\rho(G) = \{s\}$ we obtain a contradiction. Thus H is abelian and case *iv*) holds.

Case 6. $B(G) \cong P_1$

In this case we have $cd(G) = \{1, p^\alpha\}$ for a prime p and a positive integer α. It follows then by Theorem 12.5 of [Isaacs, 1976] that either $G \cong P \times A$, where A is abelian and P is a p-group, so case iv) holds, or G has a normal abelian subgroup of index p^α and case i) occurs. $\square$

7. NONSOLVABLE GROUPS WHOSE BIPARTITE DIVISOR GRAPHS ARE UNION OF PATHS

In this chapter, we consider G to be a finite nonsolvable group such that each connected component of $B(G)$ is a path. By Theorem 6.4 (1) of [Lewis, 2008], we know that $n(\Delta(G)) \leq 3$. Thus, $n(B(G)) \leq 3$ by Lemma 2.65. We start by discussing the nonabelian simple groups, then we consider a general nonsolvable group.

We follow in this chapter lemmas and theorems of paper [Hafezieh, 2017].

Lemma 7.1. *Let S be a nonabelian simple group. Then $B(S)$ is a disconnected graph whose all connected components are paths if and only if S is isomorphic to one of the following:*

i) $PSL_2(2^n)$ where n is a positive integer and $|\pi(2^n \pm 1)| \leq 2$;

ii) $PSL_2(p^n)$ where p is an odd prime, n is a positive integer and $|\pi(p^n \pm 1)| \leq 2$.

Proof. As mentioned before, we know that $n(B(S)) = n(\Delta(S)) = n(\Gamma(S))$. If the connected components of $B(S)$ are paths, then by Lemma 2.65 ii) we can see that the connected components of $\Delta(S)$ are paths too. Thus $\Delta(S)$ has no triangles. It follows then by Lemma 5.9 that one of the following cases occurs:

i) $S \cong PSL_2(2^n)$ where n is a positive integer and $|\pi(2^n \pm 1)| \leq 2$, and so $|\pi(S)| \leq 5$;

ii) $S \cong PSL_2(p^n)$ where p is an odd prime, n is a positive integer and $|\pi(p^n \pm 1)| \leq 2$, and so $|\pi(S)| \leq 4$;

which completes the proof of the first implication. Suppose now that S is the group described in i). Since $cd(PSL_2(2^n)) = \{1, 2^n, 2^n - 1, 2^n + 1\}$ is a pairwise relatively prime set, we conclude that $B(S)$ has the following three connected components:

i) The path of length one: $2 - 2^n$,

ii) The path linking $2^n + 1$ with its prime divisors, which is of length one if $|\pi(2^n + 1)| = 1$ and of length two if $|\pi(2^n + 1)| = 2$,

iii) The path linking $2^n - 1$ with its prime divisors, which is of length one if $|\pi(2^n - 1)| = 1$ and of length two if $|\pi(2^n - 1)| = 2$.

Thus, all the connected components are paths in this case. Suppose next that $S \cong PSL_2(p^n)$ for an odd prime p. As $cd(PSL_2(p^n)) = \{1, p^n, p^n - 1, p^n + 1, \frac{p^n + \varepsilon}{2}\}$

where $\varepsilon = (-1)^{\frac{p^n-1}{2}}$, $(p^n, p^n-1) = (p^n, p^n+1) = 1$ and $2 \in \pi(p^n \pm 1)$, we deduce that $B(S)$ has the following two connected components:

i) The path of length one: $p - p^n$,

ii) The path linking p^n+1, p^n-1 and $\frac{p^n+\varepsilon}{2}$ with their prime divisors.

Hence each component is a path and the proof is completed. □

Lemma 7.2. *If G is a finite group whose $B(G)$ is a union of paths and $|\rho(G)| = 5$, then $G \cong PSL_2(2^n) \times A$ where A is an abelian subgroup and $|\pi(2^n \pm 1)| = 2$.*

Proof. Since all connected components of $B(G)$ are paths and so are those of $\Delta(G)$ (Lemma 2.65 ii)), we conclude that $\Delta(G)$ has no triangles. We claim that $B(G)$ is disconnected. If $n(B(G)) = 1$, then $B(G)$ is a path. It follows then from Proposition 6.2 that G is solvable and the length of $B(G)$ is less or equal than 6, which means that $|\rho(B(G))| \leq 4$, a contradiction as $|\rho(B(G))| = 5$. Hence $B(G)$ is disconnected and so is $\Delta(G)$ by Lemma 2.65 iii). By Theorem 5.2 i), we can see that $G \cong PSL_2(2^n) \times A$, where A is an abelian group and $|\pi(2^n \pm 1)| = 2$.

As described in Lemma 7.1, the connected components of $B(G)$ are precisely the following:

$$2 - 2^n, \;\; s - (2^n - 1) - t \;\; and \;\; f - (2^n + 1) - g, \tag{7.1}$$

where $\pi(2^n - 1) = \{s, t\}$ and $\pi(2^n + 1) = \{f, g\}$. □

Theorem 7.3. *Let G be a finite nonsolvable group whose solvable radical is N. If $B(G)$ is a union of paths, then $n(B(G)) > 1$ and G has a normal subgroup M such that G/N is an almost simple group with socle M/N. Furthermore, $\rho(G) = \rho(M)$ and one of the following holds:*

i) If $B(G)$ has two connected components , then either:

- *$|cd(G)| = 5$ or,*
- *$|cd(G)| = 4$, $G/N \in \{M_{10}, PGL_2(q) : q > 3$ odd$\}$ and either $cd(G) = \{1, q-1, q, q+1\}$ or $cd(G) = cd(M_{10}) = \{1, 9, 10, 16\}$.*

In addition, if C_1 and C_2 are the connected components of $B(G)$, then $C_1 \cong P_1$ and $C_2 \cong P_n$, where either $n = |\rho(G)|$ or $n = |\rho(G)| + 1$.

ii) If $B(G)$ has three connected components, then $G \cong PSL_2(2^n) \times A$, where $n \geq 2$ and A is an abelian group.

Proof. As $B(G)$ is a union of paths, then so is $\Delta(G)$. Precisely, If $n(B(G)) = i$, then $n(\Delta(G)) = i$ for every $i \in \{1,2,3\}$. Since G is nonsolvable, it follows from Proposition 6.2 that $B(G)$ is not a path. Thus, $B(G)$ and $\Delta(G)$ are the union of either two or three paths, which implies that $\Delta(G)$ is triangle-free. It follows then by Theorem 5.1 that $|\rho(G)| \leq 5$. By applying Lemma 5.11, we can see that G has a normal subgroup M such that $\rho(G) = \rho(M)$ and G/N is an almost simple group with socle M/N. Recall that any group whose order is divisible by less than three primes is solvable (Burnside's Theorem), hence $3 \leq |\rho(M/N)| \leq |\rho(G)|$. Therefore, $3 \leq |\rho(G)| \leq 5$. By Lemma 7.2, if $|\rho(G)| = 5$, then $G \cong PSL_2(2^n) \times A$, where A is an abelian group and $|\pi(2^n \pm 1)| = 2$. Thus $B(G)$ has three connected components (see Proof of Lemma 7.2). On the other hand, as G is nonsolvable, we conclude by Theorem 6.4 in [Lewis, 2008] that $n(B(G)) = 3$ if and only if $G \cong PSL_2(2^n) \times A$ where A is abelian and $n \geq 2$. Thus if either $|\rho(G)| = 5$ or $n(B(G)) = 3$, case ii) occurs. So we may assume that $n(B(G)) = 2$ and $|\rho(G)| \leq 4$. Since G is nonsolvable and $n(B(G)) = n(\Gamma(G)) = 2$, we can see by Theorem 7.1 (3) in [Lewis, 2008] that one of the connected components of $\Gamma(G)$ is an isolated vertex and the other has diameter at most 2. Precisely, if the other component has diameter one (resp. two), then $|cd(G)^*| = 3$ (resp. 4). Therefore, $|cd(G)| \in \{4,5\}$.

Suppose next that $|cd(G)| = 4$. If $|cd(G/N)| \leq 3$, then by Corollary 12.6 and Theorem 12.15 of [Isaacs, 1976] we can deduce that G/N is solvable. Thus, M/N is solvable, a contradiction as by definition M/N is a nonabelian simple subgroup of G/N. Hence $|cd(G/N)| \geq 4$ and since $cd(G/N) \subseteq cd(G)$, we conclude that $|cd(G/N)| = |cd(G)| = 4$. By applying Theorem 1 in [Malle and Moretó, 2005] to the almost simple group G/N, we obtain that $G/N \in \{M_{10}, PGL_2(q), PSL_2(2^n) : q > 3$ is odd and $n \in \mathbb{N}^*\}$. But since $n(B(PSL_2(2^n))) = 3$ and $n(B(G)) = 2$ we deduce that $G/N \in \{M_{10}, PGL_2(q)\}$. Now, Corollary B of [Malle and Moretó, 2005] implies that either $cd(G) = \{1, q-1, q, q+1\}$ for $q > 3$ odd or $cd(G) = \{1,9,10,16\}$.

Assume the first case holds, then $cd(G) = \{1, q-1, q, q+1\}$ for an odd integer $q > 3$. As $B(G)$ is a union of at most two paths and $2 \in \pi(q \pm 1)$, we can deduce that $|\pi(q \pm 1)| \leq 2$. Thus $2 \leq |\pi(q^2 - 1)| \leq 3$. If exactly one of $q \pm 1$ is a power of 2, then

$|\pi(q^2-1)|=2$. Otherwise $|\pi(q\pm 1)|=2$, thus $|\pi(q^2-1)|=3$ (remark that the case where both of $q\pm 1$ are powers of 2 fails trivially as $q>3$). It follows then by Lemma 3.7 that q verifies one of the following:

i) $q \in \{3^4, 5^2, 7^2\}$,

ii) $q = 3^s$, where s is an odd prime,

iii) $q = p \geq 11$ is a prime.

On the other hand, since $2 \in \pi(q\pm 1)$, we conclude that the isolated vertex in $\Gamma(G)$ is q. So if $q = p^f$ for a positive integer f, then $p-q$ is a path of length one in $B(G)$, call this component C_1. Consider now the other component of $B(G)$, say C_2. If $|\rho(G)|=3$, then one of $q\pm 1$ is a power of 2 and the other is divisible by $\{2,r\}$ where r is an odd prime. Thus C_2 is a path of length 3 in this case. If $|\rho(G)|=4$, and since $B(G)$ is a union of paths, we conclude that $\pi(q+1)=\{2,m\}$ and $\pi(q-1)=\{2,k\}$ for some distinct odd primes m and k. Hence C_2 is a path of length 4. Notice that $\{2,p\} \subseteq \rho(G)$, thus $|\rho(G)|>1$. Besides, the case where $\rho(G)=\{2,p\}$ cannot happen as in this case both of $q\pm 1$ are powers of 2 which is impossible as $q>3$.

If $cd(G)=\{1,9,10,16\}$, then $B(G)$ has the following connected components:

$$C_1 : 3-9 \quad and \quad C_2 : 5-10-2-16 \tag{7.2}$$

Therefore, if $|cd(G)|=4$, then $C_1 \cong P_1$ and $C_2 \cong P_{|\rho(G)|}$.

Suppose now $|cd(G)|=5$. Since $n(B(G))=2$ and $|cd(G)^*|=4$, we can conclude the following:

- If $|\rho(G)|=3$, then the isolated vertex in $\Gamma(G)$ can be divisible by at most one prime, since otherwise we obtain an isolated vertex in $B(G)$ which contradicts its structure. Thus the isolated vertex in $\Gamma(G)$ generates a path of length one in $B(G)$, call it C_1. It follows then that the other component must be an alternating path between the other three character degrees in $cd(G)^*$ and the other two primes in $\rho(G)$. Hence $C_2 \cong P_4$.
- If $|\rho(G)|=4$. Then similar to previous, $B(G)$ has two components $C_1 \cong P_1$ generated by the isolated vertex in $\Gamma(G)$ and C_2 which is a path of length 5 alternates between the remaining character degrees and vertices in $B(G)$.

Remark that, since $\Gamma(G)$ has an isolated vertex (see Theorem 7.1 (3) in [Lewis, 2008]), $n(B(G)) = 2$ and $B(G)$ is a union of paths, we can see that if $|cd(G)| = 5$, then $|\rho(G)| > 2$, $C_1 \cong P_1$ and $C_2 \cong P_{|\rho(G)|+1}$, which completes the proof. $\square$

Example 7.4. *If* $G = PSL_2(25)$, *then* $cd(G) = \{1, 25-1, 25, 25+1, (25+1)/2\} = \{1, 13, 24, 25, 26\}$. *By a simple construction we can see that* $B(G)$ *is the graph in Figure 7.1.*

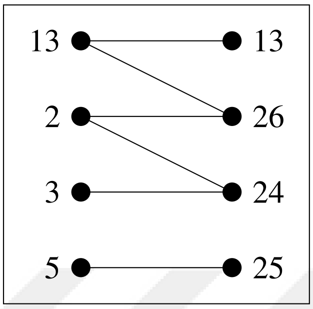

Figure 7.1: $B(PSL_2(25))$.

Hence, the components of $B(G)$ *are a path of length one:* $5-25$, *and a path of length five:* $13-13-26-2-24-3$.

8. GROUPS WHOSE BIPARTITE DIVISOR GRAPHS ARE CYCLES

We consider in this chapter finite groups G whose $B(G)$ are cycles. We start by claiming that if $B(G)$ is a cycle, then G is solvable and the length of $B(G)$ is either four or six. Moreover, we discuss some group theoretical properties of G when $B(G)$ is a C_4.

We follow in this chapter lemmas and theorems of paper [Hafezieh, 2017].

Lemma 8.1. *If G is a finite group whose $B(G) \cong C_n$ for a positive integer $n \geq 6$, then both $\Delta(G)$ and $\Gamma(G)$ are cycles.*

Proof. Let G be a finite group such that $B(G) \cong C_n$ for a positive integer $n \geq 6$ and consider Φ to be either $\Delta(G)$ or $\Gamma(G)$. By definition, we know that Φ is a cycle if and only if it is connected and $deg_{\Phi}(\alpha) = 2$ for every $\alpha \in V(\Phi)$. Now, Since $n(B(G)) = n(\Delta(G)) = n(\Gamma(G))$ (Lemma 2.65 iii)) and $B(G)$ is connected, we conclude that Φ is connected. As $B(G)$ is a cycle with bipartition parts $\rho(G)$ and $cd(G)^*$, we can see that vertices of $B(G)$ alternate in $\rho(G)$ and $cd(G)^*$, and since $deg_{B(G)}(\alpha) = 2$ for every $\alpha \in V(B(G))$, we conclude that n must be even. Indeed, if $n = 2k$, then $|\rho(G)| = |cd(G)^*| = k$ and each $\alpha \in X \in \{\rho(G), cd(G)^*\}$ is adjacent to exactly two vertices in $Y = V(B(G)) \setminus X$. To illustrate more, consider $p \in \rho(G)$. Since $deg_{B(G)}(p) = 2$, then there exist $x \neq y \in cd(G)^*$ such that $p \mid x, y$. Notice that $|\pi(x)| = deg_{B(G)}(x) = deg_{B(G)}(y) = |\pi(y)| = 2$ and since $n \geq 6$, there exist $r \neq s \in \rho(G)$ such that $r \mid x$ and $s \mid y$. Hence p is adjacent to exactly r and s in $\Delta(G)$. Therefore $deg_{\Delta(G)}(p) = 2$ for every $p \in \rho(G)$. By a similar discussion we can deduce that $deg_{\Gamma(G)}(x) = 2$ for every $x \in cd(G)^*$, which completes the proof. □

Theorem 8.2. *If G is a finite group whose $B(G) \cong C_n$ for a positive integer n, then either $n = 4$ or $n = 6$.*

Proof. Notice that a cycle must contain at least three vertices, thus $n \geq 3$. As mentioned in the previous proof, since $B(G)$ is a cycle of length n, then n is even. Furthermore, by Theorem 3 in [Iranmanesh and Praeger, 2010] we know that both of $\Delta(G)$ and $\Gamma(G)$ are acyclic (have no cycles) if and only if $B(G) \cong C_4$. In this case, both $\Delta(G)$ and $\Gamma(G)$ are paths of length one. On the other hand, if $n \geq 6$, we can deduce by Lemma 8.1 that $\Delta(G)$ and $\Gamma(G)$ are cycles. Therefore, $\Delta(G)$ is a cycle if $n \geq 6$ and it is a path of length one if $n = 4$. It follows then by Theorem 5.3 that $|\rho(G)| \leq 4$. And since $|\rho(G)| = |cd(G)^*|$ (see Proof of Lemma 8.1), we conclude that $B(G)$ is isomorphic to C_4, C_6 or C_8. We claim now that $B(G)$ cannot be a C_8. If $B(G) \cong C_8$, then both $\Delta(G)$ and $\Gamma(G)$ are cycles of length 4. Suppose first that G is solvable. By the main theorem of [Lewis and Meng, 2012], we have $G = H \times K$ where $\rho(H) = \{p, q\}$, $\rho(K) = \{r, s\}$ and both $\Delta(H)$ and $\Delta(K)$ are disconnected. In other words, each of $\Delta(H)$ and $\Delta(K)$ is composed of two isolated vertices. This implies that there exist l, $k \in cd(K)^*$ and m, $n \in cd(H)^*$ such that $l = r^\alpha$, $k = s^\beta$, $m = p^a$ and $n = q^b$, for some positive integers α, β, a and b. As G is a direct product of H and K, we have $cd(G) = cd(H) \times cd(K)$. Thus $\{1, m, n, l, k, ml, mk, nl, nk\} \subseteq cd(G)$, which contradicts the structure of $B(G) \cong C_8$ as $deg_{B(G)}(p)$, $deg_{B(G)}(q)$, $deg_{B(G)}(r)$ and $deg_{B(G)}(s)$ are greater or equal than 3. So, $B(G)$ cannot be a C_8 if G is solvable. Suppose next that G is nonsolvable. By Theorem 3.17 we know that a square cannot be the prime vertex graph of a nonsolvable group. Thus, if G is nonsolvable, then $G \ncong C_8$, which completes the proof. □

Corollary 8.3. *Let G be a finite group whose $B(G)$ is a cycle. Then G is solvable and $dl(G) \leq |cd(G)| \leq 4$.*

Proof. By Theorem 8.2, we know that $n = 4$ or $n = 6$. This implies that, $\Gamma(G)$ is either P_1 or C_3. Thus $\Gamma(G)$ is a complete graph. It follows then from Theorem 7.3 in [Lewis, 2008] that G is solvable. Now, as $|cd(G)| = 3$ if $n = 4$ and $|cd(G)| = 4$ if $n = 6$, we have $|cd(G)| \leq 4$. Since G is solvable, we conclude by [Garrison, 1973] that $dl(G) \leq |cd(G)| \leq 4$ (Taketa inequality). □

Example 8.4. *It was proved in Section 6 of [Wolf et al., 2005] that for every non-Mersenne prime p which is congruent to* 1 *(mod* 3*), we can find an odd prime q such that $q \mid p+1$. Furthermore, there exists a solvable group G associated with (p, q) of order $3p^7q$ where $cd(G) = \{1, 3q, p^2q, 3p^3\}$. This gives an example of a solvable group G whose $B(G) \cong C_6$ (see Figure 8.1).*

Example 8.5. *By using GAP (a system for computational discrete algebra), we can see*

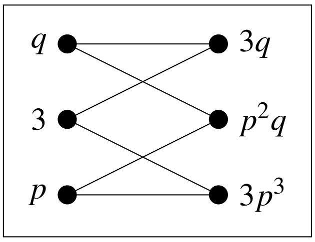

Figure 8.1: $B(G)$ where G is the group associated with (p,q) described in Example 8.4.

that there are exactly two nonabelian groups among the 66 *solvable groups of order* 588. *They have* $\{1,6,12\}$ *as their character degree sets. Thus, these groups provide an example of a finite solvable group whose bipartite divisor graph is a* C_4.

Example 8.6. *Let* $P \in Syl_p(G)$ *where G is nonabelian,* $|P| = p^2$, $p \geq 7$, $p \neq 11$ *and* $|G : P| = 12$. *Since* $p \neq 11$ *and the number of Sylow p-subgroups in G,* $n_p(G)$, *divides* $|G : P| = 12$ *and is congruent to* 1 *(mod p), we conclude that* $n_p(G) = 1$. *Thus* $P \lhd G$. *On the other hand, since* $|P| = p^2$, *we deduce that P is abelian. It follows then by Theorem 6.15 in [Isaacs, 1976] that every degree in* $cd(G)$ *divides* $|G : P|$. *Thus* $m \mid 12$ *for every* $m \in cd(G)$.

If $B(G)$ *is a cycle, then* $deg_{B(G)}(\alpha) = 2$ *for every* $\alpha \in V(B)$. *So no prime powers can occur in* $B(G)$. *Hence, the only possible degrees in* $cd(G)^*$ *are* 6 *and* 12 *where both must occur as* $B(G)$ *is a cycle.*

Therefore, $B(G)$ *is a cycle if and only if* $cd(G) = \{1,6,12\}$.

Remark 8.7. *Let G be a finite group whose* $B(G) \cong C_4$ *and* $\rho(G) = \{p,q\}$. *We claim that* $\rho(G) \neq \pi(G)$. *Suppose on the contrary that* $\pi(G) = \rho(G) = \{p,q\}$ *and let* $|G| = p^\alpha q^\beta$ *for some positive integers* α *and* β. *Since* $p \mid m$ *for every* $m \in cd(G)^*$, *we conclude by Corollary 12.2 in [Isaacs, 1976] that G has a normal p-complement Q. As* $\pi(G) = \{p,q\}$ *and by the definition of a p-complement being a subgroup whose order is a power of q and index is a power of p, we deduce that Q is the normal Sylow q-subgroup of G. Similarly, since q divides every degree in* $cd(G)^*$, *we can see that G has a normal q-complement P which is the normal Sylow p-subgroup of G. Thus G is nilpotent as all its Sylow subgroups are normal. Hence,* $G = P \times Q$. *Notice that G is solvable by Corollary 8.3 and since every degree in* $cd(G)^*$ *is divisible by p and q, we conclude by Ito-Michler's Theorem that neither P nor Q is abelian. Thus* $\{1, p^a\} \subset cd(P)$ *and*

$\{1,q^b\} \subset cd(Q)$ for some positive integers $a \le \alpha$ and $b \le \beta$. This implies that G has prime power character degrees (p^α, q^β) which is impossible as $B(G)$ is a cycle. Therefore if $B(G) \cong C_4$, then $\rho(G) \subset \pi(G)$.

However, if $B(G) \cong C_6$, then $\rho(G)$ can be equal to $\pi(G)$. For example, consider the group G associated with (p,q) where p, q and G are as described in Example 8.4 , then $|G| = 3p^7q$ and $\rho(G) = \{3,p,q\}$. Thus $\pi(G) = \rho(G)$ in this case.

Lemma 8.8. *Suppose $G = N \rtimes H$, where N is an abelian normal subgroup of G. Then $cd(G) = \{\beta(1)|G : I_G(\lambda)| : \lambda \in Irr(N),\ \beta \in Irr(I_G(\lambda)/N)\}$.*

Proof. Let $\psi \in Irr(G)$ and λ be an irreducible constituent of ψ_N. By Clifford's Theorem we can write: $\psi_N = e_\psi \sum_{i=1}^t \lambda_i$, where $t = |G : I_G(\lambda)|$ and $e_\psi \neq 0$. By Frobenius Reciprocity, we know that $e_\psi = [\psi_N, \lambda] = [\psi, \lambda^G]$. If $\lambda_0 \in Irr(I_G(\lambda))$ such that $(\lambda_0)_N = \lambda$, we conclude by Gallagher's Theorem that the characters $\beta_j\lambda_0 \in Irr(I_G(\lambda))$ for every $\beta_j \in Irr(I_G(\lambda)/N)$. Furthermore, they are all of the irreducible constituents of $\lambda^{I_G(\lambda)}$ and we have: $\lambda^{I_G(\lambda)} = \beta_j(1)\sum_j(\lambda_0\beta_j)$. Thus $(\lambda^{I_G(\lambda)})^G = \beta_j(1)\sum_j(\lambda_0\beta_j)^G$. As ψ is an irreducible constituent of λ^G, we deduce that $\psi = (\lambda_0\beta_{j_0})^G$ for some $\beta_{j_0} \in Irr(I_G(\lambda)/N)$. By the definition of induced characters we have $\psi(1) = (\lambda_0\beta_{j0})^G(1) = \lambda_0(1)\beta_{j_0}(1)|G : I_G(\lambda)| = \lambda(1)\beta_{j_0}(1)|G : I_G(\lambda)| = \beta_{j_0}(1)|G : I_G(\lambda)|$. Remark that since N is abelian, $\lambda(1) = 1$. □

The following theorem shows that if G is a finite group whose $B(G) \cong C_4$, then it has an abelian normal Hall subgroup N, whose irreducible character degree set is sufficient to specify the character degrees of G.

Theorem 8.9. *Let G be a finite group whose $B(G)$ is a C_4. Then, there exists an abelian normal Hall subgroup N of G such that $cd(G) = \{|G : I_G(\lambda)| : \lambda \in Irr(N)\}$.*

Proof. Let G be a finite group of order $p_1^{a_1}p_2^{a_2}...p_l^{a_l}$. Suppose without loss of generality that $p_1 = p$, $p_2 = q$ and $B(G) : p - m - q - n - p$. Since $(m, p_i) = (n, p_i) = 1$ for every $p_i \in \pi(G) \setminus \{p,q\}$, we conclude by Ito-Michler's Theorem that the Sylow p_i-subgroup of G is normal abelian for every $3 \le i \le l$. Let $N = P_3P_4...P_l$ where $P_i \in Syl_{p_i}(G)$ for every $3 \le i \le l$. As $B(G) \cong C_4$, we deduce by Remark 8.7 that G is not a $\{p,q\}$-group. Thus N is a nontrivial abelian normal Hall subgroup of G. Consider now G/N which is a $\{p,q\}$-

group. As $\rho(G/N) \subseteq \pi(G/N) = \{p,q\}$, we conclude that either $\rho(G/N)$ is a singleton or $\rho(G/N) = \pi(G/N)$. In both cases the bipartite divisor graph of G/N is not a square (C_4). Recall that $cd(G/N)^* \subseteq cd(G)^*$ and since $deg_{B(G)}(\alpha) = 2$ for every $\alpha \in V(B(G))$, we deduce that $cd(G/N)$ has no prime power degrees. Thus for each nonlinear character $\chi \in Irr(G/N)$, we have $\chi(1) = p^a q^b$, for some positive integers a and b. Hence each character degree in $cd(G/N)$ is divisible by p and q. As a consequence, G/N has a normal abelian p-complement Q which is the normal Sylow q-subgroup of G and a normal abelian q-complement P which is the normal Sylow p-subgroup (see Remark 8.7). This implies that $G/N = P \times Q$, which is impossible as in this case G/N has prime power character degrees (see Remark 8.7). Thus the set $cd(G/N)$ must be trivial, in other words G/N must be abelian. Furthermore, as $G = NH$, $N \trianglelefteq G$ and $H \cap N = 1$ where $H \cong G/N$, we conclude that $G = N \rtimes H$ (Theorem 2.21). It follows then from Lemma 8.8 that $cd(G) = \{\beta(1)|G : I_G(\lambda)| : \lambda \in Irr(N), \beta \in Irr(I_G(\lambda)/N)\}$. But since G/N is abelian, $I_G(\lambda)/N$ is abelian. Thus $cd(G) = \{|G : I_G(\lambda)| : \lambda \in Irr(N)\}$. $\square$

Example 8.10. *Let $G = S_3 \times N$, where $N \cong \mathbb{Z}_5$. It is clear that $cd(G) = cd(S_3) = \{1,2\}$ (see Example 2 p.153 in [Alperin and Bell, 1991]). Thus $B(G)$ is a path of length one ($B(G) : 2-2$), which is not a cycle. Furthermore, as G is the direct product of S_3 and N, we can see that N is an abelian normal Hall subgroup of G. Consider now x where $N =< x >$ and let ε be a primitive 5^{th} root of unity. We know that $Irr(N)$ has exactly five linear characters $\{\lambda_1, ..., \lambda_5\}$, where $\lambda_i(x^a) = \varepsilon^{a(i-1)}$ for every $i \in \{1, ..., 5\}$ and for every $a \in \{0, ..., 4\}$ (see Example 1 p.153 in [Alperin and Bell, 1991]). Since each λ_i is linear, we have: $I_G(\lambda_i) = \{g \in G : \lambda_i^g = \lambda_i\} = \{g \in G : \lambda_i(g^{-1}xg) = \lambda_i(x)\} = \{g \in G : 1 = 1\} = G$. Therefore, $|G : I_G(\lambda_i)| = 1$ for every $i \in \{1, ..., 5\}$, which implies that $\{1\} = \{|G : I_G(\lambda_i)| : 1 \leq i \leq 5\} \neq cd(G) = \{1,2\}$.*

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