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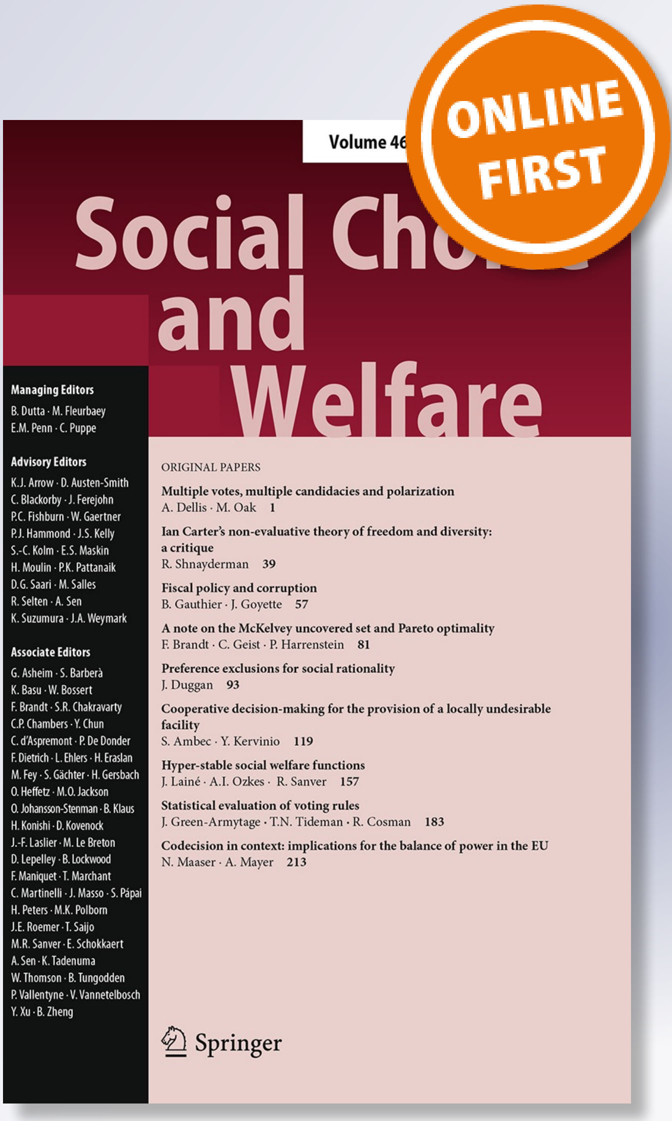

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ORIGINAL PAPER

On surplus-sharing in partnerships

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Abstract For investment or professional service partnerships (in general, for partnerships where measures of the partners' contributions are available), we consider a family of partnership agreements commonly used in real life. They allocate a fixed fraction of the surplus equally and the remains, proportional to contributions; and they allow this fraction to depend on whether the surplus is positive or negative. We analyze the implications of such partnership agreements on (i) whether the *partnership forms* in the first place, and if it does, (ii) the partners' *contributions* as well as (iii) their *welfare*. We then inquire which partnership agreements are *productively efficient* (*i.e.* maximizes the partners' total contributions) and which are *socially efficient*, (*i.e.* maximizes the partners' social welfare as formulated by the two seminal measures of *egalitarianism* and *utilitarianism*).

1 Introduction

Imagine a group of lawyers forming a partnership or a group of investors partnering up to undertake a financial endeavor. As a first step, the partners need to agree on (i) how to allocate positive surplus in case of profits and (ii) how to allocate negative

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surplus in case of losses. This is a very important choice for the partnership since it in turn affects the partners' contributions as well as their welfare from the partnership. In this paper, we focus on the implications of this choice. More specifically, we analyze the (dis)advantages of some *partnership agreements* that are commonly used in real life *professional service partnerships* (such as in law, accounting, medicine, or real estate) as well as in *investment partnerships*.

Farrell and Scotchmer (1988) and Lang and Gordon (1995) describe three basic systems law partnerships use to allocate surplus. In the first one, called the *lock-step system*, all partners of the same seniority receive the same surplus share. The lock-step system is used by most law firms with 2 or 3 partners, which approximately constitute 2/3 of all law firms in the US, though less than half the lawyers (also see Curran 1985; Flood 1985). In the second system, called the *objective performance-related system*, an explicit formula (using variables such as the number of hours billed, cases won, or business brought in) is used to determine each partner's contribution. The partners' surplus shares are then determined in proportion to these contributions. The third basic system, called the *subjective performance-related system*, is only different from the second in the sense that it is now the firm's founders who evaluate each partner's contribution.

The above case of law partnerships demonstrates the two most common surplus sharing methods in real life: *equal* (or in general, fixed) shares versus shares *proportional* to contributions (also called the piece-rate). Gaynor and Pauly (1990) mention that it is also common in professional service partnerships to combine these two methods by allocating a fixed fraction of the surplus equally and allocating the rest proportional to contributions as a bonus. A partnership agreement can additionally fix different fractions in cases of positive and negative surplus.[1] The following is an example of such a partnership agreement:

> Partners Johnson and Smith agree that (i) if their partnership makes a positive surplus, 60 % of this positive surplus will be allocated equally while the remaining 40 % will be allocated in proportion to each partner's contribution and (ii) if their partnership makes a negative surplus, all of this negative surplus will be allocated equally.

The framework of our study is as follows. First, we take a partnership agreement as a pair of surplus allocation rules (used for positive and negative surplus respectively), focusing on the class of rules discussed in the previous paragraph. Second, we analyze environments where measures of the partners' contributions are available. As already exemplified for law partnerships, such measures are commonly used in professional service partnerships. Similarly, partners' monetary contributions are routinely used to allocate surplus in investment partnerships. Third and last, we assume that there is a stochastic component to the success of the partnership. Whether the partnership

[1] Legal regulations on partnerships recognize the usage of different surplus sharing rules (as in fixing different fractions) in cases of positive and negative surplus. For example see "The practice managers' guide to co-ownership agreements, partnerships, and associateships", a guide for medical practices in Australia, prepared by "McMasters' Training Pty Ltd", available online at http://www.medicalpracticemanagement.com.au/practice_manager_s_guides/guide5/guide_5.

makes positive or negative surplus depends on some factors external to it, such as the state of the economy or the performance of the competitors. Several previous studies on partnerships make a similar assumption (e.g. Morrison and Wilhelm 2004; Comino et al. 2010; Li and Wolfstetter 2010).

In the confines of this framework, we model a simple *"partnership game"* and using it, we analyze the implications of a partnership agreement on (i) whether the *partnership forms* in the first place, and if it does, (ii) the partners' *contribution choices* as well as (iii) their resulting *welfare*. Armed with these answers, we then inquire which partnership agreements are *productively efficient* (Gaynor and Pauly 1990), that is, maximize the partners' total contributions. We also inquire which partnership agreements are *socially efficient*, that is, maximize the partners' social welfare as formulated by the two seminal measures of *egalitarianism* and *utilitarianism*. Overall, we observe a trade-off between maximizing total contributions and social welfare. To maximize total contributions, two "opposite" surplus allocation rules need to be used in cases of positive and negative surplus. However, (i) maximizing egalitarian social welfare requires choosing the same surplus-allocation rule in cases of both positive and negative surplus, and furthermore (ii) a numerical analysis detailed in Sect. 3.4 obtains a similar result for utilitarian social welfare on a significant part of our parameter space. In the Conclusion, we present a short discussion of the mechanism behind this trade-off.

In many countries, legal regulations include a *partnership act*, that is, a statutory agreement that applies to any partnership that does not have a written agreement. This default agreement typically allocates both positive and negative surplus equally. Also, if the partners have only specified the surplus-sharing rule to be used in case of positive or negative surplus, the legal default is that the same surplus-sharing rule is used in the other case as well. Our analysis thus shows that the state partnership acts have picked the welfare side of the trade-off mentioned in the previous paragraph.

The paper is organized as follows. In Sect. 1.1, we discuss the related literature. In Sect. 2, we present the model. Section 3 contains our findings. In Sect. 3.1, we analyze acceptable partnership agreements (which we argue to be intimately linked to partnership formation). In Sect. 3.2, we characterize the equilibrium contributions in a formed partnership. In Sect. 3.3, we compare partnership agreements in terms of the total contributions and in Sect. 3.4, we compare them in terms of individual and social welfare. We summarize our findings and conclude in Sect. 4. The proofs are relegated to Sect. 1.

1.1 Literature

There are two strands of theoretical literature related to our paper. The first follows the seminal papers by Alchian and Demsetz (1972) and Holmström (1982) to discuss the design of incentives in partnerships where the partners' contributions are not observable (and thus, contribution-sensitive sharing schemes like proportionality are not available). In contrast to Alchian and Demsetz (1972) who argue that efficiency can only be restored by bringing in a principle who monitors the agents, Holmström

(1982) shows that group incentives can remove the free-rider problem.[2] The following literature focuses on the same question under alternative assumptions. Kandel and Lazear (1992) analyze the effect of peer pressure, Legros and Matthews (1993) analyze the effect of limited liability, Miller (1997) and Strausz (1999) analyze cases where a partner can observe the effort exerted by a subset of other partners, and Morrison and Wilhelm (2004) discuss moral hazard problems associated with intergenerational transfer of human capital. Hart and Holmström (2010) and Hart (2011) adopt the "contracts as reference points" approach to discuss shading and efficient partnership contracts. Farrell and Scotchmer (1988) analyze the efficiency costs of equal-sharing in a theoretical model of partnership formation.

The above literature focuses on a stylized asymmetric information environment where contribution-sensitive surplus allocation rules such as the proportional are not available; the common intuition being that if informational constraints permitted it, proportional surplus sharing would solve incentive problems. We contribute to this literature by providing a formal discussion of this intuition in an environment where there is possibility of negative as well as positive surplus. Our results show that (i) a move towards more proportional surplus-shares does not necessarily increase a partner's contributions (e.g. see. Figs. 1, 2) and (ii) by using different surplus sharing rules in cases of positive and negative surplus, a partnership can improve total contributions over simple proportionality.

The second strand of theoretical literature related to our paper is on axiomatic resource allocation. The partnership agreements that we consider are based on two principles (proportional versus equal sharing) central in the surplus sharing literature. See O'Neill (1982) and Aumann and Maschler (1985) and the following literature (reviewed in Thomson 2003, 2008) for axiomatic studies on allocating negative surplus (referred to as claims or bankruptcy problems by this literature). On the other hand, Moulin (1987) and the following literature provides an axiomatic study for positive surplus. There also is a smaller literature that covers both cases simultaneously. For example, Chun (1988) proposes characterizations of classes of rules that mix the proportionality and equal awards principles in both cases of positive and negative surplus. Herrero et al. (1999) propose and analyze a "rights-egalitarian solution" which uses the equal awards principle in case of positive surplus and the equal losses principle in case of negative surplus.

The axiomatic literature analyzes a much larger class of rules in comparison to the one following Holmström (1982). However, studies in this literature focus on normative questions and typically remain silent on strategic issues, particularly the role of incentives in the formation of surplus. By focusing on this latter question and by analyzing the structure of productively and socially efficient partnership agreements, our paper contributes to this literature.

[2] While we work under different informational assumptions, Holmström's question is similar to this paper. Quoting (p. 326): "The question is whether there is a way of fully allocating the joint outcome so that the resulting noncooperative game among the agents has a Pareto optimal Nash equilibrium." Holmström shows that the free rider problem can be solved as follows. One sets an output objective (by utilizing the observable information about the agents' costs of effort). If it is not met, all partners receive zero as punishment. Otherwise, they share the produced value.

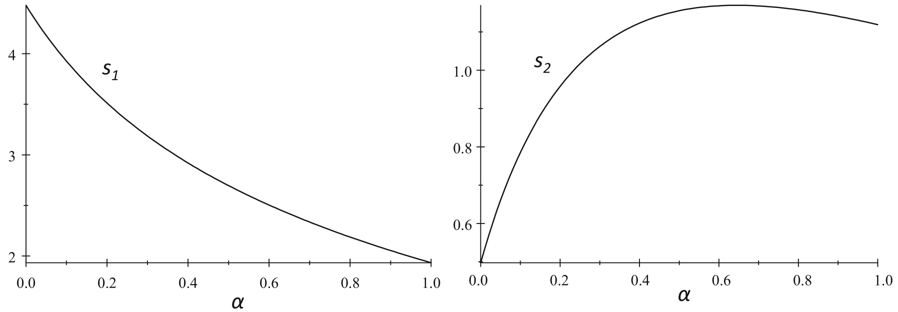

Fig. 1 Partner 1's (*left*) and Partner 2's (*right*) equilibrium contributions, as a function of α. Parameter values are $r = 0.3$, $p = 0.8$, $\beta = 0.7$, $a_1 = 1$, $a_2 = 1.5$. Also $\alpha = 0.3$ and $\gamma = 0.5$ when not a variable

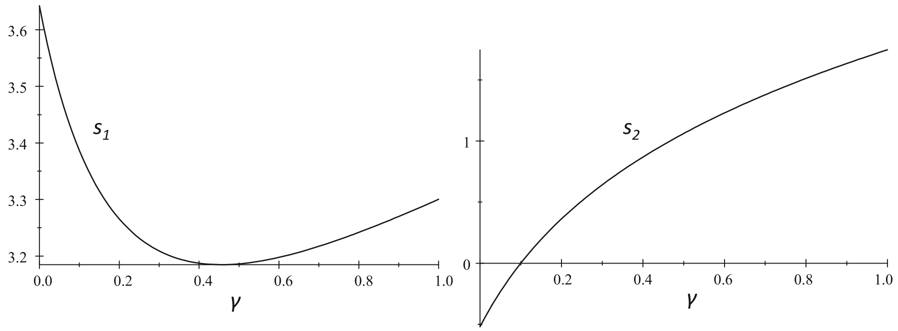

Fig. 2 Partner 1's (*left*) and Partner 2's (*right*) equilibrium contributions, as a function of γ. Parameter values are the same as in Fig. 1

Some of our modeling choices are related to earlier studies as follows. First, there are many earlier papers that, like us, model output as stochastic. For example, see Huddart and Liang (2003), Comino et al. (2010). Again similar to us, several earlier studies argue that the partners' expectations on their shares in case the partnership fails will have an effect on the partners' effort choices. For example, see Comino et al. (2010) or Li and Wolfstetter (2010). Finally, almost all the theoretical literature following Holmström (1982) uses additively separable utility functions (quasilinear preferences). Similar to those studies, we measure contributions in monetary units. But we alternatively assume that the agents have constant absolute risk aversion (*CARA*) utilities. Since we consider a stochastic production function, the CARA family provides us a good way to measure the effect of the agents' risk attitudes on the outcome.

Finally it is useful to mention Kıbrıs and Kıbrıs (2013), where a similar modeling approach is used to analyze the investment implications of bankruptcy laws. While the two studies consider two separate economic institutions and contribute to two distinct strands of literature, they both analyze the incentive implications of resource allocation mechanisms in an environment with uncertainty and, in that sense, are technically related to each other. In terms of this relation, it is useful to note that this paper analyzes a more complicated problem than Kıbrıs and Kıbrıs (2013). In that study, the allocation problem is restricted only to the "bad outcome" (in that case, bankruptcy) whereas here, it concerns both outcomes. Thus, while for the special case where the positive surplus-sharing rule is purely proportional the findings of Kıbrıs and Kıbrıs (2013) can be adapted to calculate individual and total contributions, they remain silent on partnership agreements that use an infinite number of other surplus-sharing rules (involving mixtures of proportionality and equal surplus-shares), all of which are analyzed here. As a result, central issues in this paper such as the effect of changes in the positive/negative surplus-sharing rule on individual/total contributions and on the acceptability of the partnership agreement, or how the size of these effects depends on the other surplus-sharing rule in use and on the number of partners, are outside the confines of the analysis carried out in Kıbrıs and Kıbrıs (2013). In terms of individual and social welfare, there is even less relationship between the two papers. Welfare comparisons in Kıbrıs and Kıbrıs (2013) are restricted to the two-agent case and can only be adapted to compare social welfare under two extreme partnership agreements. This paper however allows an arbitrary number of agents and involves both individual and social welfare comparisons for all of the continuum of rules that we consider. For additional discussion, please see Remark 1 at the end of Sect. 2.

2 Model

The set of *partners* is $N = \{1, \ldots, n\}$. Each partner $i \in N$ has a *Constant Absolute Risk Aversion (CARA) utility function* $u_i : \mathbb{R} \to \mathbb{R}$ on money: $u_i(x) = -e^{-a_i x}$. Assume the partners are risk averse and are ordered according to risk aversion: $0 < a_1 \leq \cdots \leq a_n$.

Each partner i chooses his contribution to the partnership, $s_i \in \mathbb{R}_+$. We measure contributions in monetary units (or equivalently assume a constant marginal cost normalized to 1). The total contribution of the partners is then $\sum_N s_j$. With *success probability* $p \in (0, 1)$, this value brings a *return* $r \in (0, 1]$ and becomes $(1+r)\sum_N s_j$, creating a *positive surplus* of $r\sum_N s_j$ for the partners. With the remaining $(1-p)$ probability, the partnership's value becomes $\beta\sum_N s_i$ where $\beta \in (0, 1)$ is the *fraction that survives failure*. In this case, the partnership makes a *negative surplus* of $(1-\beta)\sum_N s_i$.

A *partnership agreement* is a pair of rules F, G to be used in case of positive and negative surplus, respectively. The *positive-surplus rule* F allocates the gross returns $(1+r)\sum s_j$ according to the vector of contributions s, partner i's share being $F_i(s, (1+r)\sum s_j)$. The *negative-surplus rule* G, on the other hand, allocates the amount that survives failure $\beta\sum s_j$ according to the vector of contributions s, partner i's share being denoted as $G_i(s, \beta\sum s_j)$.

The following partnership agreements are based on two central surplus-sharing rules commonly used in real life. Suppose the partnership creates value V. (From previous discussion, we know V is either $(1+r)\sum s_j$ or $\beta\sum s_j$. But the next two definitions will be independent of what V is.) The *proportional surplus-sharing rule*, P, allocates the surplus proportional to the partners' contributions. The share of a typical agent is then $P_i(s, V) = \frac{s_i}{\sum s_j} V = s_i + \frac{s_i}{\sum s_j}\left(V - \sum s_j\right)$ (where $V - \sum s_j$ is the surplus). The *equal surplus-sharing rule*, E, allocates the surplus equally. The share of an agent is then $E_i(s, V) = s_i + \frac{V - \sum s_j}{n}$.

Gaynor and Pauly (1990) mention that the following "mixtures" of P and E are also commonly used, especially in professional service partnerships. For each $\rho \in [0, 1]$, the PE $[\rho]$ *rule* first reimburses each partner for his contributions. Then, it allocates $(1-\rho)$ part of the surplus equally among the partners and uses the remaining fraction ρ to give bonuses in proportion to contributions:

$$\begin{aligned} PE[\rho]_i\,(s, V) &= \rho P_i\,(s, V) + (1-\rho)\, E_i\,(s, V) \\ &= s_i + \left(V - \sum s_j\right)\left(\rho \frac{s_i}{\sum s_j} + (1-\rho)\,\frac{1}{n}\right). \end{aligned}$$

Geometrically, these rules span all convex combinations of the proportional and equal surplus-share allocations.

As noted in the introduction, a partnership agreement can specify different rules to be used in cases of positive and negative surplus. The class of partnership agreements that we analyze, therefore combine a positive-surplus rule $PE\left[\gamma\right]$ and a negative surplus rule $PE\,[\alpha]$ where $\alpha, \gamma \in [0, 1]$ and $\alpha \neq \gamma$ is allowed.[3] We will refer to such a *partnership agreement* as $PE\left[\gamma, \alpha\right]$.

Given the partnership agreement $PE\left[\gamma, \alpha\right]$, the partners simultaneously choose their contributions. Agent i's (expected) payoff from a contribution profile $s \in \mathbb{R}^n_+$ is

$$U_i^{PE[\gamma,\alpha]}(s) = pu_i\left(F_i\left(s, (1+r)\sum s_j\right) - s_i\right) + (1-p)u_i\left(G_i\left(s, \beta \sum s_j\right) - s_i\right)$$

where $F_i\left(s, (1+r)\sum s_j\right) - s_i$ and $G_i\left(s, \beta \sum s_j\right) - s_i$ are his surplus shares in cases of positive and negative surplus, respectively. Let $U^{PE[\gamma,\alpha]} = \left(U_1^{PE[\gamma,\alpha]}, \ldots, U_n^{PE[\gamma,\alpha]}\right)$. The *partnership game induced by* $PE\left[\gamma, \alpha\right]$ is then defined as

$$\mathcal{G}^{PE[\gamma,\alpha]} = \langle \mathbb{R}^N_+, U^{PE[\gamma,\alpha]} \rangle.$$

Let $\epsilon(\mathcal{G}^{PE[\gamma,\alpha]})$ denote the set of Nash equilibria of $\mathcal{G}^{PE[\gamma,\alpha]}$.

To measure the partners' social welfare from a partnership agreement, we will resort to two leading measures in the literature. The *egalitarian social welfare induced by* $PE\left[\gamma, \alpha\right]$ is the minimum utility an agent obtains at the Nash equilibrium of the partnership game induced by $PE\left[\gamma, \alpha\right]$:

$$\mathcal{E}\mathcal{G}^{PE[\gamma,\alpha]}\,(p, r, \beta, a_1, \ldots, a_n) = \min_{i \in N} U_i\left(\epsilon\left(G^{PE[\gamma,\alpha]}\right)\right).$$

[3] The parameter γ (respectively, α) determines which fraction of positive (respectively, negative) surplus is allocated proportionally.

The *utilitarian social welfare induced by* $PE\left[\gamma, \alpha\right]$ is the total utility the agents obtain at the Nash equilibrium of the partnership game induced by $PE\left[\gamma, \alpha\right]$:

$$\mathcal{UT}^{PE[\gamma,\alpha]}\left(p, r, \beta, a_1, \ldots, a_n\right) = \sum_{i \in N} U_i\left(\epsilon\left(G^{PE[\gamma,\alpha]}\right)\right).$$

Remark 1 As noted at the end of the previous section, the findings of Kıbrıs and Kıbrıs (2013) can be adapted to calculate individual and total contributions for $\gamma = 1$ (though they remain silent on partnership agreements where $\gamma \in [0, 1)$). This is a special case that contains no interaction among the agents via positive returns. (This can be verified in Eq. (2) in the "Appendix" where taking $\gamma = 1$ makes the first part of the utility function of agent i independent of the other agents' contributions.) In this paper on the other hand, with the exception of boundary cases where $\alpha = 1$ or $\gamma = 1$, strategic interaction takes place via both positive and negative returns. As will be detailed in the next section, this enriches the analysis and leads to a number of interesting conclusions. It is also useful to reiterate that welfare comparisons in Kıbrıs and Kıbrıs (2013) are restricted to the two-agent case and can only be adapted to compare social welfare under the two extreme partnership agreements $PE\,[1, 1]$ versus $PE\,[1, 0]$. This paper however allows an arbitrary number of agents and involves both individual and social welfare comparisons for all of the continuum of rules that we consider.

3 Results

As defined in the previous section, each partnership agreement $PE\left[\gamma, \alpha\right]$ induces a partnership game among the agents. We next analyze the Nash equilibria of these games to discuss equilibrium contributions and productive as well as social efficiency.

3.1 Acceptable agreements

In this section, we characterize partnership agreements that induce all partners to contribute to the partnership. To this end, we say that a partnership agreement $PE\left[\gamma, \alpha\right]$ is *acceptable* for N if at the Nash equilibrium of the partnership game, all partners choose a positive contribution level.

Acceptable partnership agreements are of special importance for two reasons. The first is technical: acceptable partnership agreements induce interior Nash equilibria at which the partners' equilibrium strategies and payoffs are differentiable with respect to the game's parameters. Therefore, they facilitate comparative statics analyses.

The second reason is empirical: real life data offers strong evidence that positive contributions by all partners is rather the norm in partnerships. Given that it is precisely the acceptable partnership agreements that induce positive contributions by all partners, this empirical regularity constitutes supportive evidence for the hypothesis that partnerships only form under acceptable partnership agreements.

The empirical evidence we offer comes from a rich dataset on legal partnerships, which are noted by Lang and Gordon (1995) to be the most common form of partnership in the US. This comprehensive dataset which has been extensively employed by the literature is based on two national surveys of lawyers in the US carried out by the American Bar Association in 1984 and 1990.[4] The data show how many billable hours each lawyer in the survey has reported in one month. The average report is 187.88 (with a standard deviation of 46.87) and the minimum report among all lawyers in the survey is 32 h per month.[5] That is, no lawyer in the dataset has chosen to make zero contributions to the partnership (s)he works for.

To determine whether a partnership agreement is acceptable, two intuitive conditions turn out to be important. The first condition, *profitability*, requires:

$$\ln\left(\frac{pr\,(n\gamma-\gamma+1)}{(1-p)\,(1-\beta)\,(n\alpha-\alpha+1)}\right)>0. \qquad \text{(Profitability)}$$

This condition, which can be rewritten as $p\left(\gamma r+(1-\gamma)\,\frac{r}{n}\right)>(1-p)\,(\alpha\,(1-\beta)$ $+\,(1-\alpha)\,\frac{(1-\beta)}{n}\Big)$, simply compares the return on unit contribution in case of positive surplus, $\left(\gamma r+(1-\gamma)\,\frac{r}{n}\right)$, weighted by the probability of success, p, with the loss incurred on unit contribution in case of negative surplus, $\left(\alpha\,(1-\beta)+(1-\alpha)\,\frac{(1-\beta)}{n}\right)$, weighted by the probability of failure, $(1-p)$. Positive contributions are optimal if the returns in case of success outweigh the losses incurred in case of failure.[6] Note that the Profitability *condition* does not make any reference to the partners' risk attitudes. That will be the concern of our next condition.

The second condition, *homogeneity*, requires that the agents are not too different in terms of their risk attitudes:

$$\frac{\frac{1}{a_n}}{\frac{1}{n}\left(\sum_N \frac{1}{a_j}\right)}>1-\frac{\gamma r+\alpha\,(1-\beta)}{r+1-\beta}. \qquad \text{(Homogeneity)}$$

The left hand side of this inequality has played an important role in previous studies such as Wilson (1968) and Huddart and Liang (2003). It is interpreted as agent n's *risk tolerance* relative to the average risk tolerance of the partnership (*e.g.* see Wilson's interpretation for the case of syndicates). Since agent n is the most risk averse partner (i.e. $a_1\leq\cdots\leq a_n$), the left hand side is less than or equal to 1 (and it is equal to 1

4 The full survey data is available from the University of Michigan based Inter-university Consortium for Political and Social Research (ICPSR) at their webpage: http://www.icpsr.umich.edu/icpsrweb/ICPSR/studies/8975.

5 Billable hours do not include administrative or clerical work or working on client development (Lang and Gordon 1995; p. 621). Therefore, the billable hours data is an understatement of a lawyer's contribution to the partnership.

6 The left hand side expression $\left(\gamma r+(1-\gamma)\,\frac{r}{n}\right)$ has two parts. The γ weighted part r is the partner's return on unit contribution under proportional surplus-sharing and the $(1-\gamma)$ weighted part $\frac{r}{n}$ is his return under equal surplus-sharing. The right hand side expression $\left(\alpha\,(1-\beta)+(1-\alpha)\,\frac{(1-\beta)}{n}\right)$ again has two parts. The α weighted part of this expression, $(1-\beta)$ is the loss incurred for unit contribution in case of proportional surplus-sharing and the $(1-\alpha)$ weighted part $\frac{1-\beta}{n}$ is the loss incurred in case of equal surplus-sharing.

precisely when $a_1 = \cdots = a_n$). For the same reason, if agent n were to be replaced with any other agent, the left hand side would increase in value, making the inequality less binding. This is why the *Homogeneity condition* is stated for agent n , even though it applies to all partners.

The right hand side of the inequality depends on how distant $PE\left[\gamma,\alpha\right]$ is from pure proportionality, $PE\,[1,1]$. The denominator of the fraction shows how $PE\,[1,1]$ allocates positive surplus (r) and negative surplus $(1-\beta)$. The numerator, on the other hand, shows that under $PE\left[\gamma,\alpha\right]$, only γ fraction of positive surplus and α fraction of negative surplus is allocated proportionally (γr and $\alpha\,(1-\beta)$). When both α and γ are 1, that is for $PE\,[1,1]$, the right hand side is zero and thus, not binding. As either of the two surplus sharing rules move towards equal shares however, that is, as α or γ goes down, the right hand side increases, becoming more binding. When $\alpha = \gamma = 0$ (*i.e.* when the partnership agreement allocates both positive and negative surplus equally), the right hand side reaches its maximum value of 1.

The reader will note an interesting distinction between α and γ. An increase in γ increases the partnership's profitability and homogeneity simultaneously. Yet, an increase in α decreases the partnership's profitability while increases its homogeneity.

Proposition 1 (Acceptable agreements) *A partnership agreement $PE[\gamma,\alpha]$ with $max\{\alpha,\gamma\} > 0$ is acceptable for N if and only if both Profitability and Homogeneity conditions are satisfied. The partnership agreement $PE[0,0]$ is acceptable for N if and only if Profitability is satisfied and the Homogeneity condition holds with a weak inequality.*

Note that when $\alpha = \gamma = 0$, the right hand side of the *Homogeneity condition* is 1. The maximum value for the left hand side, achieved when $a_1 = \cdots = a_n$, is also 1. Thus, when $\alpha = \gamma = 0$, the *Homogeneity condition* holds with a weak inequality if and only if all agents have identical risk attitudes. This is precisely the case when the partnership game has multiple Nash equilibria and for that reason, it will require special attention, as can be seen below.

3.2 Equilibrium contributions

In this section, we analyze the equilibrium contributions of partners in a formed partnership. As can be seen in the following proposition, equilibrium contributions are unique under all partnership agreements but $PE[0,0]$.

Proposition 2 (Equilibrium contributions under $PE\left[\gamma,\alpha\right]$) *If the agreement $PE[\gamma,\alpha]$ with $max\{\alpha,\gamma\} > 0$ is acceptable for N, the resulting partnership game has a unique Nash equilibrium s^* where*

$$s_i^* = \frac{\left(n\,(r+1-\beta)\,\frac{1}{a_i} - ((1-\gamma)\,r + (1-\alpha)\,(1-\beta))\left(\sum_N \frac{1}{a_j}\right)\right)\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-p)(1-\beta)(n\alpha-\alpha+1)}\right)}{n\,(r+1-\beta)\,(\gamma r + \alpha\,(1-\beta))} \tag{1}$$

for each $i \in N$*. On the other hand, if* $PE\,[0,0]$ *is acceptable for* N*, the partnership game has a continuum of Nash equilibria: any contribution profile* $s^* \geq 0$ *such that*

$$\sum_N s_i^* = \frac{n \ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{a_n\,(1-\beta+r)}$$

is a Nash equilibrium.

Note that the ln term in Eq. (1) is the one used in the Profitability *condition*. Also, as can easily be checked, the denominator of the first term in Eq. (1) is always positive. The *Homogeneity condition* guarantees that the numerator is of positive sign as well.

As stated in Proposition 1, under $PE\,[0,0]$ a partnership forms if and only if $a_1 = \cdots = a_n$. Proposition 2 then tells us that this symmetric game has a continuum of Nash equilibria. Nevertheless, the symmetric equilibrium among them (where for each $i \in N$, $s_i^* = \frac{\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{a_i(1-\beta+r)}$) is more robust than the rest in the following sense. Imagine a sequence of partnership agreements, each satisfying $max\{\alpha, \gamma\} > 0$, but converging to $PE\,[0,0]$. As can be seen from Proposition 2, the corresponding sequence of unique equilibrium contributions will also be converging, and it will converge precisely to this symmetric equilibrium under $PE\,[0,0]$. No other equilibrium under $PE\,[0,0]$ satisfies this property. Therefore, in welfare comparisons, we will focus on this symmetric equilibrium when analyzing $PE\,[0,0]$ and $a_1 = \cdots = a_n$.

Since $a_1 \leq \cdots \leq a_n$, Eq. (1) implies $s_1^* \geq \cdots \geq s_n^*$. That is, agent i is a "bigger partner" than agent j whenever $i \leq j$.

A corollary of Proposition 2 identifies conditions under which the partnership game has a dominant strategy equilibrium.[7] Partnership agreements that induce dominant strategy equilibria are advantageous to those that do not since it is possible to make a stronger prediction about how the partners will behave.

Corollary 3 (Dominant strategy equilibrium under $PE\,[1,1]$) *The partnership game induced by the agreement* $PE\,[1,1]$ *has a dominant strategy equilibrium (in strictly dominant strategies). No other partnership agreement induces dominant strategy equilibria.*

To provide the reader with a better understanding of the above propositions, we conclude this section with a two-partner numerical example that demonstrates how individual contributions depend on the partnership agreement $PE\left[\gamma, \alpha\right]$. In the example, the parameter values are $r = 0.3$, $p = 0.8$, $\beta = 0.7$, $a_1 = 1$, $a_2 = 1.5$, $\gamma = 0.5$.

Figure 1 plots how individual contributions change as a function of α, the percentage of negative surplus allocated proportionally. As can be seen, an increase in α decreases Partner 1's contribution. This might seem surprising at first glance, since it is commonly argued in the literature that a shift from equal to proportional surplus

[7] It follows from Eq. (3) in the "Appendix" that the partnership games induced by $PE\left[\gamma, \alpha\right]$ agreements admit dominant strategy equilibria if and only if $(1-\gamma)\,r + (1-\alpha)\,(1-\beta) = 0$ (in which case, partner i's best response is independent of the others' strategies). This equality holds if and only if $\alpha = \gamma = 1$. In Eq. (1), this equality ensures that partner i 's equilibrium strategy is independent of the others' risk attitudes.

shares will increase individual contributions. However, the reader will note after a closer inspection that an increase in α decreases the marginal return on contributions in case of negative surplus (by making losses more sensitive to contributions, as can be seen in Footnote 5). It thereby induces both partners to contribute less.

Maybe more surprisingly, Fig. 1 shows that α has a non-monotonic effect on the contribution of the smaller partner, Partner 2, who first increases and then decreases his contribution. This nonmonotonicity is caused by two competing effects. The first, direct effect is already mentioned in the previous paragraph. The second, indirect effect is due to the fact that the two partners' contributions are strategic substitutes. Thus, as Partner 1 decreases his contribution in response to an increase in α, partner 2 is inclined to increase his own contribution in response. The figure shows that the latter affect is dominant for small values of α. But for high α values, the first direct effect overtakes the second.[8]

The nonmonotonicity of s_2^* in α is not a knife-edge case. In this example, unilateral changes in γ or r do not disturb this nonmonotonicity at all; a unilateral change in p disturbs it only when $p > 0.87$ (making s_2^* an increasing function) and a unilateral change in β disturbs it only when $\beta > 0.95$ (making s_2^* a decreasing function). It is also useful to note that, for the above parameter values, the value of α that maximizes s_2 is decreasing in γ (the percentage of positive surplus allocated proportionally). This shows that the incentives Partner 2 faces are not straightforward, but are determined through an interplay of the positive-surplus and negative-surplus rules.

In the same example, we next fix $\alpha = 0.3$ and let γ vary. Figure 2 demonstrates that, as claimed by the previous literature, an increase in γ (the percentage of positive surplus allocated proportionally) in turn increases Partner 2's contributions.[9] However, it also shows that the effect of γ on Partner 1 is non-monotonic (the discussion, similar to the case of α, is omitted). Thus, contrary to what the previous literature suggests, even when allocating positive surplus, moving from fixed surplus-shares towards proportionality (the piece-rate) does not necessarily increase individual contributions of all partners.

3.3 Productive efficiency

In this section, we compare partnership agreements in terms of the total contribution that they induce in equilibrium, that is, in terms of their *productive efficiency*. As demonstrated in the previous section, a look at individual contributions suggests no clear prediction as to how total contributions would be affected from changes in the underlying partnership agreement. On the other hand, Fig. 3 suggests a clear ordering in our numerical example. First, an increase in γ in turn increases the partners' total

8 As can be more formally seen in Eq. (3) in the "Appendix", both partners have linear best response functions (with a positive intercept and a negative slope). An increase in α affects both best response functions in the same way: it decreases the intercept and decreases the slope in absolute value, making best responses less sensitive to the other partner's choices. It is because of this that the strategic substitutes property matters less at high values of α.

9 Figure 2 also demonstrates that, for $\gamma \leq 0.1$, the partnership agreement $PE[\gamma, \alpha]$ is not acceptable and, as discussed in the previous section, the partnership does not form.

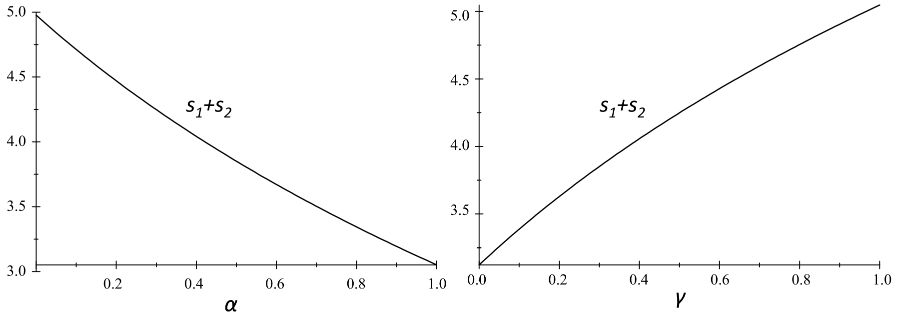

Fig. 3 The effect of α (*left*) and γ (*right*) on total contributions. Parameter values are the same as in Fig. 1

contributions. This confirms the common belief that a move from equal surplus-sharing towards proportionality increases total contributions. However, the figure also shows that a similar move in the allocation of negative surplus has exactly the opposite effect.

The following theorem generalizes what we observe in this numerical example to the whole parameter space.

Theorem 1 *Equilibrium total contributions under* $PE\left[\gamma, \alpha\right]$ *is* (*i*) *increasing in* γ *(the fraction of positive surplus allocated proportionally) and* (*ii*) *decreasing in* α *(the fraction of negative surplus allocated proportionally). Furthermore, both effects are increasing in the number of partners in the partnership.*

In terms of what it says regarding the positive-surplus rule, the theorem supports the general view that moving from fixed surplus shares to proportionality increases total contributions. For the negative-surplus rule, however, the theorem identifies that a move towards proportionality now decreases total contributions. The theorem, thus, shows us that a way to improve over the commonly-used piece rate agreement is to change the surplus-sharing rule used in case of negative-surplus; a move towards equal surplus-shares helps productive efficiency. While such a change does not incentivize every partner to contribute more (e.g. see Partner 2 in Fig. 1), its aggregate effect is certain.

It is interesting to note that, even in symmetric partnerships (i.e. when all partners have identical risk attitudes), the ordering of partnership agreements in terms of total contributions is still as above. Particularly, $PE\,[1, 0]$ still remains as the unique productively efficient agreement. It is also important to reiterate that the effect of the agreement on total contributions is emphasized in partnerships with a greater number of partners. Thus, bigger partnerships would be more likely to pick greater γ and smaller α parameters.

Theorem 1 implies that the partnership agreement $PE\,[1, 0]$ is the unique productively efficient agreement in the $PE\left[\gamma, \alpha\right]$ family. However, as our findings in Sect. 3.1 demonstrate, there are partnerships where this agreement will not be acceptable. In such partnerships, $PE\,[1, 0]$ violates either the Profitability or the *Homogeneity* condition. First, it is straightforward to see that if $PE\,[1, 0]$ violates Profitability, every other partnership agreement also does so. Thus, in such cases the partnership will not form under any $PE\left[\gamma, \alpha\right]$ agreement. The more interesting case is when $PE\,[1, 0]$

violates *Homogeneity*. Then, an increase in α helps to satisfy the inequality while a decrease in γ does not. Thus, keeping $\gamma = 1$, there is a critical value

$$\alpha^* = 1 - \frac{r+1-\beta}{1-\beta}\frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}}$$

where the set of acceptable agreements are $PE[1, \alpha]$ such that $\alpha > \alpha^*$.[10] As α decreases, productive efficiency increases and simultaneously the most risk averse Partner n's contribution decreases. At the limit $\alpha = \alpha^*$, Partner n picks a zero contribution making $PE\left[1, \alpha^*\right]$ unacceptable.

3.4 Individual and social welfare

In this section, we look at the individual and social welfare levels induced by alternative partnership agreements. We make an analytical comparison in terms of egalitarian social welfare. Additionally, we carry out a numerical analysis in terms of utilitarian social welfare.

Figure 4 demonstrates how equilibrium welfare of the two partners in our example changes in response to $PE\left[\gamma, \alpha\right]$. The following observations are in order. First, in both pictures Partner 1 (the bigger partner) receives a greater utility than Partner 2 if and only if $\alpha < \gamma$, that is, when a higher proportion of positive than negative surplus is allocated proportionally. Thus, egalitarian social welfare is equal to the utility of Partner 1 when $\alpha > \gamma$ and to the utility of Partner 2 when $\alpha < \gamma$. As both pictures demonstrate, when $\alpha = \gamma$, the two partners receive equal payoff. Second, this egalitarian social welfare increases as α and γ gets closer to each other, and is maximized at $\alpha = \gamma$.

Surprisingly, both of the above points are generalizable to an arbitrary number of agents and to all parameter values we consider. The following proposition orders the agents according to their equilibrium welfare.

Proposition 4 *Under $PE\left[\gamma, \alpha\right]$, the partners are ordered according to their equilibrium utilities as 1, 2,. . ., n. If $\alpha > \gamma$, the least risk-averse Partner 1 always receives the smallest utility and the most risk-averse Partner n always receives the highest utility. If $\alpha < \gamma$, the ordering is reversed, Partner 1 now receiving the highest utility and Partner n, the smallest. If $\alpha = \gamma$, all partners receive the same utility level.*

The above proposition implies that the egalitarian social welfare is equal to the equilibrium payoff of either the most or the least risk averse partner, depending on the α-γ relationship in their partnership agreement. The following theorem shows that this egalitarian social welfare is maximized at $\alpha = \gamma$.

10 To see this, note that $\frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}} > 1 - \frac{r+\alpha(1-\beta)}{r+1-\beta}$ iff $\alpha > \frac{(r+1-\beta)}{(1-\beta)}\left(1 - \frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}}\right) - \frac{r}{(1-\beta)}$ iff $\alpha > 1 - \frac{r+1-\beta}{1-\beta}\frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}}$.

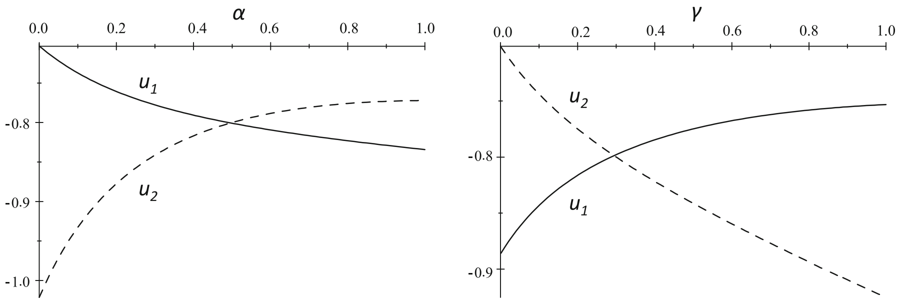

Fig. 4 Utility of Partner 1 (*solid line*) and Partner 2 (*dashed line*) as a function of α (*left*) and γ (*right*). Parameter values are the same as in Fig. 1

Theorem 2 *Under $PE\left[\gamma, \alpha\right]$, egalitarian social welfare is decreasing in $|\alpha - \gamma|$, being maximized when $\alpha = \gamma = x$. In this case, all partners' payoffs are equal and this common payoff, which is also the egalitarian social welfare under the $PE\,[x, x]$ partnership agreement, is independent of x.*

While all $PE\,[x, x]$ partnership agreements induce the same egalitarian social welfare level, they might be different in other aspects. The first that comes to mind is the agents' contribution choices. It turns out that all $PE\,[x, x]$ partnership agreements induce the same total contribution in equilibrium. (Thus, maximizing total contributions among $PE\,[x, x]$ agreements does not restrict x at all.) These agreements, however, differ in terms of the individual contributions that they induce in equilibrium. Partners who are less (more) risk averse than the average decrease (increase) their contributions in response to an increase in the common x, keeping total contributions constant (for a proof, please see Claim 1 in Sect. 1).

Due to differences in individual contribution choices, it might be that some $PE\,[x, x]$ agreements are acceptable while the others are not (as discussed in Sect. 3.1). It is straightforward to check that the Profitability condition does not distinguish among the $PE\,[x, x]$ agreements; either they all satisfy or violate it. The *Homogeneity condition*, on the other hand, partitions the set of $PE\,[x, x]$ agreements. There is a critical value

$$x^* = 1 - \frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}}$$

where an agreement $PE\,[x, x]$ is acceptable if and only if $x > x^*$. As the common x decreases in an acceptable agreement, the most risk averse Partner n's contribution will also decrease, reaching zero at $x = x^*$.

We conclude this section with a discussion of utilitarian social welfare. For this case, the ordering of partnership agreements in terms of utilitarian social welfare depends on the underlying parameter values. This makes a general analytical result as in the case of egalitarian social welfare not possible.

We first carry out a numerical analysis for the case of two partners. We allow the following parameter values:

$$\beta, p, r, \alpha, \gamma \in \{0.01, 0.11, 0.21, 0.31, 0.41, 0.51, 0.61, 0.71, 0.81, 0.91\},$$
$$a_1, a_2 \in \{0.01, 0.51, 1.01, 1.51, 2.01, 2.51, 3.01, 3.51, 4.01, 4.51\} \text{ and } a_2 \geq a_1.$$

This grid produces (1) 23,469 combinations of β, p, r, a_1, a_2 at which both the Profitability and Homogeneity conditions are satisfied under some α-γ combinations, that is, the partnership has acceptable agreements available[11] and (2) 53,721 combinations of $\alpha, \gamma, \beta, p, r, a_1, a_2$ where the α-γ combination maximizes utilitarian social welfare under β, p, r, a_1, a_2. Surprisingly, at 36,280 (that is, 67.5 %) of these parameter combinations, utilitarian social welfare is maximized when $\gamma = \alpha$. And, at 40,794 (that is, 75.9 %) of these combinations, utilitarian social welfare is maximized when α and γ differ by at most one grid point. It is also interesting to note that, among the remaining parameter combinations, $\alpha > \gamma$ is observed more than twice as much as $\gamma > \alpha$ (precisely, at 8981 versus 3946 combinations).

We also carried out a numerical analysis for the case of three partners. Since the computer could not handle the above grid, we switched to a slightly coarser grid of

$$\beta, p, r \in \{0.01, 0.16, 0.31, 0.46, 0.51, 0.66, 0.71, 0.86, 0.91\},$$
$$\alpha, \gamma \in \{0.01, 0.11, 0.21, 0.31, 0.41, 0.51, 0.61, 0.71, 0.81, 0.91\},$$
$$a_1 \in \{0.1, 0.7, 1.3, 1.9, 2.5, 3.1, 3.7, 4.3, 4.9\} \text{ and } a_3 \geq a_2 \geq a_1.$$

This grid produces (1) 24,286 combinations of $\beta, p, r, a_1, a_2, a_3$ at which both the Profitability and Homogeneity conditions are satisfied under some α-γ combinations, that is, the partnership has acceptable agreements available[12] and (2) 44,442 combinations of $\alpha, \gamma, \beta, p, r, a_1, a_2, a_3$ where the α-γ combination maximizes utilitarian social welfare under $\beta, p, r, a_1, a_2, a_3$. At more than half (specifically 25,680, that is, 57.8 %) of these parameter combinations, utilitarian social welfare is maximized when $\gamma = \alpha$. And, at more than two thirds (specifically 31,244, that is, 70.3%) of these combinations, utilitarian social welfare is maximized when α and γ differ by at most one grid point. Finally, among the remaining parameter combinations, $\alpha > \gamma$ is observed twice as much as $\gamma > \alpha$ (precisely, at 8785 versus 4413 combinations).

These numerical findings should be interpreted with caution. The $\gamma = \alpha$ finding, in a significant number of the cases, is due to the grid that we impose on the parameter space. Thus, we can only deduce from this analysis that quite frequently, utilitarian social welfare is maximized at α, γ values that are close to each other and that, maximizing utilitarian social welfare does not create agreements that systematically differ from those that maximize egalitarian social welfare.

[11] This corresponds to 1,324,692 combinations of $\alpha, \gamma, \beta, p, r, a_1, a_2$.

[12] This corresponds to 1,029,914 combinations of $\alpha, \gamma, \beta, p, r, a_1, a_2, a_3$.

4 Conclusion

Our analysis compares a family of partnership agreements (i.e. surplus allocation rules) in terms of total contributions and social welfare that they induce in equilibrium of a noncooperative partnership game. Our findings are as follows:

(i) Equilibrium total contributions induced by a partnership agreement increases as the positive-surplus rule gets closer to proportionality and the negative-surplus rule gets closer to equal surplus-shares. Using proportionality in case of positive surplus and equal-surplus shares in case of negative surplus (*i.e.* $\gamma = 1, \alpha = 0$) maximizes total contributions whenever this agreement is acceptable. Otherwise, the partners pick $\gamma = 1$ and α as small as acceptability permits.

(ii) Egalitarian social welfare increases as the percentages of positive and negative surplus allocated proportionality (i.e. γ and α) get closer to each other. Partnership agreements where $\gamma = \alpha$ all maximize egalitarian social welfare. Such agreements give all agents the same welfare and produce the same amount of total contributions. They, however, differ in terms of individual contribution choices that they induce.

(iii) The ordering of partnership agreements in terms of utilitarian social welfare depends on the parameter values. Thus a general statement as in egalitarian social welfare or total contributions can not be made. However, a numerical analysis shows that the utilitarian optimal partnership agreements are not systematically different from egalitarian optimal ones. Simulations for two and three agent partnerships show that 60 to 70 % of the utilitarian optimal partnership agreements exhibit $\gamma = \alpha$.

(iv) In symmetric games (where $a_1 = \cdots = a_n$), the egalitarian optimal agreements described in (ii) additionally Pareto dominate all other agreements.

(v) There always is a unique dominant strategy equilibrium under the purely proportional agreement $PE[1, 1]$. No other partnership agreement induces dominant strategies.

Overall, we observe a trade-off between maximizing total contributions and social welfare. To maximize total contributions, two opposite surplus allocation rules needs to be used in cases of positive and negative surplus (*i.e.* $\gamma = 1$ and $\alpha = 0$). However, (i) maximizing egalitarian social welfare requires choosing the same surplus-allocation rule in cases of both positive and negative surplus (i.e. $\gamma = \alpha$), and furthermore (ii) a numerical analysis detailed in Sect. 3.4 obtains a similar result for utilitarian social welfare on a significant part of our parameter space. The mechanism behind this trade-off can be explained as follows. An increase in γ increases the marginal return of contributing by making returns in case of positive surplus more sensitive to contributions. A decrease in α has a similar effect by decreasing the sensitivity of losses (made in case of negative surplus) to contributions. Due to this reason, an increase in γ or a decrease in α provides all agents with a direct incentive to contribute more, thus leading to an increase in total contributions, as demonstrated in

Fig. 3.[13] However, an increase in γ or a decrease in α also transfers wealth from the smaller (than average) partners to the bigger ones and the high γ low α combinations which induce high total contribution levels thus make the bigger partners much better off than the smaller ones as can be seen in Fig. 4. As the figure also shows, bringing the two parameters closer to each other (by decreasing γ or increasing α) makes the bigger partners worse off and the smaller partners better off, thereby bringing the partners' welfare levels closer to each other and increasing egalitarian social welfare.

Appendix

We will start this section by calculating the Nash equilibrium of the partnership game. Under the family $PE\left[\gamma,\alpha\right]$, the utility function of partner i is

$$U_i^{PE[\gamma,\alpha]}(s) = -pe^{-a_i\left(\gamma r s_i + (1-\gamma)\frac{rs_i + r\sum_{N\setminus i} s_j}{n}\right)} -(1-p)e^{\left(\frac{(1-\beta)(1+(n-1)\alpha)}{n}\right)a_i s_i + \frac{(1-\alpha)(1-\beta)}{n} a_i \sum_{N\setminus i} s_j}. \tag{2}$$

The unconstrained maximizer of this expression is $s_i = \sigma_i(s_{-i}) =$

$$\frac{n\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-p)(1-\beta)(n\alpha-\alpha+1)}\right)}{a_i\left((1-\alpha+n\alpha)(1-\beta)+(n-1)r\gamma+r\right)} - \frac{r(1-\gamma)+(1-\beta)(1-\alpha)}{(1-\alpha+n\alpha)(1-\beta)+(n-1)r\gamma+r}\left(\sum_{N\setminus i} s_j\right). \tag{3}$$

Since $\frac{r(1-\gamma)+(1-\beta)(1-\alpha)}{(1-\alpha+n\alpha)(1-\beta)+(n-1)r\gamma+r} \in [0,1]$, the slope of this expression is negative.[14] Also, the sign of its constant term is determined by the sign of $\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)$. This ln term is nothing but the left hand side of the Profitability condition. Partner i's best response is $b_i(s_{-i}) = \max\{0, \sigma_i(s_{-i})\}$.

Solving the system in Expression 3 gives for each $i \in N$ (Expression 1 of Proposition 2)

[13] There is a second indirect effect on individual contributions stemming from the fact that the partners' contributions are strategic substitutes. A partner increasing his contribution incentivizes the other partners to decrease their contributions in return. A combination of these two effects can thus create nonmonotonic individual contribution responses to changes in α and γ as seen in Figs. 1 and 2. Yet, as our theorem shows, when aggregated over agents, this first effect overrides the second.

[14] This expression is equal to 0 if and only if $\alpha = \gamma = 1$ and equal to 1 if and only if $\alpha = \gamma = 0$. The former is trivial. To see the latter, note that $\frac{r(1-\gamma)+(1-\beta)(1-\alpha)}{(1-\alpha+n\alpha)(1-\beta)+(n-1)r\gamma+r} \leq 1$ simplifies to $0 \leq n\alpha(1-\beta) + nr\gamma$, achieved with equality if and only if $\alpha = \gamma = 0$.

$$s_i^* = \frac{\left(n\left(r-\beta+1\right)\frac{1}{a_i} - \left(r-\alpha-\beta-r\gamma+\alpha\beta+1\right)\left(\sum_N \frac{1}{a_j}\right)\right)}{n\left(r-\beta+1\right)\left(\alpha+r\gamma-\alpha\beta\right)} \ln\left(\frac{pr\left(n\gamma-\gamma+1\right)}{\left(1-\beta\right)\left(1-p\right)\left(n\alpha-\alpha+1\right)}\right) \tag{4}$$

which, under certain conditions, will give us the unique Nash equilibrium of the partnership game.

Proof of Proposition 1 **Case 1:** The partnership agreement is $PE\left[\gamma,\alpha\right]$ such that $\max\{\alpha,\gamma\} > 0$.

($\Longrightarrow$) Assume $PE\left[\gamma,\alpha\right]$ is acceptable for N. To see that Profitability and Homogeneity are satisfied, first suppose Profitability is violated. Then, for each $i \in N$ and for all $s_{-i}, \sigma_i\left(s_{-i}\right) < 0$. Thus, the unique Nash equilibrium is $s = (0, \ldots, 0)$, contradicting acceptability of $PE\left[\gamma,\alpha\right]$. Next, suppose Profitability is satisfied but Homogeneity is violated. Then, $(n\left(r-\beta+1\right)\frac{1}{a_n} - \left(r-\alpha-\beta-r\gamma+\alpha\beta+1\right)\left(\sum_N \frac{1}{a_j}\right)) \leq 0$ and thus $s_n^* < 0$, contradicting acceptability of $PE\left[\gamma,\alpha\right]$.

($\Longleftarrow$) Assume Profitability and Homogeneity conditions are satisfied. By Profitability, we have $\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right) > 0$ and by Homogeneity, we have $(n\left(r-\beta+1\right)\frac{1}{a_n} - \left(r-\alpha-\beta-r\gamma+\alpha\beta+1\right)\left(\sum_N \frac{1}{a_j}\right)) > 0$. This guarantees $s^* > 0$ where s^* (Expression 4) is then the unique Nash equilibrium. Thus, $PE\left[\gamma,\alpha\right]$ is acceptable.

Case 2: The partnership agreement is $PE\left[0,0\right]$.

($\Longrightarrow$) Assume $PE\left[0,0\right]$ is acceptable for N. We want to show that Profitability holds and Homogeneity holds with a weak inequality. First suppose Profitability is violated. Then, as noted above, the unique Nash equilibrium is $s = (0, \ldots, 0)$, contradicting acceptability of $PE\left[0,0\right]$. Next, suppose Profitability is satisfied but Homogeneity is violated. Since $\alpha = \gamma = 0$, we then have $\frac{1}{a_n} < \frac{1}{n}\left(\sum_N \frac{1}{a_j}\right)$. This implies, $a_1 < a_n$. Again due to $\alpha = \gamma = 0$, Expression 3 simplifies to

$$s_i = \sigma_i\left(s_{-i}\right) = \frac{n\ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_i\left(1-\beta+r\right)} - \left(\sum_{N\setminus i} s_j\right). \tag{5}$$

Since $a_1 < a_n$ and since each agent i's best response is the maximum of zero and $\sigma_i\left(s_{-i}\right)$, agent n picks zero contributions in equilibrium, contradicting acceptability of $PE\left[0,0\right]$.

($\Longleftarrow$) Assume Profitability and the weaker form of Homogeneity are satisfied. We want to show that $PE\left[0,0\right]$ is acceptable for N. By the weaker form of Homogeneity, $\frac{1}{a_n} \geq \frac{1}{n}\left(\sum_N \frac{1}{a_j}\right)$ which in turn implies $a_1 = \cdots = a_n$. By Profitability, we have $\frac{n\ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_n(1-\beta+r)} > 0$. Thus, the best response expression of every agent i can be written as $s_i = \sigma_i\left(s_{-i}\right) = \frac{n\ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_n(1-\beta+r)} - \left(\sum_{N\setminus i} s_j\right)$. Thus, all $s^* \geq 0$ such that

$\sum_N s_i^* = \frac{n \ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_n(1-\beta+r)}$ is a Nash equilibrium. Since a continuum among these equilibria satisfy $s^* > 0$, we conclude that $PE\,[0,0]$ is acceptable. □

Proof of Proposition 2 **Case 1:** The partnership agreement is $PE\left[\gamma,\alpha\right]$ such that $\max\{\alpha,\gamma\} > 0$.

Assume that $PE[\gamma,\alpha]$ is acceptable for N. By Proposition 1, both Profitability and Homogeneityconditions hold. To see that the resulting partnership game has a unique Nash equilibrium s^* which is given by Expression 1, note that s^* solves the system in Expression 3. By Profitability, we have $\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right) > 0$ and by Homogeneity, we have $(n\,(r-\beta+1)\,\frac{1}{a_i} - (r-\alpha-\beta-r\gamma+\alpha\beta+1)\left(\sum_N \frac{1}{a_j}\right)) > 0$. This guarantees that that $s^* > 0$. It is thus the unique Nash equilibrium of the partnership game under $PE\left[\gamma,\alpha\right]$.

Case 2: The partnership agreement is $PE\,[0,0]$.

Assume $PE\,[0,0]$ is acceptable for N. By Proposition 1, Profitability and the weaker form of Homogeneity are satisfied. The ($\Longleftarrow$) part in Case 2 of the previous proof then shows that all $s^* \geq 0$ such that $\sum_N s_i^* = \frac{n \ln\left(\frac{pr}{(1-\beta)(1-p)}\right)}{a_n(1-\beta+r)}$ is a Nash equilibrium of the partnership game. □

Proof of Corollary 3 In the Expression 3, the slope is: $-\frac{r(1-\gamma)+(1-\beta)(1-\alpha)}{(1-\alpha+n\alpha)(1-\beta)+(n-1)r\gamma+r}$. If this expression is zero, the best response of partner i is independent of s_{-i}, making it a strictly dominant strategy. Now note that the denominator of this expression is always positive. And its numerator $r\,(1-\gamma)+(1-\beta)\,(1-\alpha) = 0$ if and only if $\alpha = \gamma = 1$. Therefore, $PE\,[1,1]$ is the only partnership agreement that always induces a dominant strategy equilibrium. □

Proof Theorem 1 Total contribution is[15]

$$\sum s_i^* = \frac{1}{r-\beta+1}\left(\sum \frac{1}{a_i}\right)\ln\left(\frac{pr\,(n\gamma-\gamma+1)}{(1-\beta)\,(1-p)\,(n\alpha-\alpha+1)}\right).$$

The derivative of this expression with respect to α is

$$\frac{\partial\left(\sum s_i^*\right)}{\partial\alpha} = \frac{-\left(\sum_N \frac{1}{a_j}\right)(n-1)}{(n\alpha-\alpha+1)\,(r-\beta+1)} < 0.$$

Thus, a decrease in α increases total contributions. Now let us look at the effect of adding a partner on this derivative. Since $\frac{(n-1)}{(n\alpha-\alpha+1)} < \frac{n}{(n\alpha+1)}$, we have $\frac{-\left(\sum_N \frac{1}{a_j}\right)(n-1)}{(n\alpha-\alpha+1)(r-\beta+1)} > \frac{-\left(\frac{1}{a_{n+1}}+\sum_N \frac{1}{a_j}\right)(n)}{(n\alpha+1)(r-\beta+1)}$, the desired conclusion. That is, $\frac{\partial(\sum s_i^*)}{\partial\alpha}$ is increasing in absolute value as the number of agents increases.

[15] Note that, this expression gives total contribution when $\alpha = \gamma = 0$ as well. Even though there is multiplicity of equilibria in this case, they all have the same total contribution level given by this expression.

Now, let us look at the derivative of total contributions respect to γ :

$$\frac{\partial\left(\sum s_i^*\right)}{\partial\gamma}=\frac{\left(\sum_N \frac{1}{a_j}\right)(n-1)}{(n\gamma-\gamma+1)(r-\beta+1)}>0.$$

So, an increase in γ increases total contributions. Now let us look at the effect of adding a partner on this derivative. As above, $\frac{(n-1)}{(n\gamma-\gamma+1)}<\frac{(n)}{(n\gamma+1)}$ implies $\frac{\left(\sum_N \frac{1}{a_j}\right)(n-1)}{(n\gamma-\gamma+1)(r-\beta+1)}<\frac{\left(\frac{1}{a_{n+1}}+\sum_N \frac{1}{a_j}\right)(n)}{(n\gamma+1)(r-\beta+1)}$, the desired conclusion. That is, $\frac{\partial(\sum s_i^*)}{\partial\gamma}$ is increasing in the number of agents. □

Prrof Proposition 4 Introducing s^* into partner i's utility function, we obtain $U_i^{PE[\gamma,\alpha]}(s^*)=$

$$\begin{pmatrix} -p\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)^{\frac{-r\gamma}{\alpha+r\gamma-\alpha\beta}} \\ -(1-p)\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)^{\frac{\alpha(1-\beta)}{\alpha+r\gamma-\alpha\beta}} \end{pmatrix}$$
$$\times\left(\frac{pr\,(n\gamma-\gamma+1)}{(1-\beta)\,(1-p)\,(n\alpha-\alpha+1)}\right)^{\frac{ra_i(1-\beta)(\gamma-\alpha)}{n(r-\beta+1)(\alpha+r\gamma-\alpha\beta)}\left(\sum\frac{1}{a_j}\right)}$$

All components of $\frac{ra_i(1-\beta)(\gamma-\alpha)}{n(r-\beta+1)(\alpha+r\gamma-\alpha\beta)}\left(\sum\frac{1}{a_j}\right)$ have determinate signs except $\gamma-\alpha$, whose sign determines the effect of a_i on $U_i^{PE[\gamma,\alpha]}(s^*)$. If $\gamma-\alpha>0$, an increase in a_i decreases $U_i^{PE[\gamma,\alpha]}(s^*)$. As a result, Partner 1 receives the highest utility and Partner n, the lowest. The welfare ordering of the partners is exactly the opposite when $\gamma-\alpha<0$. And if $\gamma-\alpha=0$, a_i does not affect $U_i^{PE[\gamma,\alpha]}(s^*)$. Thus, all agents receive the same utility. □

Proof of Theorem 2 Proposition 4 establishes that

$$\mathcal{EG}^{PE[\gamma,\alpha]}(p,r,\beta,a_1,\ldots,a_n)$$
$$=\begin{cases} U_1^{PE[\gamma,\alpha]}(s^*) & if\ \gamma<\alpha \\ U_n^{PE[\gamma,\alpha]}(s^*) & if\ \gamma>\alpha \\ U_1^{PE[\gamma,\alpha]}(s^*)=\cdots=U_n^{PE[\gamma,\alpha]}(s^*) & if\ \gamma=\alpha \end{cases}\min_{i\in N}U_i(\epsilon(G^{PE[\gamma,\alpha]})).$$

We will treat each case separately. First, assume $\gamma=\alpha$. In this case, the individual utility functions simplify to

$$U_i^{PE[\gamma,\alpha]}(s^*)=\left(-p\left(\frac{pr}{(1-\beta)(1-p)}\right)^{\frac{-r}{1+r-\beta}}-(1-p)\left(\frac{pr}{(1-\beta)(1-p)}\right)^{\frac{(1-\beta)}{1+r-\beta}}\right)$$

Two observations are in line. First, Partner i's equilibrium utility is independent of a_i. Therefore, all partners receive identical utility. Second, the expression is independent of the common value of $\gamma = \alpha$. That is, all $\gamma = \alpha$ partnership agreements produce the same level of egalitarian social welfare. This establishes the second sentence of the theorem.

To see the first sentence, first assume $\gamma > \alpha$ and $i = n$. Then $\frac{\partial U_n^{PE[\gamma,\alpha]}(s^*)}{\partial \alpha} = \gamma r$ $(1-\beta)(r-\beta+1)(n\alpha-\alpha+1)((n-1)p\alpha+1)((\sum_j \frac{a_n}{a_j})-n)\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)$ $-\quad (n-1)(\alpha+r\gamma-\alpha\beta)(pr(\alpha-\gamma)(\sum_j \frac{a_n}{a_j})(r(n\gamma-\gamma+1)+(1-\beta)(1-\alpha$ $+n\alpha))-n(r-\beta+1)pr(\alpha-\gamma))$. Since $\left(\left(\sum_j \frac{a_n}{a_j}\right)-n\right) \geq 0$, the first term is nonnegative. And it is strictly positive unless $a_1 = \cdots = a_n$. The second term is also positive since[16]

$$\left(pr(\alpha-\gamma)\left(\sum_j \frac{a_n}{a_j}\right)(r(n\gamma-\gamma+1)\right.$$
$$\left.+(1-\beta)(1-\alpha+n\alpha))-n(r-\beta+1)pr(\alpha-\gamma)\right)<0.$$

This establishes that egalitarian social welfare is increasing in α when $\gamma > \alpha$.

Next assume $\gamma < \alpha$ and $i = 1$. Then, $\frac{\partial U_1^{PE[\gamma,\alpha]}(s^*)}{\partial \alpha} = \gamma r(1-\beta)(r-\beta+1)$ $(n\alpha-\alpha+1)((n-1)p\alpha+1)((\sum_j \frac{a_1}{a_j})-n)\ln\left(\frac{pr(n\gamma-\gamma+1)}{(1-\beta)(1-p)(n\alpha-\alpha+1)}\right)-(n-1)$ $(\alpha+r\gamma-\alpha\beta)(pr(\alpha-\gamma)(\sum_j \frac{a_1}{a_j})(r(n\gamma-\gamma+1)+(1-\beta)(1-\alpha+n\alpha))-n$ $(r-\beta+1)pr(\alpha-\gamma))$. The first term is nonpositive since $\left(\left(\sum_j \frac{a_1}{a_j}\right)-n\right) \leq 0$. And it is strictly negative unless $a_1 = \cdots = a_n$. The second term is also negative since

$$\left(pr(\alpha-\gamma)\left(\sum_j \frac{a_1}{a_j}\right)(r(n\gamma-\gamma+1)+(1-\beta)(1-\alpha+n\alpha))\right.$$
$$\left.\times -n(r-\beta+1)pr(\alpha-\gamma)\right)>0.$$

This establishes that egalitarian social welfare is decreasing in α when $\gamma < \alpha$.

Similar calculations show that the egalitarian social welfare is decreasing in γ when $\gamma > \alpha$ and increasing in γ otherwise. □

Claim 1 *All $PE[x,x]$ agreements induce the same amount of total contributions. However, a partner more (less) risk averse than average responds to an increase in x by increasing (decreasing) his contributions.*

[16] For brevity of presentation, calculations that prove this and similar secondary claims have been skipped. However, they all are available from the authors upon request.

Proof Under $PE[x,x]$, the total contribution expression (used in the proof of Theorem 1) simplifies to $\sum s_i^* = \frac{1}{r-\beta+1}\left(\sum \frac{1}{a_i}\right)\ln\left(\frac{pr}{(1-\beta)(1-p)}\right)$. Note that the expression is independent of x, proving the first claim. For the second claim, note that Expression 1 simplifies to $s_i^* = \frac{\left(n\frac{1}{a_i}-(1-x)\left(\sum_N \frac{1}{a_j}\right)\right)\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{nx(r+1-\beta)}$ under $PE[x,x]$. Taking the derivative of this expression with respect to x, we obtain $\left(\frac{1}{n}\left(\sum_N \frac{1}{a_j}\right)-\frac{1}{a_i}\right)\frac{\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)}{x^2(r-\beta+1)}$. The second part of the expression is positive (by Profitability). Thus, the sign is determined by the first part. If agent i is more (less) risk averse than average, this term is positive (negative), the desired conclusion. □

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