MIXED STRATEGIES

when there are

MORE THAN TWO ACTIONS
The best response analysis gets complicated

Get help from the concept of domination
Given a strategic form game with mixed strategies

\[ G = (N, (\Pi(S_1), \ldots, \Pi(S_n)), (U_1, \ldots, U_n)) \]

for an Agent \(i\), \(\pi_i\) strictly dominates \(\pi'_i\) if

1. for every strategy profile
   \[ \pi_{-i} = (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n) \]
   of the agents other than \(i\),

   \[ U_i(\pi_i, \pi_{-i}) > U_i(\pi'_i, \pi_{-i}) \]
Given a strategic form game with mixed strategies

\[ G = (N, (\Pi(S_1), \ldots, \Pi(S_n)), (U_1, \ldots, U_n)) \]

Agent \( i \)'s pure strategy \( s_i \), is strictly dominated if

he has a mixed strategy that strictly dominates it, that is,

- There is a mixed strategy \( \pi_i \) of agent \( i \) such that for every strategy profile
  \[ \pi_{-i} = (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n) \]
  of the agents other than \( i \),

  \[ u_i(\pi_i, \pi_{-i}) > u_i(s_i, \pi_{-i}) \]
A pure strategy that is not dominated by pure strategies can be strictly dominated by a mixed strategy:

<table>
<thead>
<tr>
<th></th>
<th>DL</th>
<th>CC</th>
<th>q-Mix</th>
</tr>
</thead>
<tbody>
<tr>
<td>SELES</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DL</td>
<td>50</td>
<td>80</td>
<td>$50q + 80(1 - q)$</td>
</tr>
<tr>
<td>CC</td>
<td>100</td>
<td>40</td>
<td>$100q + 40(1 - q)$</td>
</tr>
<tr>
<td>Lob</td>
<td>40</td>
<td>100</td>
<td>$40q + 100(1 - q)$</td>
</tr>
<tr>
<td>$p$-Mix</td>
<td>$50p_1 + 100p_2 + 40(1 - p_1 - p_2)$</td>
<td>$80p_1 + 40p_2 + 100(1 - p_1 - p_2)$</td>
<td></td>
</tr>
</tbody>
</table>
Mix the graphs of CC and Lob to obtain a line that is always above the graph of DL.
Strictly dominated strategies are never played.
Using this knowledge, you can eliminate some pure strategies.

The simplified game:

<table>
<thead>
<tr>
<th>SELES</th>
<th>HINGIS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DL</td>
</tr>
<tr>
<td>CC</td>
<td>100</td>
</tr>
<tr>
<td>Lob</td>
<td>40</td>
</tr>
<tr>
<td>$p$-Mix</td>
<td>$100p_2 + 40(1 - p_2)$</td>
</tr>
</tbody>
</table>
Weak Domination

Does not really help. Players can assign positive probability to weakly dominated actions and still play a best response.

What if there is no domination?

**Very Useful Proposition:**

If a player is playing a mixed strategy as a best response and if he assigns positive probability to two his actions (say A and B) then his expected payoffs from these two actions are equal.
FIGURE 5.17 Payoff Table for Tennis Point with Lob

<table>
<thead>
<tr>
<th></th>
<th>HINGIS</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DL</td>
<td>CC</td>
<td>q-Mix</td>
</tr>
<tr>
<td>SELES</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DL</td>
<td>50</td>
<td>80</td>
<td>50$q$ + 80$(1 - q)$</td>
</tr>
<tr>
<td>CC</td>
<td>90</td>
<td>20</td>
<td>90$q$ + 20$(1 - q)$</td>
</tr>
<tr>
<td>Lob</td>
<td>70</td>
<td>60</td>
<td>70$q$ + 60$(1 - q)$</td>
</tr>
<tr>
<td>p-Mix</td>
<td>$50p_1 + 90p_2 + 70(1 - p_1 - p_2)$</td>
<td>$80p_1 + 20p_2 + 60(1 - p_1 - p_2)$</td>
<td></td>
</tr>
</tbody>
</table>
When Seles plays DL, Lob, and CC

FIGURE 5.18 Diagrammatic Solution for Hingis's q-Mix

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Seles’ best responses to different q choices of Hingis:

Against $q < 0.5$ Seles plays DL \[ p_1 = 1 \quad p_2 = 0 \]

Against $q = 0.5$ Seles mixes DL and Lob \[ \text{any } p_1 \text{ in } [0,1] \quad p_2 = 0 \]

Against $0.5 < q < 0.667$ Seles plays Lob \[ p_1 = 0 \quad p_2 = 0 \]

Against $q = 0.667$ Seles mixes CC and Lob \[ p_1 = 0 \quad \text{any } p_2 \text{ in } [0,1] \]

Against $q > 0.667$ Seles plays CC \[ p_1 = 0 \quad p_2 = 1 \]
Claims:

In a Nash equilibrium, Hingis does NOT play

1. $q < 0.5$
   
   then Seles plays DL, then Hingis plays DL, contradiction

2. $0.5 < q < 0.667$
   
   then Seles plays Lob, then Hingis plays CC, contradiction

3. $q = 0.667$
   
   then Seles mixes Lob and CC, then Hingis plays CC, contradiction

4. $q > 0.667$
   
   then Seles plays CC, then Hingis plays CC, contradiction
When $q = 0.5$:

<table>
<thead>
<tr>
<th></th>
<th>DL</th>
<th>CC</th>
<th>$q$-Mix</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SELES</strong></td>
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<td>50</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>Lob</td>
<td>70</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>$p$-Mix</td>
<td>$50p_1 + 70(1 - p_1)$</td>
<td>$80p_1 + 60(1 - p_1)$</td>
</tr>
</tbody>
</table>

Nash equilibrium: $q = 0.5$, $p_2 = 0$, $p_1 = ?$
100 minus Hingis’ payoffs from DL and CC (as a function of $p_1$)

FIGURE 5.20 Diagrammatic Solution for Seles’s $p$-Mix
## FIGURE 5.21 Payoff Table for Tennis Point with Lob: The Coincidence Case

<table>
<thead>
<tr>
<th></th>
<th>HINGIS</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DL</td>
<td>CC</td>
<td>q-Mix</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SELES</td>
<td>50</td>
<td>80</td>
<td>$50q + 80(1 - q)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>90</td>
<td>20</td>
<td>$90q + 20(1 - q)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>50</td>
<td>$70q + 50(1 - q)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$50p_1 + 90p_2 + 70(1 - p_1 - p_2)$</td>
<td>$80p_1 + 20p_2 + 50(1 - p_1 - p_2)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Question: Is Lob weakly dominated?
Seles’ best responses to different $q$ choices of Hingis:

Against $q < 0.6$  Seles plays DL  \[ p_1 = 1 \quad p_2 = 0 \]

Against $q = 0.6$  Seles mixes DL, CC and Lob  any $p_1 \; p_2$ in $[0,1]$

Against $q > 0.6$  Seles plays CC  \[ p_1 = 0 \quad p_2 = 1 \]
Claims:

In a Nash equilibrium, **Hingis does NOT play**

1. $q < 0.6$
   
   then Seles plays DL, then Hingis plays DL, contradiction

4. $q > 0.6$
   
   then Seles plays CC, then Hingis plays CC, contradiction

---

How to make $q = 0.6$ part of an equilibrium?

Seles must choose a mixed strategy such that Hingis will receive the same payoff from DL and CC
How to make \( q = 0.6 \) part of an equilibrium?

Seles must choose a mixed strategy such that Hingis will receive the same payoff from DL and CC

For this, solve

\[
50 p_1 + 90 p_2 + 70 (1 - p_1 - p_2) = 80 p_1 + 20 p_2 + 50 (1 - p_1 - p_2)
\]

\[
=>
50 p_2 + 20 = 50 p_1
\]

\[
=>
 p_2 = p_1 - 0.4
\]

Now remember that \( p_2 + p_1 \leq 1 \)
FIGURE 5.23 Seles’s Indeterminate $p$-Mix in the Coincidence Case

$p_2 = p_1 - 0.4$

$D = (0.7, 0.3)$

$C = (0.4, 0)$
The mixed strategy Nash equilibria are

\[ \left( \left( p_1, p_2, 1-p_1-p_2 \right), \left( 0.6, 0.4 \right) \right) \]

where \( p_1 \) is between 0.4 and 0.7 and

\[ p_2 = p_1 - 0.4 \]

**NOTE:** Even though there is an infinite number of equilibria, the resulting payoff profile is unique: (62, 38)
A 3x3 game:

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>Mixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>5, 9</td>
<td>3, 7</td>
<td>4, 14</td>
<td>4+q₁ - q₂, 14 - 5q₁ - 7q₂</td>
</tr>
<tr>
<td>M</td>
<td>4, 3</td>
<td>6, 4</td>
<td>4, 5</td>
<td>4+2q₂, 5 - 2q₁ - q₂</td>
</tr>
<tr>
<td>D</td>
<td>3, 4</td>
<td>4, 6</td>
<td>5, 1</td>
<td>5 - 2q₁ - q₂, 1 + 3q₁ + 5q₂</td>
</tr>
<tr>
<td>Mixed</td>
<td>3+2p₁+p₂, 4+5p₁-p₂</td>
<td>4-p₁+2p₂, 6+p₁-2p₂</td>
<td>5-p₁-p₂, 1+13p₁+4p₂</td>
<td></td>
</tr>
</tbody>
</table>
1. Is there a pure strategy equilibrium? No

2. Is there a mixed equilibrium where all pure strategies are assigned positive probabilities? That is, where

\[ p_1 > 0 \quad p_2 > 0 \quad 1 - p_1 - p_2 > 0 \quad \text{and} \]
\[ q_1 > 0 \quad q_2 > 0 \quad 1 - q_1 - q_2 > 0 \]
\[ p_1 > 0 \quad p_2 > 0 \quad 1 - p_1 - p_2 > 0 \quad \text{implies (by our proposition)} \]

\[ 4 + q_1 - q_2 = 4 + 2q_2 \quad \text{and} \]
\[ 4 + q_1 - q_2 = 5 - 2q_1 - q_2 \]

Solving this, one gets \[ q_1 = 1/3 \quad \text{and} \quad q_2 = 1/9 \]

The proposition tells us that if Column’s q-mix is not this one, Row can’t play a mixed strategy of the above kind as a best response.

That is, if there is an equilibrium where Row plays a mixed strategy that satisfies \[ p_1 > 0 \quad p_2 > 0 \quad 1 - p_1 - p_2 > 0 \], then in that equilibrium Col must play \[ q_1 = 1/3 \] and \[ q_2 = 1/9 \].
\[ q_1 > 0 \quad q_2 > 0 \quad 1 - q_1 - q_2 > 0 \quad \text{implies (by our proposition)} \]

\[ 4 + 5p_1 - p_2 = 6 + p_1 - 2p_2 \quad \text{and} \]

\[ 4 + 5p_1 - p_2 = 1 + 13p_1 + 4p_2 \]

Solving this, one gets \[ p_1 = \frac{7}{12} \quad \text{and} \quad p_2 = -\frac{1}{3} \]

This is a contradiction. So there is no equilibrium in which Col plays a mixed strategy where \[ q_1 > 0 \quad q_2 > 0 \quad 1 - q_1 - q_2 > 0 \]
What about a mixed strategy where

\[ 1 - q_1 - q_2 = 0 \]  
but \( q_1 > 0 \) and \( q_2 > 0 \)?

Then

\[ 4 + 5 p_1 - p_2 = 6 + p_1 - 2p_2 \]

and

\[ 4 + 5 p_1 - p_2 > 1 + 13 p_1 + 4 p_2 \]

This can be rewritten as

\[ p_2 = 2 - 4 p_1 \]

and

\[ p_2 < 3/5 - 8/5 p_1 \]

No \( p_1 \) and \( p_2 \) value simultaneously satisfies these conditions.

So there is no equilibrium where \( 1 - q_1 - q_2 = 0 \) but \( q_1 > 0 \) and \( q_2 > 0 \)
What about a mixed strategy where

\[ q_2 = 0 \text{ but } q_1 > 0 \text{ and } 1 - q_1 - q_2 > 0 \] ?

Then

\[ 4 + 5 p_1 - p_2 = 1 + 13 p_1 + 4 p_2 \]

and

\[ 4 + 5 p_1 - p_2 > 6 + p_1 - 2 p_2 \]

This can be rewritten as

\[ p_2 > 2 - 4 p_1 \]

and

\[ p_2 = 3/5 - 8/5 p_1 \]

No \( p_1 \) and \( p_2 \) value simultaneously satisfies these conditions.

So there is no equilibrium where \( q_2 = 0 \) but \( q_1 > 0 \) and \( 1 - q_1 - q_2 > 0 \)
What about a mixed strategy where
\[ q_1 = 0 \text{ but } q_2 > 0 \text{ and } 1 - q_1 - q_2 > 0 \]?

Then
\[ 4 + 5 p_1 - p_2 < 6 + p_1 - 2 p_2 \]

and
\[ 6 + p_1 - 2 p_2 = 1 + 13 p_1 + 4 p_2 \]

This can be rewritten as
\[ p_2 < 2 - 4 p_1 \]

and
\[ p_2 = \frac{5}{6} - 2 p_1 \]

Any \( p_1 \) and \( p_2 \) value such that
\[ 0 \leq p_1 \leq \frac{5}{12} \]

and
\[ p_2 = \frac{5}{6} - 2 p_1 \]
satisfies this condition.
If Col chooses $q_1 = 0$ but $q_2 > 0$ and $1 - q_1 - q_2 > 0$

Then for Row, U is a dominated strategy.

Thus, in equilibrium, $p_1 = 0$.

Since $p_2 = 5/6 - 2p_1$, this implies $p_2 = 5/6$.

This means, Row is assigning a positive probability to both M and D.

For this to be part of an equilibrium, our proposition says

$$4 + 2q_2 = 5 - 2q_1 - q_2 = 5 - q_2$$

Solving, we get $q_2 = 1/3$. 
So the equilibrium is as follows:

\[(0, \frac{5}{6}, \frac{1}{6}), (0, \frac{1}{3}, \frac{2}{3})\]
Collective Action Games

+ Games with a very large number of players
COLLECTIVE ACTION GAMES

- Social problems concerning collective action
  - multiple-person games with too many players
  - unsatisfactory outcomes
  - social interest vs. private incentives

Societies usually have problems in implementing outcomes that are considered to be good for everybody.

- Helping the poor
- Planting trees and not burning them later
- Keeping the environment clean
- Obeying the traffic laws
A Simple Example

• Two farmers: need an irrigation project
  • it is a pure public good (nonexcludable and nonrival)
    like national defense
    compare it to a private good (like a sandwich)
  • who is going to build it?

• Strategies: participate or shirk
  • $b_1$ and $c_1$: benefit and cost of project when 1 person builds
  • $b_2$ and $c_2$: benefit and cost of project when 2 persons build
What is an individual’s payoff?

The benefit minus the cost   (if she participated)

or

The benefit   (if she shirked)

What is the best for the society?

The outcome that maximizes

the sum of individual payoffs

Why sum? (utilitarianism)

Why utilitarianism? Because the Dixit-Skeath book uses it.
What is the best for the society?

• **Pareto optimality** (Vilfredo Pareto)
  An outcome is Pareto-optimal if there is no alternative outcome which gives all agents an at least as high payoff and some agents a higher payoff.

• **Egalitarianism** (John Rawls)
  The Egalitarian-optimal outcome maximizes the smallest payoff in the society.

• **Utilitarianism** (John Stuart Mill)
  The Utilitarian-optimal outcome maximizes the total payoff in the society.
\[ b_1 \quad c_1 \]
\[ b_2 \quad c_2 \]

**General Case**

<table>
<thead>
<tr>
<th></th>
<th><strong>YOUR NEIGHBOR</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>YOU</strong></td>
<td>Build</td>
<td>Not</td>
</tr>
<tr>
<td>Build</td>
<td>( b_2 - c_2, b_2 - c_2 )</td>
<td>( b_1 - c_1, b_1 )</td>
</tr>
<tr>
<td>Not</td>
<td>( b_1, b_1 - c_1 )</td>
<td>( 0, 0 )</td>
</tr>
</tbody>
</table>

**Utilitarian payoff:** your payoff + your neighbor’s payoff

**Utilitarian optimum:** maximizes the utilitarian payoff

**Egalitarian payoff:** minimum \{your payoff, your neighbor’s payoff\}

**Egalitarian optimum:** maximizes the egalitarian payoff
Prisoners’ Dilemma Game

\[ b_1 = 6 \quad c_1 = 7 \]
\[ b_2 = 8 \quad c_2 = 4 \]

**YOUR NEIGHBOR**

<table>
<thead>
<tr>
<th></th>
<th>Build</th>
<th>Not</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>YOU</strong></td>
<td>Build</td>
<td>4, 4</td>
</tr>
<tr>
<td></td>
<td>Not</td>
<td>6, -1</td>
</tr>
</tbody>
</table>

Utilitarian optimum: (Build, Build)

Egalitarian optimum: (Build, Build)

Pareto optima: (Build, Build), (Not, Build), (Build, Not)

Nash equilibrium: (Not, Not)
Prisoners’ Dilemma Game

\[ b_1 = 6 \quad c_1 = 7 \]
\[ b_2 = 6.3 \quad c_2 = 4 \]

Utilitarian optimum: (Build, Not) or (Not, Build)

Egalitarian optimum: (Build, Build)

Pareto optima: (Build, Build), (Not, Build), (Build, Not)

Nash equilibrium: (Not, Not)
\[ b_1 = 6 \quad c_1 = 4 \]  Chicken Game
\[ b_2 = 8 \quad c_2 = 3 \]

<table>
<thead>
<tr>
<th></th>
<th>Build</th>
<th>Not</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>YOU</strong></td>
<td>Build</td>
<td>5, 5</td>
</tr>
<tr>
<td></td>
<td>Not</td>
<td>6, 2</td>
</tr>
</tbody>
</table>

Utilitarian optimum: (build, build)

Egalitarian optimum: (build, build)

Pareto optima: (Build, Build), (Not, Build), (Build, Not)

Nash equilibrium: (build, not) and (not, build)
\[ b_1 = 3 \quad c_1 = 7 \]
\[ b_2 = 8 \quad c_2 = 4 \]

**Assurance Game**

<table>
<thead>
<tr>
<th>YOU</th>
<th>Build</th>
<th>Not</th>
</tr>
</thead>
<tbody>
<tr>
<td>Build</td>
<td>(4, 4)</td>
<td>(-4, 3)</td>
</tr>
<tr>
<td>Not</td>
<td>(3, -4)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

- **Utilitarian optimum:** (build, build)
- **Egalitarian optimum:** (build, build)
- **Pareto optima:** (Build, Build)
- **Nash equilibrium:** (build, build) and (not, not)

**FIGURE 11.4** Collective Action as an Assurance Game
What about in a large group?

N agents for a public project

An agent’s benefit if n people participates: $b(n)$

Cost of participating if n people participates: $c(n)$

Two strategies: Shirk or Participate

An agent’s payoff depends on what the others are doing

If n people are participating:

payoff of a shirking agent: $s(n) = b(n)$

payoff of a participating agent: $p(n) = b(n) - c(n)$

An agent compares $s(n)$ and $p(n+1)$
Social payoff from \( n \) participants

\[
T(n) = n \, p(n) + (N - n) \, s(n)
\]

The marginal social gain from a one person increase in participants

\[
T(n+1) - T(n) = p(n+1) - s(n) + n \, (p(n+1) - p(n)) + (N - n - 1) \, (s(n+1) - s(n))
\]

- Marginal private gain (the part that derives individual choice)
- Externality on participants
- Externality on shirkers
FIGURE 11.6 Multiperson Prisoners' Dilemma
FIGURE 11.7  Multiperson Chicken

Payoff

\[ p(n+1) \]

\[ s(n) \]

\[ n \rightarrow \]

\[ N - 1 \]
FIGURE 11.8 Multiperson Assurance Game
The route choice from home to work:

6000 drivers

Two routes from the suburbs to the city

Local route: always takes 45 minutes

Expressway: takes 15 minutes if there are not more than 2000 drivers
After that, increases 0.01 minutes with every additional driver

Want to

model it as a collective-action game

find Nash equilibria

find the social optimum
The route choice game:

Payoff: gain from traffic out of an hour

Shirkers’ payoffs (from the local route)

\[ s(n) = 15 \]

Participants’ payoffs (from the expressway)

\[ p(n) = \begin{cases} 
45 & \text{if } n \leq 2000 \\
45 - 0.01(n - 2000) & \text{if } n > 2000 
\end{cases} \]
Nash equilibria: $n = 4999, 5000$
Finding the social optimum

\[ T(n) = n p(n) + (6000 - n) s(n) \]

\[ T(n) = \begin{cases} 
  n \cdot 45 + (6000 - n) \cdot 15 & \text{if } n \leq 2000 \\
  n \left( 45 - 0.01 (n - 2000) \right) + (6000 - n) \cdot 15 & \text{if } n > 2000
\end{cases} \]

\[ T(n) = \begin{cases} 
  90000 + 30n & \text{if } n \leq 2000 \\
  90000 + 50n - 0.01n^2 & \text{if } n > 2000
\end{cases} \]
\[ T'(n) = \begin{cases} 
30 & \text{if } n \leq 2000 \\
50 - 0.02n & \text{if } n > 2000 
\end{cases} \]

\[ T''(n) = \begin{cases} 
0 & \text{if } n \leq 2000 \\
-0.02 & \text{if } n > 2000 
\end{cases} \]

\[ 50 - 0.02n = 0 \implies n = 2500 \]
FIGURE 11.10 Payoffs in Computer Choice Game
The differential version of marginal social gain:

\[ T'(n) = p(n) - s(n) + np'(n) + (N - n)s'(n) \]

- Marginal private gain (the part that derives individual choice)
- Externality on participants
- Externality on shirkers