On Algorithmic Solutions to Simple Allocation Problems

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Abstract

We study the implications of rationality on a class of solutions to simple allocation problems: an endowment $E$ is to be allocated among $n$ agents (or commodities) described by a characteristic vector $c$. We introduce a class of algorithmic rules that generalize the “Equal Gains algorithm” and show that it uniquely satisfies rationality, continuity in characteristic values, and other-characteristic monotonicity. The Equal Gains rule is the only anonymous member of this class. We show that it uniquely satisfies rationality, continuity in characteristic values, and equal treatment of equals. Its dual, the Equal Losses rule, uniquely satisfies continuity, equal treatment of equals, and two properties that constitute the dual of rationality: translation down and translation up.

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1 Introduction

*Revealed preference theory* studies conditions under which by observing the choice behavior of an agent, one can discover the underlying preferences that govern it. Choice rules for which this is possible are called *rational*. Most of the earlier work on rationality analyzes consumers’ demand choices from budget sets (e.g. see Samuelson, 1938, 1948). The underlying premise that choices reveal information about preferences, however, is applicable to a wide range of choice situations. For example, applications of the theory to bargaining games (Nash, 1950) characterize bargaining rules which can be “rationalized” as maximizing the underlying preferences of an impartial arbitrator (Peters and Wakker, 1991; Bossert, 1994; Ok and Zhou, 2000; Sánchez, 2000).

In this paper, we study a class of *rational* solutions to *simple allocation problems*. We also study the dual properties of *rationality*. A simple allocation problem for a society $N$ is an $|N| + 1$ dimensional nonnegative real vector $(c_1, ..., c_{|N|}, E)$ satisfying $\sum_{N} c_i \geq E$ where $E$, the *endowment* has to be allocated among agents (or commodities) in $N$ who are characterized by $c$, the *characteristic vector*.

Simple allocation problems have a wide range of applications discussed in detail in Subsection 1.1 (including consumer choice under fixed prices). We will use the following “permit allocation problem” as our running example in the paper: the Environmental Protection Agency (EPA) is to allocate an amount $E$ of pollution permits among firms in $N$. Each firm $i$ is imposed by the local authority an *emission constraint* $c_i$ on its pollution level. A “permit allocation rule” is *rational* if its choices are consistent with maximization of a binary relation on the allocation space. This binary relation can be interpreted as either a social welfare ordering or individual preferences of a social planner (such as the EPA official who designed the rule). From a positive standpoint (*i.e.* interpreting a rule as data on a decision-maker’s choices), rationality is desirable since it allows modeling choice behavior as preference maximization. Rationality can also be interpreted as a desirable normative property since the maximized relation serves as a rationale to explain and motivate the rule’s choices to the public and to endow it with a certain “consistency”.

In a companion paper (Kubris, 2008), we study *rational, transitive-rational, and representable* solutions to simple allocation problems.\(^1\) There, we observe that, among the four well-known solutions to simple allocation problems, only the *Equal Gains rule* is *rational* while the *Proportional*\(^1\)\textsuperscript{A rule is transitive-rational (respectively, representable) if its choices coincide with maximization of a transitive binary relation (respectively, utility function) on the allocation space.}
rule, Equal Losses rule, and the Talmudic rule are not. The Equal Gains rule has other nice characteristics discussed in more detail later in this section. Anonymity, which requires that the rule should not discriminate between the agents, is one of them. This property makes Equal Gains an inappropriate rule when there are asymmetries among the agents one wishes to respect. For example in permit allocation, the EPA might want to favor firms from sectors that face higher import competition.

It is then useful to understand how one can generalize the Equal Gains rule by dropping the anonymity requirement but preserving rationality. It is also interesting to know if there are other anonymous rules that satisfy rationality. This paper provides an answer to these questions. Also, the concept of duality has been extensively used in the analysis of certain simple allocation problems (e.g. see Thomson, 2007). In relation to that literature, we also study the dual properties of rationality and their implications on the Equal Losses rule, the dual of the Equal Gains rule.

We start with the observation that the Equal Gains rule can be interpreted as following an algorithm that starts from equal division, but then corrects it to take feasibility into account. Let us iterate this algorithm on a permit allocation problem. Initially propose an equal division of the permits among the firms. If there are firms whose local constraint is less than this equal amount, allot them their local constraints. This frees up additional permits for the remaining firms. Now divide this remaining total equally among the remaining firms. If any of them has a local constraint less than this equal amount, allot them their local constraints. Continue iterating this algorithm until each firm is allotted a feasible share.

In Section 3, we introduce a class of algorithmic rules that generalize the Equal Gains algorithm. We show in Theorem 1 that algorithmic rules uniquely satisfy rationality, continuity in characteristics, and the following monotonicity property. Monotonicity in others’ characteristics (other-c monotonicity) applies to situations where one individual’s characteristic value changes and this leads to a compensating change in the remaining agents’ allotments. This property requires that no two of the remaining agents’ allotments move in opposite directions. It is a basic solidarity property in the same spirit as resource monotonicity (Thomson, 2007). Other-c monotonicity is a weak property in the sense that it does not specify how much the share of each agent will change or how these changes will be related to the agents’ characteristics (so for instance, among two agents with identical characteristics, one’s share may remain the same while the other’s share increases).

Kibris (2008) shows that every rule that satisfies rationality, continuity in characteristics, and other-c monotonicity is representable (i.e. maximizes a utility function on the allocation space).
Therefore, a simple corollary to Theorem 1 is that every algorithmic rule is representable.

A similar class of algorithmic rules are introduced for allocation problems with single peaked preferences by Barberà, Jackson, and Neme (1997) who analyzed the implications of strategic considerations. The authors showed on that domain that algorithmic rules uniquely satisfy strategy proofness, efficiency, and a monotonicity property similar to ours.

In Section 4, we inquire if there are other anonymous rules that are rational. To this end, we analyze the implications of a weak anonymity property, called equal treatment of equals. This property requires that agents with identical characteristics should receive identical shares. In Theorem 2, we show that the Equal Gains rule is the only rational rule that satisfies equal treatment of equals and continuity in characteristics.

As a characterization of the Equal Gains rule, Theorem 2 is related to the previous literature as follows. Dagan (1996) shows that the Equal Gains rule uniquely satisfies equal treatment of equals, “truncation invariance”, and composition up. Schummer and Thomson (1997) show that the Equal Gains rule minimizes (i) the difference between the largest and the smallest share and (ii) the variance of the shares. In a related result, Bosmans and Lauwers (2006) show that the Equal Gains solution Lorenz dominates every other allocation. Herrero and Villar (2002) and Yeh (2004) show that the Equal Gains rule uniquely satisfies conditional full compensation and composition down. Finally, Yeh (2006) shows that the Equal Gains rule uniquely satisfies conditional full compensation and “own-claim monotonicity”. Our characterization is logically independent from these previous results. Furthermore, the main principles employed in these characterizations (such as “composition”, full compensation, or Lorenz domination) are quite different than our main axiom: rationality. Also, with the exception of Schummer and Thomson (1997) and Bosmans and Lauwers (2006), the above characterizations use properties that relate the rule’s behavior at different social endowment levels. This is not the case for Theorem 2.

Our final results are in Section 5 where we first introduce two properties: translation down and translation up. Both are concerned with the implications of translating a problem by simultaneously

\[\text{Composition up requires that dividing the social endowment in two, first allocating one part, revising the characteristic vector accordingly, and then allocating the rest produces the same final allocation as allocating all the social endowment at once.}\]

\[\text{Conditional full compensation roughly requires agents with sufficiently small characteristic values to receive their characteristic values. Composition down deals with the following scenario: after the social endowment is allocated, we discover that the actual social endowment is smaller; then, it requires that using the original characteristic vector or the initially chosen allocation should produce the same final outcome.}\]
changing, at the same amount, the characteristic value of an agent and the endowment. For such
translations, these properties require that the initial allocation be translated the same way. We
show, in Lemma 4, that a rule satisfies translation down and translation up if and only if its dual
rule is rational. In Theorem 3, we then use this lemma and Theorem 2 to show that the Equal
Losses rule uniquely satisfies translation down, translation up, continuity, and equal treatment of
equals.

In Kibris (2008), we show that an allocation rule is rational if and only if it satisfies a standard
property called contraction independence (also called independence of irrelevant alternatives in the
context of bargaining by Nash (1950) and Property α in the context of consumer choice by Sen
(1971)). In this paper, we make extensive use of this equivalence.

In the next subsection, we discuss the various applications of our analysis. In Section 2, we
present our model and further discuss rational rules. In the following sections, we present our
results as summarized above.

1.1 Examples and Applications

A simple allocation problem for a society N is an $|N| + 1$ dimensional nonnegative real vector
$c_1, \ldots, c_{|N|}, E$ which is interpreted as follows. An endowment $E$ of a perfectly divisible commodity
is to be allocated among members of $N$ (which are either agents or commodities, depending on the
application). Each $i \in N$ is characterized by an amount $c_i$ of the commodity to be allocated. Next,
we discuss the alternative interpretations of $c$ and $E$ at various applications.

1. Permit Allocation: The Environmental Protection Agency is to allocate an amount $E$ of
pollution permits among firms in $N$ (such as $CO_2$ emission permits allocated among energy
producers). Each firm $i$, depending on its location, is imposed by the local authority an
emission constraint $c_i$ on its pollution level. For more on this application, see Kibris (2003)
and the literature cited therein.

2. Demand Rationing: A supplier is to allocate its production $E$ among demanders in $N$. Each
demander $i$ demands $c_i$ units of the commodity. The supply-chain management literature
contains detailed analysis of this problem. For example, see Cachon and Lariviere (1999)
and the literature cited therein.

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4We would like to thank Rakesh Vohra for bringing this application to our attention.
3. **Consumer Choice under fixed prices and rationing:** A consumer has to allocate his income $E$ among a set $N$ of commodities. The prices of the commodities are fixed and thus, do not change from one problem to another. (With appropriate choice of consumption units, normalize the price vector so that all commodities have the same price.) As typical in the fixed-price literature, the consumer also faces “rationing constraints” on how much he can consume of each commodity. Let $c_i$ be the agent’s *consumption constraint* on commodity $i$. See Bénassy (1993) or Kbris and Kütçüşenel (2008) for more on rationing rules.

4. **Bargaining with Quasilinear Preferences and Claims:** An arbitrator is to allocate $E$ units of a *numeriare good* among agents who have quasilinear preferences with respect to it. Each agent holds a *claim* $c_i$ on what he should receive. For examples of bargaining problems with claims, see Chun and Thomson (1992) and the following literature. For bargaining problems with quasilinear preferences, see Moulin (1985) and the following literature.

5. **Taxation:** A public authority is to collect an amount $E$ of tax from a society $N$. Each agent $i$ has *income* $c_i$. This is a central and very old problem in public finance. For example, see Edgeworth (1898) and the following literature. Young (1987) proposes a class of “parametric solutions” to this problem.

6. **Bankruptcy:** A bankruptcy judge is to allocate the remaining *assets* $E$ of a bankrupt firm among its creditors, $N$. Each agent $i$ has *credited* $c_i$ to the bankrupt firm and now, claims this amount. For example, see O’Neill (1982) and the following literature. For a detailed review of the extensive literature on taxation and bankruptcy problems, see Thomson (2003 and 2007).

7. **Surplus Sharing:** A social planner is to allocate the *return* $E$ of a project among its investors in $N$. Each investor $i$ has invested $s_i$. The project is profitable, that is, $\sum_N s_i \leq E$. Using the principal that no agent should receive less than his investment, define the *maximal share of an agent $i$* as $c_i = E - \sum_{N \setminus \{i\}} s_j$. Note that $\sum_N c_i \geq E$. The surplus sharing problem can now be analyzed as a simple allocation problem. For more on surplus-sharing problems, see Moulin (1985 and 1987) and the following literature.

8. **Single-peaked or Saturated Preferences:** A social planner is to allocate $E$ units of a perfectly divisible commodity among members of $N$. Each agent $i$ is known to have preferences with *peak* *(saturation point)* $c_i$. The rest of the preference information is disregarded as typical
in several well-known solutions to this problem, such as the Uniform rule or the Proportional rule. For example, see Sprumont (1991) and the following literature.

### 2 Model

Let \( N = \{1, \ldots, n\} \) be the set of agents (or commodities). For \( i \in N \), let \( e_i \) be the \( i^{th} \) unit vector in \( \mathbb{R}^N_+ \). Let \( e = \sum_N e_i \). We use the vector inequalities \( \leq, \leq, < \). For \( c \in \mathbb{R}^N_+, \alpha \in \mathbb{R}_+, \) and \( S \subseteq N \), with an abuse of notation, we write \( (c_S, \alpha_{N\setminus S}) \) to denote the vector which coincides with \( c \) on \( S \) and which chooses \( \alpha \) for every coordinate in \( N \setminus S \).

A **simple allocation problem** for \( N \) is a pair \( (c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+ \) such that \( \sum_N c_i \geq E \) (please see Figure 1). We call \( E \) the **endowment** and \( c \) the **characteristic vector**. As discussed in Subsection 1.1, depending on the application, \( E \) can be an asset or a liability and \( c \) can be a vector of incomes, claims, demands, preference peaks, or consumption constraints. Let \( \mathcal{C} \) be the set of all simple allocation problems for \( N \). Given a simple allocation problem \( (c, E) \in \mathcal{C} \), let \( X(c, E) = \{ x \in \mathbb{R}^N_+ | x \leq c \) and \( \sum_N x_i \leq E \} \) be the **choice set of** \( (c, E) \).

An allocation **rule** \( F : \mathcal{C} \to \mathbb{R}^N_+ \) assigns each simple allocation problem \( (c, E) \) to an allocation \( F(c, E) \in X(c, E) \) such that \( \sum_N F_1(c, E) = E \). Each rule \( F \) satisfies \( F(c, E) \leq c \) which, depending on the application, might be interpreted as a consumption constraint (as in permit allocation) or an efficiency requirement (as in single-peaked preferences). Also, \( \sum_N F_1(c, E) = E \) can be interpreted as an efficiency property (as in permit allocation) or a feasibility requirement (as in taxation). In
consumer choice, it is the Walras law.

The Equal Gains rule allocates the endowment in each problem equally, subject to no agent receiving more than his characteristic value: for each \( i \in N \), \( EG_i (c, E) = \min \{ c_i, \lambda \} \) where \( \lambda \in \mathbb{R}_+ \) satisfies \( \sum_N \min \{ c_i, \lambda \} = E \). In the single-peaked allocation literature, this rule is called the Uniform rule, in the bankruptcy literature it is called the Constrained Equal Awards rule, and in the taxation literature, it is called the Leveling Tax. The Equal Losses rule equalizes the losses agents incur, subject to no agent receiving a negative share: for each \( i \in N \), \( EL_i (c, E) = \max \{ 0, c_i - \lambda \} \) where \( \lambda \in \mathbb{R}_+ \) satisfies \( \sum_N \max \{ 0, c_i - \lambda \} = E \). In the single-peaked allocation literature, this rule is called the Equal Distance rule, in the bankruptcy literature it is called the Constrained Equal Losses rule, and in the taxation literature, it is called the Head Tax.

A rule \( F \) is continuous in characteristics (c-continuous) if for each \( E \in \mathbb{R}_+ \), \( F (., E) \) is a continuous function. It is monotonic in others’ characteristics (other-c monotonic) if a change in agent \( i \)'s characteristic value affects any other two agents in the same way: for each \( (c, E), (c', E) \in \mathcal{C} \) and \( i \in N \) such that \( c_{-i} = c'_{-i} \) and \( c_i \neq c'_i \), we have for each \( j, k \in N \setminus \{ i \} \), \( F_j (c, E) > F_j (c'_i, c_{-i}, E) \) implies \( F_k (c, E) \geq F_k (c'_i, c_{-i}, E) \).

A rule \( F \) is rational if there is a binary relation \( B \subseteq \mathbb{R}_+^N \times \mathbb{R}_+^N \) such that for each \( (c, E) \in \mathcal{C} \), \( F (c, E) = \{ x \in X (c, E) \mid \text{for each } y \in X (c, E), \ xBy \} \). Kibris (2008) shows that rationality is equivalent to the following property. A rule \( F \) satisfies contraction independence if a chosen alternative from a set is still chosen from subsets (contractions) that contain it: for each pair \( (c, E), (c', E) \in \mathcal{C} \), \( F (c, E) \in X (c', E) \subseteq X (c, E) \) implies \( F (c', E) = F (c, E) \). In the literature, this property is also referred to as independence of irrelevant alternatives (Nash, 1950) or Sen's property \( \alpha \) (Sen, 1971).

**Theorem A. (Kibris, 2008)** A rule \( F \) is rational if and only if it is contraction independent.

The following lemma is from Kibris (2008). We include the simple proof.

**Lemma 1** A rule \( F \) is rational if and only if for each \( (c, E), (c', E) \in \mathcal{C} \) it satisfies the following properties

**Property (i).** if for each \( i \in N \), \( \min \{ c_i, E \} = \min \{ c'_i, E \} \), then \( F (c, E) = F (c', E) \).

**Property (ii).** if \( F (c, E) \leq c' \leq c \), then \( F (c', E) = F (c, E) \).

**Proof.** \( (\Rightarrow) \) Assume that \( F \) is rational. Then, by Theorem A, it satisfies contraction independence. Let \( (c, E), (c', E) \in \mathcal{C} \). First, assume that for each \( i \in N \), \( \min \{ c_i, E \} = \min \{ c'_i, E \} \). Let \( x \in \mathbb{R}_+^N \)
satisfy $\sum_N x_i \leq E$. Then, $x \leq c$ if and only if $x \leq c'$. Thus $X(c,E) = X(c',E)$. This implies $F(c,E) = F(c',E)$. Next, assume that $F(c,E) \leq c' \leq c$. Then, $F(c,E) \in X(c',E) \subseteq X(c,E)$, which by contraction independence, implies $F(c,E) = F(c',E)$.

$(\Rightarrow)$ Assume that (i) and (ii) are satisfied. Let $(c,E), (c',E) \in C$ be such that $F(c,E) \in X(c',E) \subseteq X(c,E)$. Then for each $i \in N$, either $c_i' \leq c_i$ or $\min\{c_i',E\} = \min\{c_i,E\}$. Let $S = \{i \in N \mid c_i' \leq c_i\}$. Let $c'' = (c'_S, c_{N\setminus S})$. Then $F(c,E) \leq c'' \leq c$ and by (ii), $F(c,E) = F(c'',E)$. Now, for each $i \in N$, $\min\{c_i',E\} = \min\{c_i'',E\}$. Thus, by (i), $F(c'',E) = F(c',E)$. Altogether, we obtain $F(c,E) = F(c',E)$. Thus, $F$ satisfies contraction independence. Then, by Theorem A, $F$ is rational. $\blacksquare$

Property (i) of Lemma 1 is called truncation-invariance for rules on bankruptcy and taxation problems (Thomson, 2003 and 2007). Property (ii) says that a decrease in characteristic values does not change the initially chosen allocation as long as it remains feasible.

In what follows, we will make extensive use of the equivalence stated in Lemma 1.

3 Algorithmic Rules

The Equal Gains rule can be alternatively defined as choosing the outcome of the following algorithm: let $(c,E) \in C$,

**Step 1.** Determine the set of agents whose characteristic value, $c_i$, is less than his share from equal division, $\frac{E}{|N|}$. If no such agent exists, pick equal division and terminate the algorithm. Otherwise, assign each such agent his characteristic value and move to the next step.

**Step 2.** Determine the remaining agents (say $N'$) to be allotted and the remaining endowment to be allotted (say $E'$). Repeat Step 1 by replacing $N$ with $N'$ and $E$ with $E'$.

This algorithm can be generalized in several ways. First, rather than starting the algorithm with equal division as a reference allocation, one can use another allocation. Second, the way the reference allocation is updated in each step can depend on the problem’s parameters $c$ and $E$. Without any regularity requirements, such a generalization can create quite undesirable rules. Consider the following example (adapted from Barberà, Jackson, Neme, 1997).

**Example 1** Let $N = \{1,2,3\}$. For each problem $(c,E) \in C$, start with the reference allocation $(E,0,0)$. If $c_1 \geq E$, pick this allocation. Otherwise, let $x_1 = c_1$ (i.e. give agent 1 his characteristic...
value). If \( c_1 \) is a rational number, let \( x_2 = \min \{ c_2, E - c_1 \} \) and \( x_3 = E - x_1 - x_2 \). Otherwise, let \( x_3 = \min \{ c_3, E - c_1 \} \) and \( x_2 = E - x_1 - x_3 \). This “serial dictatorship rule” is rational since it satisfies both properties of Lemma 1. However, it violates two desirable properties. First of all, it is not \( c \)-continuous: a small change in \( c_1 \) has a big effect on the shares of the other agents. Second, it is not other-\( c \) monotonic: a change in \( c_1 \) affects agents 2 and 3 in opposite directions.

To rule out such undesirable rules, we will require our generalized algorithm to produce rational rules that are \( c \)-continuous and other-\( c \) monotonic. The following class of “algorithmic rules” generalize the Equal Gains algorithm in this way. Algorithmic rules are closely related to a family of rules introduced and analyzed by Barberà, Jackson and Neme (1997) on the domain of allocation problems with single-peaked preferences.

To define an “algorithmic rule”, we use two functions. The first function picks an initial allocation for each endowment level: the reference function \( r : \mathbb{R}_+ \to \mathbb{R}_N^+ \) assigns each endowment level \( E \) to an allocation \( r (E) \in \mathbb{R}_N^+ \) such that \( \sum_N r_i (E) = E \). The second function \( g \) adjusts the reference shares at each step of the algorithm. That is, \( g \) assigns each \( (c, E) \in C \) and \( x \in \mathbb{R}_N^+ \) with \( \sum_N x_i = E \) to an adjusted allocation \( g (x, c, E) = x' \in \mathbb{R}_N^+ \) with \( \sum_N x'_i = E \). For each \( t \in \{ 1, ..., n \} \), let \( g^t (x, c, E) = g (g^{t-1} (x, c, E), c, E) \) with the convention that \( g^0 (x, c, E) = x \). (That is, \( g^t \) represents \( g \) composed itself with \( t \) times.) The function \( g \) is a sequential adjustment function with respect to \( r \) if \( g^n \) is continuous in \( c \) and the following are true for any \( t \in \{ 1, ..., n \} \) and \( x' = g (x^{t-1}, c, E) = g^t (r (E), c, E) \):

\[
\begin{align*}
1. & \ x'^t - c_i \geq 0 \implies x'_i = c_i. \\
2. & \ x'^t - c_i < 0 \implies x'_i \geq x'^{t-1}_i. \\
3. & \text{Let } x'^t - c_i < 0 \implies g (x'^{t-1}, c_i, c_{-i}, E) = g (r (E), c_i, c_{-i}, E). \text{ Then, } x'^{t-1}_i < c_i < c'_i \implies g (x'^{t-1}, c_i, c_{-i}, E) = g (x'^{t-1}, c_i, c_{-i}, E). \\
4. & \text{Let } x = g^n (r (E), c_i, c_{-i}, E) \text{ and } \bar{x} = g^n (r (E), c_i, c_{-i}, E). \text{ Then, } c_i < \bar{c}_i \implies x_j \geq \bar{x}_j \text{ for each } j \neq i.
\end{align*}
\]

Property 1 requires that if at any stage, an agent’s reference share is greater than or equal to his characteristic value, his reference share is adjusted in the next step to his characteristic value. This

\[5\] Similar to Barberà et al (1997), we will only require these four properties to be satisfied at an allocation obtained at some step of the sequential adjustment process. (And this is why \( g \) is defined in reference to \( r \).) The function \( g \) could be arbitrary on other parts of the domain and induce a perfectly well-behaved allocation rule.

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property is intimately linked to \textit{c-continuity} (see \textit{Lemma 2}). It also guarantees that once an agent is served his characteristic value, his share is fixed in the rest of the algorithm. \textit{Property 2} requires that, if the reference share of an agent is smaller than his characteristic value, the adjustment can not decrease it. This property guarantees that, as some additional endowment is freed up in a step (due to some agents receiving their \(c_i\) and leaving the algorithm), none of the remaining agents are worse off. \textit{Property 3} requires that a change in \(c_i\) that does not affect the feasibility of agent \(i\)'s reference share does not affect the adjustment. This property is intimately linked to \textit{rationality} (which, by \textit{Property (ii)} of \textit{Lemma 1} makes a similar requirement). \textit{Property 4} requires that a decrease in \(c_i\) does not decrease the final share of any other agent. This property is intimately linked to \textit{other-c monotonicity}. By rationality, decreasing \(c_i\) does not increase \(x_i\). So the remaining agents consume at least as much in total. If some agent’s share decreases in response to a decrease in \(c_i\), there should be another agent whose share increases, violating \textit{other-c monotonicity}.

Given a \textit{reference function} \(r\) and a \textit{sequential adjustment function} \(g\) with respect to \(r\), an \textbf{algorithmic rule with respect to} \(g\) and \(r\), \(F^{g,r}\), is defined for each \((c, E) \in C\) as \(F^{g,r} (c, E) = g^n (r(E), c, E)\). That is \(F^{g,r} (c, E)\) is the allocation obtained at the end of \(n\) steps of the sequential adjustment algorithm.\footnote{The sequential adjustment algorithm in fact obtains the final allocation in at most \((n - 1)\) steps. Thus \(g^n = g^{n-1}\). We use \(g^n\) since it slightly simplifies the notation as well as the argument in Step 3 of the proof of \textit{Theorem 1}.}

Algorithmic rules constitute a large class that, for example, contains rules that partition agents into priority classes or rules that adjust agents’ shares proportionally to their shares (as in Bayesian updating). The following example illustrates an algorithmic rule.

\textbf{Example 2} \textit{EPA is to allocate} \(E = 140\) \textit{permits among five pollutant firms} \(N = \{1, 2, 3, 4, 5\}\) \textit{with local constraints} \(c = (55, 10, 26, 25, 25)\). \textit{Assume that the EPA algorithm follows the following rules:} (i) \textit{Firms 1 and 2 operate in an import-competing sector so they are initially offered twice as much shares as the other firms (which receive equal shares).} (ii) \textit{Factories of firms 1 and 3 are more distant to urban centers.} \textit{Thus, any additional permits that become available at a step will be offered to these two first and divided equally.} (iii) \textit{When the shares of firms 4 and 5 are updated, it is agreed that firm 4 receives twice as much as firm 5.} \textit{Let us calculate the solution:}

\begin{quote}
The algorithm starts with the reference allocation \(r(140) = (40, 40, 20, 20, 20)\) (by (i), firms 1 and 2 receive twice the rest). Since \(c_2 < 40\), firm 2 receives \(c_2\) in the next step. By (ii), the additional amount that becomes available is equally divided between firms 1 and 3. Thus \(g(r(E), c, E) = x^1 = (55, 10, 35, 20, 20)\). Since \(c_1 \geq 55\) and \(c_3 < 35\), these firms receive \(c_1\) and \(c_3\) in the next step. The
\end{quote}
Figure 2: The reference function $r$ completely defines a two-agent algorithmic rule. In the figure, since $r_1(E) > c_1$, $x_1 = c_1$ and $x_2 = E - c_1$.

additional amount that becomes available is divided between 4 and 5 (and by (iii), 4 gets twice the share of 5). Thus $g(x^1, c, E) = x^2 = (55, 10, 26, 26, 23)$. Now $c_4 < 26$. Thus firm 4 receives $c_4$ in the next step and the remains go to firm 5. That is, $g(x^2, c, E) = x^3 = (55, 10, 26, 25, 24)$. Since $x^3 \leq c$, the algorithm does not update the allocation any more. That is, $F(c,E) = g^5(r(E), c, E) = x^3$.

Two-agent algorithmic rules can be completely defined via the reference function $r$ (please see Figure 2). If $r(E)$ is feasible at a problem $(c,E)$, it is chosen as the final allocation. Otherwise, there is at most one agent $i$ such that $c_i < r_i(E)$. The adjusted (and final) allocation is then $x_i = c_i$ and $x_j = E - c_i$.

We next present the main result of this section.

**Theorem 1** A rule $F$ is rational, c-continuous and other-c monotonic if and only if $F$ is an algorithmic rule, that is, there is a reference function $r$ and a sequential adjustment function $g$ with respect to $r$ such that $F = F^{g,r}$.

The following lemmata are useful in the proof of Theorem 1. The first lemma shows under rationality and c-continuity that, if $c_i$ decreases below agent $i$’s current share, agent $i$’s updated share should be his new characteristic value.

**Lemma 2** Assume that $F$ is rational and c-continuous. Let $(c,E), (c', E) \in \mathcal{C}$, and $i \in N$ be such that $c_{-i} = c'_{-i}$, $c_i > c'_i$, and $F_i (c,E) > c'_i$. Then $F_i (c', E) = c'_i$. 


Proof. Suppose \( F_i(c', E) < c'_i \). By \( c \)-continuity, there is \( c''_i \in \mathbb{R}_+ \) such that \( c'_i < c''_i < c_i \) and \( F_i(c''_i, c_{-i}, E) = F_i(c'_i, c'_{-i}, E) = c'_i \). Then, \( F(c''_i, c'_{-i}, E) \leq c'_i \leq (c''_i, c'_{-i}) \), by Property (ii) of Lemma 1, implies \( F(c', E) = F(c''_i, c'_{-i}, E) \), a contradiction. ■

The following lemma states that if \( c_i \) decreases, the share of agent \( i \) can not increase and the shares of other agents can not decrease in response.

Lemma 3 Assume that \( F \) is rational, \( c \)-continuous, and \( other-c \) monotonic. Let \((c, E), (c', E) \in C\), and \( i \in N \) be such that \( c_{-i} = c'_{-i} \) and \( c_i > c'_i \). Then \( F_i(c, E) \geq F_i(c', E) \) and for each \( j \in N \setminus \{i\} \), \( F_j(c', E) \geq F_j(c, E) \).

Proof. If \( F_i(c, E) \leq c'_i \), by Property (ii) of Lemma 1, we have \( F(c', E) = F(c, E) \). Thus the result trivially holds. Alternatively assume \( F_i(c, E) > c'_i \). Then by Lemma 2, \( F_i(c', E) = c'_i < F_i(c, E) \). Thus, \( \sum_{N \setminus \{i\}} F_j(c, E) < \sum_{N \setminus \{i\}} F_j(c', E) \). Therefore, there is \( k \in N \setminus \{i\} \) such that \( F_k(c, E) < F_k(c', E) \). By \( other-c \) monotonicity, this implies for each \( j \in N \setminus \{i\} \), \( F_j(c, E) \leq F_j(c', E) \). ■

We next present the proof of Theorem 1.

Proof. (Theorem 1) \((\Leftarrow)\) Let \( F^{g,r} \) be an algorithmic rule. Since \( g^a \) is continuous in \( c \), \( F^{g,r} \) is \( c \)-continuous. Let \((c, E), (c', E) \in C \) and \( i \in N \) be such that \( c_{-i} = c'_{-i} \).

Claim 1: If \( F^{g,r}_i(c, E) \leq c'_i \leq c_i \) or \( F^{g,r}_i(c, E) < c_i \leq c'_i \), we have \( F^{g,r}_i(c, E) = F^{g,r}_i(c', E) \). If \( c_i = c'_i \), the claim trivially holds. If \( F^{g,r}_i(c, E) < c'_i < c_i \) or \( F^{g,r}_i(c, E) < c_i < c'_i \), the claim follows from Property 3 of \( g \). Finally if \( F^{g,r}_i(c, E) = c'_i < c_i \), the claim follows from the previous case and \( c \)-continuity of \( F^{g,r} \).

Claim 2: If \( \min \{c_i, c'_i\} = \min \{c'_i, E\} \), we have \( F^{g,r}_i(c, E) = F^{g,r}_i(c', E) \). Now \( F^{g,r}_i(c, E) \leq E \leq \min \{c_i, c'_i\} \). If \( F^{g,r}_i(c, E) \leq c'_i \leq c_i \) or \( F^{g,r}_i(c, E) < c_i \leq c'_i \), Claim 1 implies \( F^{g,r}_i(c, E) = F^{g,r}_i(c', E) \). If \( F^{g,r}_i(c, E) = c_i < c'_i \), then \( F^{g,r}_i(c, E) \leq E \leq c_i \) implies \( c_i = E \). Thus \( F^{g,r}_i(c', E) \leq c_i < c'_i \). This, by Claim 1, implies \( F^{g,r}_i(c, E) = F^{g,r}_i(c', E) \).

Claim 3: \( F^{g,r} \) is rational. Property (i) of Lemma 1 follows from the application of Claim 2 to each \( i \in N \). Property (ii) of Lemma 1 follows from the application of the first part of Claim 1 to each \( i \in N \).

Claim 4: \( F^{g,r} \) is \( other-c \) monotonic. Assume \( c_i \neq c'_i \). Then by Property 4, either [for each \( j \in N \setminus \{i\} \), \( F^{g,r}_j(c, E) \geq F^{g,r}_j(c', E) \)] or [for each \( j \in N \setminus \{i\} \), \( F^{g,r}_j(c, E) \leq F^{g,r}_j(c', E) \)].

\((\Rightarrow)\) Let \( F \) satisfy the given properties.
Step 1: defining $g$ and $r$. For each $E \in \mathbb{R}_+$, let $r(E) = F(E_N, E)$. For each $(c, E) \in \mathcal{C}$ and $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = E$, let $g(x, c, E) = F(cM(x, c), E)$ where $M(x, c) = \{i \in N \mid c_i \leq x_i\}$.

Let $x^0 = r(E)$ and for $t \in \{1, \ldots, n\}$, let $x^t = g(x^{t-1}, c, E) = g^t(r(E), c, E)$. Let $M^{-1} = \emptyset$ and for each $t \in \{0, \ldots, n\}$, let $M^t = M(x^t, c)$.

Step 2: if $t \in \{0, \ldots, n\}$ and $i \in M^{t-1}$, then $i \in M^t$ and $x^t_i = c_i$. The proof is by induction. For $t = 0$, $M^{-1} = \emptyset$ implies the desired conclusion. Now assume $M^{-1} \subseteq \ldots \subseteq M^{t-1}$. Let $i \in M^{t-1}$. Then $x^{t-1}_i \geq c_i$. Let $K = M^{t-2} \cup \{i\}$ and note that $K \subseteq M^{t-1}$. If $x^{t-1}_i = c_i$, by Property (ii) of Lemma 1, $x^{t-1}_i = F(c_K, E_{N \setminus K}, E) = c_i$. Alternatively, if $x^{t-1}_i > c_i$, by Lemma 2, $F_i(c_K, E_{N \setminus K}, E) = c_i$. If $K = M^{t-1}$, $i \in M^t$ and $x^t_i = c_i$. Otherwise, $F_i(c_K, E_{N \setminus K}, E) = c_i$ and Lemma 3 imply $x^t_i \geq c_i$. Thus $i \in M^t$. Since $i \in M^{t-1}$, by definition, $x^t_i \leq c_i$. Thus overall, $x^t_i = c_i$.

Step 3: $x^n \leq c$. First assume $M^{t-1} = M^t$ for some $t \in \{0, \ldots, n-1\}$. Then by definition, $x^t = x^{t+1}$ and thus, $M^t = M^{t+1}$. Iterating, $x^t = x^n$. Also, by Step 2, for each $i \in M^t$, $x^t_i = c_i$. Thus, $x^t = x^n \leq c$. Alternatively, assume $M^{t-1} \neq M^t$ for each $t \in \{0, \ldots, n-1\}$. By Step 2, $M^{n-1} = N$. Thus $x^n = F(c, E) \leq c$.

Step 4: $F = F^{g,r}$. Let $(c, E) \in \mathcal{C}$. By Property (i) of Lemma 1, assume $c \subseteq E_N$. Note that $F^{g,r}(c, E) = x^n = F(c_{M^{n-1}}, E_{N \setminus M^{n-1}}, E)$. By Step 3, $x^n \leq c \subseteq (c_{M^{n-1}}, E_{N \setminus M^{n-1}})$. Then, by Property (ii) of Lemma 1, $x^n = F(c, E)$.

Step 5: $g$ is a sequential adjustment function. Since $F$ is $c$-continuous, $g^n$ is continuous in $c$. Also, Step 2 above proves Property 1. Now let $i \in N$ and $t \in \{1, \ldots, n\}$.

For Property 2, assume $x^{t-1}_i < c_i$. Then $i \notin M^{t-1}$ implies $i \notin M^{t-2}$. If $M^{t-2} = M^{t-1}$, by definition, $x^t = x^{t-1}$ and thus, $x^t_i \geq x^{t-1}_i$. Otherwise, by Lemma 3, $x^t_i \geq x^{t-1}_i$.

For Property 3, assume $x^{t-1}_i < \tilde{c}_i < c_i$. Let $\tilde{c} = (\tilde{c}_i, c_{-i})$. Then $i \notin M(x^{t-1}, c) = M(x^{t-1}, \tilde{c})$. Thus, by definition of $g$, we have $g(x^{t-1}, c, E) = g(x^{t-1}, \tilde{c}, E)$.

For Property 4, assume $c_i < \tilde{c}_i$. Let $\tilde{E} = (\tilde{c}_i, c_{-i})$, $x = g^n(r(E), c, E)$ and $\tilde{x} = g^n(r(E), \tilde{c}, E)$. Then, $x = F(c, E)$ and $\tilde{x} = F(\tilde{c}, E)$. By Lemma 3, $x_i \leq \tilde{x}_i$ and for each $j \in N \setminus \{i\}$, $x_j \geq \tilde{x}_j$.

The above characterization is tight. Without rationality, Proportional rule becomes admissible.

The following example presents a rule that violates other-$c$ monotonicity but satisfies the other properties. Finally, Example 4, at the end of the next section, presents a rule that violates $c$-continuity but satisfies the other properties.

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$^7$If $c_i > E$, $\min\{c_i, E\} = E$, and by Property (i) of Lemma 1, both $F(c, E) = F(c_{N \setminus i}, E_i, E)$ and $F^{g,r}(c, E) = F^{g,r}(c_{N \setminus i}, E_i, E)$. Then $F^{g,r}(c_{N \setminus i}, E_i, E) = F(c_{N \setminus i}, E_i, E)$ implies $F^{g,r}(c, E) = F(c, E)$. 

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Example 3 Let \( N = \{1, 2, 3\} \). Let

\[
F(c, E) = \begin{cases} 
\left( \frac{E}{3}, \frac{E}{3}, \frac{E}{3} \right) & \text{if } (\frac{E}{3}, \frac{E}{3}, \frac{E}{3}) \leq c \\
(c_1, c_1, E - 2c_1) & \text{else if } c_1 < \frac{E}{3} \text{ and } (c_1, c_1, E - 2c_1) \leq c \\
(E - 2c_2, c_2, c_2) & \text{else if } c_2 < \frac{E}{3} \text{ and } (E - 2c_2, c_2, c_2) \leq c \\
(c_3, E - 2c_3, c_3) & \text{else if } c_3 < \frac{E}{3} \text{ and } (c_3, E - 2c_3, c_3) \leq c \\
(c_1, c_2, E - c_1 - c_2) & \text{else if } E - c_1 - c_2 > c_2 \text{ and } c_1 > c_2 \\
(c_1, E - c_1 - c_3, c_3) & \text{else if } E - c_1 - c_3 > c_1 \text{ and } c_3 > c_1 \\
(E - c_2 - c_3, c_2, c_3) & \text{else if } E - c_2 - c_3 > c_3 \text{ and } c_2 > c_3.
\end{cases}
\]

Using Lemma 1, it is straightforward to check that this rule is rational. It is also \( c \)-continuous. It is not other-\( c \) monotonic since, for \( x = (c_1, c_1, E - 2c_1) \), \( x_2 \) is increasing in \( c_1 \) while \( x_3 \) is decreasing. We will next demonstrate that this rule is not algorithmic. Suppose otherwise. Let \( E = 9, c = (9, 2, 1) \). The initial reference allocation should then be \( r(E) = F(E_N, E) = (3, 3, 3) \). Since agents 2 and 3 have smaller characteristic values than 3, they must receive their characteristic values: \( x_2 = 2 \) and \( x_3 = 1 \). Thus \( x_1 = 6 \). However, \( F(c, E) = (7, 1, 1) \), a contradiction.

4 Equal Gains Rule

In this section, we analyze the implications on rational rules of a weak anonymity property: a rule \( F \) satisfies **equal treatment of equals** if two agents with identical characteristics are awarded equal shares: for each \((c, E) \in C \) and \( i, j \in N, c_i = c_j \) implies \( F_i(c, E) = F_j(c, E) \). A large class of rules, including the four central ones (i.e. Equal Gains, Equal Losses, Proportional, and Talmudic rules) satisfy this property (e.g. see Young, 1987). Among them, however, the Equal Gains rule is the only **rational** rule.

**Theorem 2** A rule \( F \) satisfies rationality, \( c \)-continuity, and equal treatment of equals if and only if it is the Equal Gains rule.

**Proof.** It is straightforward to show that \( EG \) satisfies the given properties. Conversely, let \( F \) be a rule that satisfies them. We next show \( F = EG \). Let \((c, E) \in C \). By Property (i) of Lemma 1, assume \( c_N \leq E_N \) (see Footnote 7). Without loss of generality, assume \( c_1 \leq \ldots \leq c_n \). Let \( c^0 = E_N \), \( c^n = c \), and for each \( k \in \{1, \ldots, n - 1\} \), let \( c^k = (c_{\{1, \ldots, k\}}, E_{\{k+1, \ldots, n\}}) \).
We inductively show that for each \( k \in \{0, ..., n\} \), we have \( F(c^k, E) = EG(c^k, E) \). For \( k = n \), this will imply the desired conclusion. Initially, let \( k = 0 \). By equal treatment of equals, \( F(c^0, E) = EG(c^0, E) \). Now let \( k \in \{1, ..., n\} \) and assume that the statement holds for each \( l < k \).

**Case 1:** There is \( l < k \) such that \( F(c^l, E) = F(c^l, E) \). Then, by our assumption, \( F(c^l, E) = EG(c^l, E) \). Thus, \( EG(c^l, E) \leq c^k \leq c^l \). This, by Property (ii) of Lemma 1, implies \( EG(c^l, E) = EG(c^k, E) \). Combining the equalities, we then have \( F(c^k, E) = EG(c^k, E) \).

**Case 2:** For each \( l < k \), \( F(c^k, E) \neq F(c^l, E) \). Thus, \( F(c^k, E) \) is first obtained at \( c^k \).

We first show that \( F_k(c^k, E) = c_k \). For this, note that \( F(c^k, E) \neq F(c^{k-1}, E) \) implies, by Property (ii) of Lemma 1, \( F_k(c^{k-1}, E) > c_k \). Thus, by Lemma 2, \( F_k(c^k, E) = c_k \).

We next show that for each \( l < k \), \( F_l(c^k, E) = c_l \). Let \( \overline{c} = c^k + (c_k - c_l) e_l \) and note that \( \overline{c} \geq c^k \). First assume \( F(\overline{c}, E) \neq F(c^k, E) \). Then \( c_l < c_k \). Thus, by Property (ii) of Lemma 1, \( F_l(\overline{c}, E) > c_l \). This, by Lemma 2, implies \( F_l(c^k, E) = c_l \). Next, assume \( F(\overline{c}, E) = F(c^k, E) \). By equal treatment of equals, \( F_l(\overline{c}, E) = F_k(\overline{c}, E) \). Then, \( F_l(c^k, E) = F_k(c^k, E) \). This implies \( F_l(c^k, E) = c_k \geq c_l \). Thus, \( F_l(c^k, E) = c_l \).

Overall, for each \( i \in \{1, ..., k\} \), \( F_l(c^k, E) = c_i \). This, by equal treatment of equals, implies for each \( i \in \{k+1, ..., n\} \), \( F_i(c^k, E) = \frac{E - \sum_{j=1}^{k} c_j}{n-k} \). Applying the same arguments to \( EG \) shows that it picks the same allocation. Thus, \( F(c^k, E) = EG(c^k, E) \).

The above characterization is tight. Without rationality, Proportional rule becomes admissible. Without equal treatment of equals, weighted versions of the Equal Gains rule, become admissible. Finally, the following example presents a rule that violates c-continuity but satisfies the other properties.

**Example 4** Let \( N = \{1, 2\} \). Let \( F \) be defined as

\[
F(c, E) = \begin{cases} 
(c, E) & \text{if } c_1 \geq \frac{E}{2} \text{ and } c_2 \geq \frac{E}{2}, \\
(0, 0) & \text{if } c_1 \geq E \text{ and } c_2 < \frac{E}{2}, \\
(c_1, E - c_1) & \text{if } E - c_2 < c_1 < E \text{ and } c_2 < \frac{E}{2}, \\
(0, E) & \text{if } c_1 < \frac{E}{2} \text{ and } c_2 \geq E, \\
(E - c_2, c_2) & \text{if } c_1 < \frac{E}{2} \text{ and } E - c_1 < c_2 < E.
\end{cases}
\]

Note that \( F \) satisfies rationality and equal treatment of equals. It is not c-continuous since for each \( \varepsilon \in (0, \frac{E}{2}) \), \( F(E_1, (\frac{E}{2} - \varepsilon)_1, E) = (E, 0) \) but \( F(E_1, (\frac{E}{2})_2, E) = (\frac{E}{2}, \frac{E}{2}) \).
5 Duality and the Equal Losses rule

We start this section by introducing two properties that require the solution to a problem to be covariant under certain translations of the problem. Both properties are concerned with the implications of translating a problem by simultaneously changing, at the same amount, the characteristic value of an agent and the endowment. For such translations, they require that the initial allocation be translated the same way. Similar properties have been analyzed in bargaining theory (e.g. see Thomson, 1981).

A rule $F$ satisfies translation down (Figure 3, left) if decreasing the endowment and the characteristic value of an agent $i$ by the same amount does not affect the shares of other agents, that is, for each $(c, E) \in C$, each $i \in N$, and each $\delta \in (0, F_i(c, E)]$, we have $F(c - \delta e_i, E - \delta) = F(c, E) - \delta e_i$.

The second property only applies to a limited set of agents that we define next. An agent $i$ is critical for a problem $(c, E) \in C$ if $\sum_{N \setminus i} c_j \leq E$, that is, reducing $i$’s characteristic value, $c_i$, to zero makes the problem either not well-defined ($\sum_{N \setminus i} c_j < E$) or trivial ($\sum_{N \setminus i} c_j = E$). A rule $F$ satisfies translation up (Figure 3, right) if increasing the endowment and the characteristic value of a critical agent $i$ by the same amount does not affect the shares of other agents, that is, for each $(c, E) \in C$, each $i \in N$ such that $\sum_{N \setminus i} c_j \leq E$, and each $\delta \in (0, \infty)$, we have $F(c + \delta e_i, E + \delta) = F(c, E) + \delta e_i$.

It turns out that these two properties are very closely related to rationality. The relationship is
through a concept called duality. The dual of a rule $F$, $F^d$, allocates what is available in the same way as $F$ allocates what is missing, that is, for each $(c, E) \in C$, $F^d(c, E) = c - F(c, \sum_N c_i - E)$.

Aumann and Maschler (1985) quote several passages from the Talmud where the notion of duality is implicitly discussed and self-duality of a rule (that is, the rule coinciding with its dual) is promoted.

The duality of rules can also be used to define a notion of duality for properties. A property $\Pi$ is the dual of another property $\Pi^d$ if whenever a rule $F$ satisfies $\Pi$, its dual rule $F^d$ satisfies $\Pi^d$. Some properties, such as $c$-continuity or equal treatment of equals, are self-dual. That is, a rule $F$ satisfies $c$-continuity (or equal treatment of equals) if and only if its dual $F^d$ satisfies the same property.

The following result shows that translation down and translation up are, together, the dual of rationality.

**Lemma 4** A rule $F$ satisfies rationality if and only if its dual $F^d$ satisfies translation up and translation down.

**Proof.** First assume that $F$ satisfies rationality. Then it satisfies the two properties of Lemma 1. Let $(c, E) \in C$, $i \in N$, and $\delta \in (0, \infty)$.

**Claim 1:** $F^d$ satisfies translation down. To see this, assume $\delta \leq F^d_i(c, E)$. Let $E = \sum_N c_i - E$. Then, $\delta \leq c_i - F_i(c, E)$ implies $F(c, E) \leq c - \delta c_i \leq c$. This, by Property (ii) of Lemma 1, implies $F(c - \delta e_i, E) = F(c, E)$. Rewriting this equality for $F^d$, $c - \delta e_i - F^d(c - \delta e_i, \sum_N c_i - E - \delta) = c - F^d(c, \sum_N c_i - E)$. Thus, $F^d(c, E) - \delta e_i = F^d(c - \delta e_i, E - \delta)$.

**Claim 2:** $F^d$ satisfies translation up. To see this, assume $\sum_{N \setminus i} c_j \leq E$. Let $\bar{E} = \sum_N c_i - E$. Then, $\min\{c_i, E\} = \bar{E} = \min\{c_i + \delta, \bar{E}\}$. Then, by Property (i) of Lemma 1, $F(c + \delta e_i, \bar{E}) = F(c, \bar{E})$. Rewriting this equality for $F^d$, we have $F^d(c + \delta e_i, E + \delta) = F^d(c, E) + \delta e_i$.

Next, assume that $F^d$ satisfies translation down and translation up. Let $(c, E), (c', E) \in C$.

**Claim 3:** $F$ satisfies Property (ii) of Lemma 1. Assume $F(c, E) \leq c' \leq c$. Let $A = \{j \in N \mid c_j < c_j\}$. If $A = \emptyset$, $F(c', E) = F(c, E)$ trivially holds. Alternatively, let $i \in A$. Let $\delta_i = c_i - c'_i$ and $\bar{E} = \sum_N c_i - E$. Then $F_i(c, E) \leq c_i - \delta_i$ implies $\delta_i \leq c_i - F_i(c, E) = F^d_i(c, E)$. Thus, $\delta_i \in (0, F^d_i(c, E))$. Then, by translation down, $F^d(c - \delta_i e_i, \bar{E} - \delta_i) = F^d(c, \bar{E}) - \delta_i e_i$. This implies $F(c - \delta_i e_i, E) = F(c, E)$. Now enumerate $A = \{i_1, ..., i_a\}$. Then, by sequentially applying the above argument we obtain $F(c, E) = F(c - \delta_i e_i, E) = F(c - \delta_i e_i - \delta_i e_i, E) = ... = F(c - \sum_A \delta_i e_i, E) = F(c', E)$, the desired conclusion.
Claim 4: $F$ satisfies Property (i) of Lemma 1. Assume for each $j \in N$, $\min \{c_j, E\} = \min \{c_j, E\}$. Let $A = \left\{ j \in N \mid c_j < c_{j'} \right\}$, $B = \left\{ j \in N \mid c_j > c_{j'} \right\}$, and $c'' = (c'_A, c_{N\setminus A})$. Now $F(c, E) \leq c'' \leq c$, by Claim 3, implies $F(c'', E) = F(c, E)$. We will next show $F(c', E) = F(c'', E)$. Let $B = \emptyset$, this trivially holds. Alternatively, let $i \in B$. Let $\delta = c'_i - c''_i$ and $\bar{E} = \sum_N c''_j - E$. Note that $c''_i = c_i$. Then $\min \{c'', E\} = \min \{c', E\}$ implies $E \leq c''_i$. Thus, $c''_i \geq \sum_N c''_j - \bar{E}$. Then, by translation up, $F^d(c'' + \delta e_i, \bar{E} + \delta) = F^d(c'', \bar{E}) + \delta e_i$. This implies $F(c'' + \delta e_i, E) = F(c'', E)$. Now enumerate $B = \{i_1, \ldots, i_b\}$. Then, by sequentially applying the above argument we obtain $F(c'', E) = F(c + \delta_{i_1} e_{i_1}, E) = F(c + \delta_{i_1} e_{i_1} + \delta_{i_2} e_{i_2}, E) = \ldots = F(c + \sum_B \delta_{i_k} e_{i_k}, E) = F(c', E)$. Combining $F(c'', E) = F(c', E)$ with the previous $F(c'', E) = F(c, E)$, we obtain $F(c', E) = F(c, E)$, the desired conclusion.

Claims 3 and 4, by Lemma 1, imply that $F$ is rational. 

The following result is a corollary of Theorem 2 and Lemma 4. It also uses the fact that the Equal Gains and Equal Losses rules are dual of each other (e.g. see Thomson, 2003).

Theorem 3 A rule $F$ satisfies translation up, translation down, c-continuity, and equal treatment of equals if and only if it is the Equal Losses rule.

Proof. It is straightforward to show that $EL$ satisfies the given properties. Conversely, let $F$ be a rule that satisfies them. We next show $F = EL$. By Lemma 4, $F^d$ satisfies rationality. Since c-continuity and equal treatment of equals are self-dual properties, $F^d$ also satisfies them. Thus, by Theorem 2, $F^d = EG$. Since $EG$ and $EL$ are dual rules, then, $F = EL$.

References


[28] Thomson, W., 2007, How to Divide When There Isn’t Enough: From the Talmud to Game Theory, book manuscript.


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