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Bargaining with nonanonymous disagreement: Decomposable rules ☆

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ABSTRACT

We analyze bargaining situations where the agents' payoffs from disagreement depend on who among them breaks down the negotiations. We model such problems as a superset of the standard domain of Nash (1950). We first show that this domain extension creates a very large number of new rules. In particular, decomposable rules (which are extensions of rules from the Nash domain) constitute a nowhere dense subset of all possible rules. For them, we analyze the process through which "good" properties of rules on the Nash domain extend to ours. We then enquire whether the counterparts of some well-known results on the Nash (1950) domain continue to hold for decomposable rules on our extended domain. We first show that an extension of the Kalai–Smorodinsky bargaining rule uniquely satisfies the Kalai and Smorodinsky (1975) properties. This uniqueness result, however, turns out to be an exception. We characterize the uncountably large classes of decomposable rules that survive the Nash (1950), Kalai (1977), and Thomson (1981) properties.

1. Introduction

A typical bargaining problem, as modeled by Nash (1950) and the vast literature that follows, is made up of two elements. The first is a set of alternative agreements on which the agents negotiate. The second element is an alternative realized in the case of disagreement. This "disagreement outcome" does not contain detailed information about the nature of the disagreement. In particular, it is assumed in the existing literature that the realized disagreement alternative is independent of who among the agents disagree(s). For a review of this literature, see Kıbrıs (2010). However, in real life examples of bargaining such as the 2004 reunion negotiations of Cyprus or wage negotiations between firms and labor unions, the identity of the agent who terminates the negotiations turns out to have a significant effect on the agents' "disagreement payoffs" (for more discussion, see Kıbrıs and Tapkı, 2010). To be able to represent them, we extend Nash's (1950) standard model to a *nonanonymous-disagreement* model of bargaining by replacing the disagreement *payoff vector* there with a disagreement *payoff matrix*. The ith row of this matrix is the payoff vector that results from agent i terminating the negotiations. The *standard* (*anonymous-disagreement*) domain of Nash (1950) is a "measure-zero" subset of ours where all rows of the disagreement matrix are identical.

Our domain extension significantly increases the richness of admissible rules. Every rule on the Nash domain has counterparts on our domain. We call such rules *decomposable* since they are a composition of a rule from the Nash domain and an *aggregator function* that transforms disagreement matrices to disagreement vectors.[2] But our domain also offers an abundance of rules that are *nondecomposable*.

In Section 3, we first show that the class of decomposable rules is a *nowhere dense* subset of all bargaining rules. That is, the interior of the class of nondecomposable rules is sufficient to approximate any rule (*i.e.* it is *dense*). The class of decomposable rules, however, contains the uncountably many extensions of each rule that has been analyzed in the literature until now. The analysis of this class is thus crucial in understanding the links between the findings of the existing literature and our model. To this end, we analyze the relationship between "good" properties of a decomposable rule and that of its components. We show that, if a decomposable

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2 Our usage of the term "decomposable" is different than the previous literature on the Nash domain (*e.g.* see Thomson and Lensberg, 1989).

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rule is "sufficiently sensitive" to the disagreement payoffs, its *scale invariance* and *symmetry* properties carry on to its *aggregator function*.

In Section 4, we enquire whether the counterparts of some well-known results on the Nash domain continue to hold for decomposable rules on our extended domain. We first show that an extension of the Kalai–Smorodinsky rule uniquely satisfies the Kalai and Smorodinsky (1975) properties. This uniqueness result, however, turns out to be an exception. An infinite number of decomposable rules survive the Nash (1950), Kalai (1977) and Thomson (1981) properties even though, on problems with anonymous disagreement, each of these results characterizes a single rule. Furthermore, we characterize the exact classes of decomposable rules that satisfy these properties.

We conclude this section with a discussion of the related literature. In a companion paper (Kıbrıs and Tapkı, 2010), we focus on the implications of monotonicity properties on our domain. Our main result is a characterization of a class of "monotone path rules" most of which are *nondecomposable*. We also characterize a subclass of monotone path rules that additionally satisfy *scale invariance* as well as a "cardinal egalitarian rule".

The literature also contains some other extensions of the Nash model. Gupta and Livne (1988) analyze bargaining problems with an additional reference point in the feasible set. They interpret it as a past agreement that the agents can refer to when negotiating. Chun and Thomson (1992) analyze an alternative model where the reference point is not feasible (and is interpreted as a vector of "incompatible" claims). Neither of these two papers, however, focuses on disagreement. Livne (1988) and Smorodinsky (2005) analyze cases where the implications of disagreement are uncertain. They thus extend the Nash (1950) model to allow probabilistic disagreement points. Finally, Basu (1996) analyzes cases where disagreement leads to a noncooperative game with multiple equilibria and to model them, he extends the Nash model to allow for a set of disagreement points over which the players do not have probability distributions. It is interesting to note that in all these extension models, the axiomatic analyses reveal rules that are "decomposable".

Chun and Thomson (1990a,b) and Peters and Van Damme (1991) use the standard Nash model but they introduce axioms to represent cases where the agents are not certain about the implications of disagreement. Some other papers that discuss disagreement-related properties but do not extend the Nash model are Dagan et al. (2002), Livne (1986) and Thomson (1987).

The common feature of all of the above papers is that the implications of disagreement are independent of the identity of the agent who causes it. On the other hand, there are noncooperative bargaining models in which agents are allowed to leave and take an outside option. Shaked and Sutton (1984) present one of the first examples. Ponsatí and Sákovics (1998) analyze a model where both agents can leave at each period (but the resulting payoffs are independent of who leaves) and Corominas-Bosch (2000) analyzes a model where the disagreement payoffs depend on who the last agent to reject an offer was (but the agents are not allowed to leave; disagreement is randomly determined by nature). Our model can be seen as providing a cooperative counterpart to these noncooperative models.

2. The model

2.1. Preliminaries

Let $N = \{1, \ldots, n\}$ be the set of agents. For each $i \in N$, let $e_i \in \mathbb{R}^N$ be the vector whose ith coordinate is 1 and for which every other coordinate is 0. Let $\mathbf{1} \in \mathbb{R}^N$ (respectively, $\mathbf{0}$) be the vector whose every coordinate is 1 (respectively, 0). For vectors in $\mathbb{R}^N$, inequalities are defined as: $x \leq y$ if and only if $x_i \leqq y_i$ for each $i \in N$; $x \leq y$ if and only if $x \leqq y$ and $x \neq y$; $x < y$ if and only if $x_i < y_i$ for each $i \in N$. For each $S \subseteq \mathbb{R}^N$, $\mathrm{Int}(S)$ denotes the interior of S and $\mathrm{Cl}(S)$ denotes the closure of S. For each $S \subseteq \mathbb{R}^N$ and $s \in S$, $\mathrm{conv}\{S\}$ is the convex hull of S and s-$\mathrm{comp}\{S\} = \{x \in \mathbb{R}^N \mid s \leqq x \leqq y \text{ for some } y \in S\}$ is the s-comprehensive hull of S. The set S is *s-comprehensive* if s-$\mathrm{comp}\{S\} \subseteq S$.

Let the *Euclidean metric* be defined as $\|x - y\| = \sqrt{\sum (x_i - y_i)^2}$ for $x, y \in \mathbb{R}^N$ and let the *Hausdorff metric* be defined as

$$\mu^H(S^1, S^2) = \max\left\{\max_{x\in S^1}\min_{y\in S^2}\|x-y\|,\ \max_{x\in S^2}\min_{y\in S^1}\|x-y\|\right\}$$

for compact sets $S^1, S^2 \subseteq \mathbb{R}^N$.

Let

$$D = \begin{bmatrix} D_{11} & \cdots & D_{1n} \\ \vdots & \ddots & \vdots \\ D_{n1} & \cdots & D_{nn} \end{bmatrix} = \begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix} \in \mathbb{R}^{N\times N}$$

be a matrix on $\mathbb{R}^N$. The ith row vector $D_i = (D_{i1}, \ldots, D_{in}) \in \mathbb{R}^N$ represents the disagreement payoff profile that arises from agent i terminating the negotiations.

For each $i \in N$, let $\overline{d}_i(D) = \max\{D_{ji} \mid j \in N\}$ be the maximum payoff that agent i can get from disagreement and let $\underline{d}_i(D) = \min\{D_{ji} \mid j \in N\}$ be the minimal payoff. Let $\overline{d}(D) = (\overline{d}_i(D))_{i\in N}$ and $\underline{d}(D) = (\underline{d}_i(D))_{i\in N}$. Let the *metric* μ^Δ *on* $\mathbb{R}^N \times N$ be defined as $\mu^\Delta(D, D') = \max_{i\in N}\left\|D_i - D'_i\right\|$ for $D, D' \in \mathbb{R}^{N\times N}$.

Let Π be the set of all permutations π on N. A function $f : \mathbb{R} \to \mathbb{R}$ is *positive affine* if there is $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $f(x) = ax+b$ for each $x \in \mathbb{R}$; f is a *translation* if it is a positive affine function with $a = 1$. Let Λ be the set of all $\lambda = (\lambda_1, \ldots, \lambda_n)$ where each $\lambda_i : \mathbb{R} \to \mathbb{R}$ is a positive affine function. Let Λ_{trans} be the set of all $\lambda = (\lambda_1, \ldots, \lambda_n)$ where each $\lambda_i : \mathbb{R} \to \mathbb{R}$ is a translation.

For $\pi \in \Pi$, $S \subseteq \mathbb{R}^N$, and $D \in \mathbb{R}^{N\times N}$, let $\pi(S) = \{y \in \mathbb{R}^N : y = (x_{\pi(i)})_{i\in N} \text{ for some } x \in S\}$ and $\pi(D) = (D_{\pi(i)\pi(j)})_{i,j\in N}$. The set S (respectively, the matrix D) is *symmetric* if for every permutation $\pi \in \Pi$, $\pi(S) = S$ (respectively, $\pi(D) = D$). Otherwise, it is *asymmetric*. For $\lambda \in \Lambda$, let $\lambda(S) = \{(\lambda_1(x_1), \ldots, \lambda_n(x_n)) \mid x \in S\}$ and

$$\lambda(D) = \begin{bmatrix} \lambda_1(D_{11}) & \cdots & \lambda_n(D_{1n}) \\ \vdots & \ddots & \vdots \\ \lambda_1(D_{n1}) & \cdots & \lambda_n(D_{nn}) \end{bmatrix} = \begin{bmatrix} \lambda(D_1) \\ \vdots \\ \lambda(D_n) \end{bmatrix} \in \mathbb{R}^{N\times N}.$$

A *(nonanonymous-disagreement bargaining) problem for* N (Fig. 1) is a pair (S, D) where $S \subseteq \mathbb{R}^N$ and $D \in \mathbb{R}^{N\times N}$ satisfy: (i) for each $i \in N$, $D_i \in S$; (ii) S is compact, convex, and $\underline{d}(D)$-comprehensive; (iii) there is $x \in S$ such that $x > \overline{d}(D)$. The feasible set S is made of payoff profiles that the bargainers can obtain only through unanimous agreement whereas D contains n payoff profiles, each resulting from an agent unilaterally terminating the negotiations. Assumptions (i), (ii) and a counterpart of (iii) are standard.[3] They come out of problems where the agents have expected utility functions on a bounded set of lotteries.

Let $\mathcal{B}$ be the class of all problems for agents in N. Let $\mathcal{B}_= = \{(S, D) \in \mathcal{B} \mid D_1 = D_2 = \cdots = D_n\}$ be the subclass of *problems with anonymous disagreement*. Let $\mathcal{B}_{\neq} = \mathcal{B} \setminus \mathcal{B}_=$ be the subclass of *problems with nonanonymous disagreement*. Finally,

[3] When Assumption (iii) is violated, the agents are not guaranteed to reach an agreement. In particular, for each alternative x, there will be an agent who receives higher payoff from someone (including himself or herself) leaving the negotiation table. It will be in the interest of this agent then to follow strategies that induce disagreement rather than to accept x.

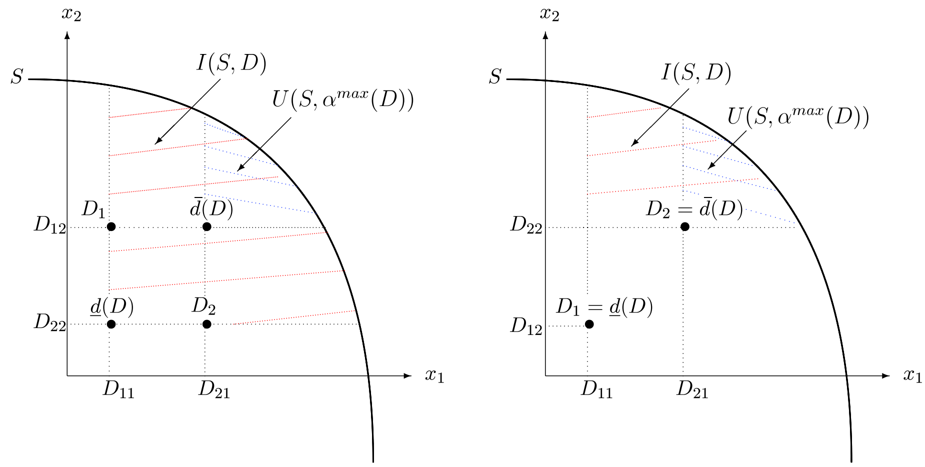

Fig. 1. A typical bargaining problem with nonanonymous disagreement. $I(S,D)$ denotes the individually rational set and $U(S,\alpha^{\max}(D))$ denotes the set of alternatives that all agents prefer to any kind of disagreement.

let $\mathcal{B}^{sc}$ (respectively, $\mathcal{B}^{sc}_{=}$) be the subset of problems (S,D) in $\mathcal{B}$ (respectively, $\mathcal{B}_{=}$) where S is strictly convex. Let the *metric* $\mu^{\mathcal{B}}$ *on* $\mathcal{B}$ be defined as $\mu^{\mathcal{B}}((S,D),(S',D')) = \max\{\mu^H(S,S'),\mu^{\Delta}(D,D')\}$ for $(S,D),(S',D') \in \mathcal{B}$.

Remark 1. We will abuse notation and use (S,d) to represent a problem with anonymous disagreement $(S,D) \in \mathcal{B}_{=}$ where $D_1 = \cdots = D_n = d$.

Two-agent problems with nonanonymous disagreement have an interesting subclass that we will use in some results and examples. This subclass represents cases where the agents disagree on their ranking of the two disagreement alternatives:

$$\mathcal{B}^2_{\gg} = \{(S,D) \in \mathcal{B}_{\neq} \mid |N| = 2 \text{ and there is no } i,j \in N \text{ such that } D_i > D_j\}.$$

The Cyprus negotiations at 2004 fall into this subclass since both parties seem to have preferred the other leaving the table to themselves doing so.

An agreement is Pareto optimal if there is no alternative to it that makes an agent better off without making another agent worse off. It is weakly Pareto optimal if there is no alternative that makes all agents better off. Formally, given $(S,D) \in \mathcal{B}$, the *Pareto optimal set* is $PO(S,D) = \{x \in S \mid y \geq x \text{ implies } y \notin S\}$; the *weakly Pareto optimal set* is $WPO(S,D) = \{x \in S \mid y > x \text{ implies } y \notin S\}$.

Agent i can always terminate the negotiations and receive the payoff D_{ii}. Thus, if an agreement x is such that $x_i < D_{ii}$, it is not individually rational for i to accept x.[4] Since this is true for all $i \in N$, the *individually rational set* is

$$I(S,D) = \{x \in S \mid x_i \geqq D_{ii} \text{ for each } i \in N\}.$$

The notion of *individual rationality* uses the diagonal of the matrix D. It can be generalized by replacing the diagonal with more general choices. For this purpose, let $\alpha : \mathbb{R}^{N\times N} \to \mathbb{R}^N$ be an arbitrary function that maps $D \in \mathbb{R}^{N\times N}$ to a vector $\alpha(D)$ such that $\underline{d}(D) \leqq \alpha(D) \leqq \bar{d}(D)$. For example, the function $\alpha^{\text{diag}}(D) = (D_{ii})_{i\in N}$ always picks the diagonal of D. In Section 3, we will call α an aggregator function and will discuss more examples. Given $(S,D) \in \mathcal{B}$, the *upper contour set of* $\alpha(D)$ *at* S is

$$U(S,\alpha(D)) = \{x \in S \mid x_i \geqq \alpha(D) \text{ for each } i \in N\}.$$

Note that for each $(S,D) \in \mathcal{B}$, $U\left(S,\alpha^{\text{diag}}(D)\right) = I(S,D)$. Given $(S,D) \in \mathcal{B}$ and $i \in N$, the *maximal payoff of* i *with respect to* α is $m_i(S,\alpha(D)) = \max\{x_i \mid x \in U(S,\alpha(D))\}$.

2.2. Bargaining rules

A *(nonanonymous-disagreement bargaining) rule* $F : \mathcal{B} \to \mathbb{R}^N$ is a function that satisfies $F(S,D) \in S$ for each $(S,D) \in \mathcal{B}$. Let $\mathcal{F}$ be the class of all rules. A rule can be interpreted as (i) a *description* of the outcome of the interaction between the bargainers (*positive interpretation*) or (ii) a *prescription* to the bargainers, very much like an arbitrator (*normative interpretation*).

Let the *metric* $\mu^{\mathcal{F}}$ *on* $\mathcal{F}$ be defined as $\mu^{\mathcal{F}}(F,F') = \sup\left\{\frac{\|F(S,D)-F'(S,D)\|}{1+\|F(S,D)-F'(S,D)\|} \mid (S,D) \in \mathcal{B}\right\}$ for $F,F' \in \mathcal{F}$. Given a rule $F \in \mathcal{F}$, a feasible set $S \subseteq \mathbb{R}^N$, and an agreement $x \in S$, the *anonymous inverse set of* F *at* (S,x), which contains all disagreement vectors d that produce x, is defined as

$$F^{-1}_{=}(S,x) = \{d \in \mathbb{R}^N \mid (S,d) \in \mathcal{B}_{=} \text{ and } F(S,d) = x\}.$$

We first introduce some standard properties. The first two are efficiency requirements. A rule F is *Pareto optimal* if for each $(S,D) \in \mathcal{B}$, $F(S,D) \in PO(S,D)$. It is *weakly Pareto optimal* if for each $(S,D) \in \mathcal{B}$, $F(S,D) \in WPO(S,D)$. Next, we introduce a weak anonymity property which requires that in symmetric problems, agents should receive the same payoff. Formally, a bargaining rule F is *symmetric* if for each $(S,D) \in \mathcal{B}$ with *symmetric* S and D, $F(S,D)$ is also *symmetric*, that is, $F_1(S,D) = \cdots = F_n(S,D)$.

The following is a central property in bargaining theory. It requires the physical bargaining outcome to be invariant under utility-representation changes as long as the underlying Von Neumann and Morgenstern (1944) preference information is unchanged. A rule F is *scale invariant* if for each $(S,D) \in \mathcal{B}$ and each $\lambda \in \Lambda$, $F(\lambda(S),\lambda(D)) = \lambda(F(S,D))$. A weaker version of the property only allows affine transformations with a unit multiplicative coefficient: F is *translation invariant* if for each $(S,D) \in \mathcal{B}$ and each $\lambda \in \Lambda_{\text{trans}}$, $F(\lambda(S),\lambda(D)) = \lambda(F(S,D))$.

The next property is introduced by Nash (1950) and requires the solution not to change as some other alternatives cease to be feasible. A rule F is *contraction independent* if for each $(S,D),(S',D') \in \mathcal{B}$, $F(S,D) \in S' \subseteq S$ and $D = D'$ imply $F(S',D') = F(S,D)$.[5] A stronger version is introduced by Thomson (1981): F is *strongly contraction independent* if for all $(S,D),(S',D') \in \mathcal{B}$, $F(S,D) \in S' \subseteq S$ implies $F(S',D') = F(S,D)$.

[4] Note that D_{ii} is special in this sense. Agent i can also possibly receive a D_{ji} via another agent j's disagreement, but does not have the power to induce this.

[5] This property is usually called *independence of irrelevant alternatives* after Nash (1950) who shows that it, together with *weak Pareto optimality*, *symmetry*, and *scale invariance*, characterizes the Nash rule.

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The following "monotonicity" properties are also standard in the literature. They require that an expansion of the set of possible agreements make no agent worse off. A rule F is *strongly monotonic* (Kalai, 1977) if for each $(S, D), (T, D) \in \mathcal{B}, T \subseteq S$ implies $F(T, D) \leqq F(S, D)$.[6] The following is a weaker version introduced by Roth (1979).[7] It has alternative formulations in our domain. The first is the original one that uses individual rationality: a rule F is *restricted monotonic with respect to individual rationality* if for each $(S, D), (T, D) \in \mathcal{B}, T \subseteq S$ and $m(S, \alpha^{\text{diag}}(D)) = m(T, \alpha^{\text{diag}}(D))$ implies $F(T, D) \leqq F(S, D)$. On our domain, this property can be generalized by replacing the diagonal of D, $\alpha^{\text{diag}}(D)$, with an arbitrary function $\alpha : \mathbb{R}^{N\times N} \to \mathbb{R}^N$ satisfying $\underline{d}(D) \leqq \alpha(D) \leqq \overline{d}(D)$. The rule F is *restricted monotonic with respect to* α if for each $(S, D), (T, D) \in \mathcal{B}, T \subseteq S$ and $m(S, \alpha(D)) = m(T, \alpha(D))$ implies $F(T, D) \leqq F(S, D)$.

3. Decomposability

The literature on the Nash (1950) model analyzes rules that are defined on *problems with anonymous disagreement*, $\phi : \mathcal{B}_= \to \mathbb{R}^N$ (hereafter, *anonymous-disagreement rules*). Two well-known examples are the rules of Nash (1950) and Kalai and Smorodinsky (1975).

Anonymous-disagreement rules can be extended to our domain via a function that aggregates the multiple disagreement points $D = (D_i)_{i\in N} \in \mathbb{R}^{N\times N}$ to a single one $d \in \mathbb{R}^N$. A function $\alpha : \mathbb{R}^{N\times N} \to \mathbb{R}^N$ is an *aggregator function* if for each $D \in \mathbb{R}^{N\times N}$, $\underline{d}(D) \leqq \alpha(D) \leqq \overline{d}(D)$. This property guarantees $(S, \alpha(D)) \in \mathcal{B}$ (see Remark 1) and for $d \in \mathbb{R}^N$, $\alpha(d, \ldots, d) = d$.

An *aggregator function* simplifies the informational content of the disagreement matrix by disposing of information. For example, $\alpha^{\text{diag}}(D) = (D_{ii})_{i\in N}$ only retains information about the diagonal of D, that is, about what each agent receives when leaving the negotiation table; $\alpha^{\max}(D) = \overline{d}(D)$ and $\alpha^{\min}(D) = \underline{d}(D)$ only retain information about the maximum and minimum payoffs from disagreement, respectively; $\alpha^{\text{avg}}(D) = \left(\frac{1}{n}\sum_{j\in N} D_{ji}\right)_{i\in N}$, on the other hand, retains information about the average payoff. It is possible to interpret each *aggregator function* as corresponding to a different (noncooperative) interaction among the bargainers. More specifically, an *aggregator function* can be interpreted as choosing from the disagreement matrix those parameters that are relevant for the equilibrium of a noncooperative game that it corresponds to. For example, α^{diag} can be interpreted as corresponding to the equilibrium of an alternating-offers game where the responder has the right to leave the negotiation table (*e.g.* see Section 3.12 in Osborne and Rubinstein (1990)). On the other hand, α^{avg} might correspond to a case where there is uncertainty regarding the resulting disagreement vector.[8] Similarly, $\alpha^{\max}$ and $\alpha^{\min}$ can be interpreted as corresponding to cases where there is ambiguity regarding the resulting disagreement vector and where the agents are either optimistic or pessimistic.

We next introduce two properties of an *aggregator function*. The first one is a weak anonymity property which requires that α should preserve symmetry: the *aggregator function* α is *symmetric* if for each *symmetric* $D \in \mathbb{R}^{N\times N}$, $\alpha(D)$ is also *symmetric*, that is, $\alpha_1(D) = \cdots = \alpha_n(D)$. The second property requires that the physical disagreement outcome that α produces be invariant under utility-representation changes as long as the underlying Von Neumann and Morgenstern (1944) preference information is unchanged: α is *scale invariant* if for each $D \in \mathbb{R}^{N\times N}$ and each $\lambda \in \Lambda$, $\alpha(\lambda(D)) = \lambda(\alpha(D))$.

A rule $F : \mathcal{B} \to \mathbb{R}^N$ is *decomposable* if there is an *anonymous-disagreement rule* $\phi : \mathcal{B}_= \to \mathbb{R}^N$ and an *aggregator function* $\alpha : \mathbb{R}^{N\times N} \to \mathbb{R}^N$ such that $F = \phi \circ \alpha$ (more precisely, for each $(S, D) \in \mathcal{B}, F(S, D) = \phi(S, \alpha(D))$ holds). Otherwise, F is called *nondecomposable*. Let $\mathcal{F}_=$ be the class of *decomposable* rules. Let $\mathcal{F}_{\neq} = \mathcal{F} \setminus \mathcal{F}_=$ be the class of *nondecomposable* rules.

Decomposable rules extend an *disagreement rule* ϕ to the whole domain $\mathcal{B}$ in a consistent manner: the α*-extension of* ϕ is defined as $(\phi \circ \alpha)(S, D) = \phi(S, \alpha(D))$. The following is a simple but useful observation on *decomposable* rules (see the Appendix for its proof). Let $F|_{\mathcal{B}_=}$ denote the restriction of F to the subdomain $\mathcal{B}_=$.

Lemma 1. *If F is* decomposable *with $F = \phi \circ \alpha$, then $\phi = F|_{\mathcal{B}_=}$.*

We first analyze the extent to which the imposition of *decomposability* restricts the class of admissible rules. Since *decomposable rules* are intimately linked to the Nash domain, this will give us an idea on how big a class of rules was excluded by the anonymous-disagreement assumption of Nash. To this end, consider the following rather technical property. Fix any disagreement matrix, $D \in \mathbb{R}^{N\times N}$. Then for each feasible set $S \subseteq \mathbb{R}^N$, consider the set of anonymous-disagreement vectors $d \in \mathbb{R}^N$ for which $F(S, d) = F(S, D)$ (note that the set of such d is the anonymous inverse set $F_=^{-1}(S, F(S, D))$). Let the correspondence $\delta^F : \mathbb{R}^{N\times N} \Longrightarrow \mathbb{R}^N$ be defined as follows: for each $D \in \mathbb{R}^{N\times N}$,

$$\delta^F(D) = \bigcap_{\substack{S\subseteq\mathbb{R}^N \text{s.t.} \\ (S,D)\in\mathcal{B}}} F_=^{-1}(S, F(S, D)).$$

That is, $\delta^F(D)$ is the set of all d that is contained in every $F_=^{-1}(S, F(S, D))$, independently of S. Note that $\underline{d}(D) \leqq d \leqq \overline{d}(D)$ holds for all $d \in \delta^F(D)$. A rule F is *disagreement-simple* if the correspondence δ^F is nonempty-valued. In other words, F is *disagreement-simple* if for each $D \in \mathbb{R}^{N\times N}$ there is $d \in \mathbb{R}^N$ such that $F(., d) = F(., D)$. We thus have the following lemma, which relates *decomposability* and *disagreement simplicity* (see the Appendix for its proof).

Lemma 2. *A rule is* decomposable *if and only if it is* disagreement-simple.

Note that *disagreement simplicity* is a very demanding property. Therefore, bargaining rules on our extended domain are mostly *nondecomposable*. We next present a result that supports this intuition. We show that the class of *decomposable rules* $\mathcal{F}_=$ is *nowhere dense* in $\mathcal{F}$, that is $\text{Int}(\text{Cl}(\mathcal{F}_=)) = \emptyset$. The class $\mathcal{F}_=$ is *nowhere dense* in $\mathcal{F}$ if and only if the interior of its complement, $\text{Int}(\mathcal{F}_{\neq})$, is *dense in* $\mathcal{F}$, that is $\text{Cl}(\text{Int}(\mathcal{F}_{\neq})) = \mathcal{F}$ (see Sutherland, 2002, pp. 63–64). This means that the interior of the class of *nondecomposable rules*, $\mathcal{F}_{\neq}$ (which by definition is an open set), is so big that it can be used to approximate any rule.

Theorem 1. *The class of* decomposable rules $\mathcal{F}_=$ *is* nowhere dense *in $\mathcal{F}$.*[9]

Proof. We first show that any *decomposable rule* can be approximated by a *nondecomposable rule*.

[6] Note that *Pareto optimal* and *strongly monotonic* rules also satisfy *contraction independence*.

[7] Roth introduces this property as a weakening of an "individual monotonicity" property by Kalai and Smorodinsky (1975). Both properties, together with *weak Pareto optimality*, *symmetry*, and *scale invariance*, characterize the Kalai–Smorodinsky rule.

[8] This, for example, happens when the agents' disagreement payoffs depend on the public opinion (as in the Cyprus case) and due to informational imperfections, it is uncertain who the public will blame for disagreement.

[9] This statement is stronger than the class of *nondecomposable rules* $\mathcal{F}_{\neq}$ being *dense* in $\mathcal{F}$. For example, the set of rational numbers $\mathbb{Q}$ is *dense* in $\mathbb{R}$ but its complement is not *nowhere dense*.

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Step 1. For each $F \in \mathcal{F}_=$ and $\varepsilon > 0$, there is $F^\varepsilon \in \mathcal{F}_{\neq}$ such that $\mu^{\mathcal{F}}(F, F^\varepsilon) < \varepsilon$.

Fix some $(S^*, D^*) \in \mathcal{B}_{\neq}$ and suppose that $x^* = F(S^*, D^*)$. Without loss of generality assume that $\varepsilon < 1$ and suppose that $0 < \delta < \frac{\varepsilon}{2(1-\varepsilon)}$. Let $x^\varepsilon \in S^* \setminus \{x^*\}$ be such that $\|x^* - x^\varepsilon\| < \delta$. Let

$$F^\varepsilon(S, D) = \begin{cases} x^\varepsilon & \text{if } S = S^* \text{ and } D \in F_=^{-1}(S^*, x) \\ & \text{for some } x \in S^* \text{ such that } \left\|x^* - x\right\| < \delta, \\ F(S, D) & \text{otherwise.} \end{cases}$$

Since $F \in \mathcal{F}_=$, $\delta^F(D^*) \neq \emptyset$ and thus, $F_=^{-1}(S^*, x^*) \neq \emptyset$. This implies that $F^\varepsilon \neq F$. Note that for $x \in S^*$ such that $\|x^* - x\| < \delta$, $\|x^* - x^\varepsilon\| < \delta$ implies $\|x - x^\varepsilon\| < 2\delta$. Thus for each $(S, D) \in \mathcal{B}$, $\|F(S, D) - F^\varepsilon(S, D)\| < 2\delta$. Therefore

$$\mu^{\mathcal{F}}(F, F^\varepsilon) \leq \frac{2\delta}{1 + 2\delta} < \varepsilon.$$

Finally, to see that F^ε is *nondecomposable*, note that for $x \in S^* \setminus \{x^\varepsilon\}$ such that $\|x^* - x\| < \delta$, we have $(F^\varepsilon)_=^{-1}(S^*, x) = \emptyset$. In particular,

$$(F^\varepsilon)_=^{-1}(S^*, x^*) = (F^\varepsilon)_=^{-1}(S^*, F^\varepsilon(S^*, D^*)) = \emptyset.$$

Thus, $\delta^{F^\varepsilon}(D^*) = \bigcap_{\substack{S \subseteq \mathbb{R}^N \text{s.t.} \\ (S, D^*) \in \mathcal{B}}} (F^\varepsilon)_=^{-1}(S, F^\varepsilon(S, D^*)) \subseteq (F^\varepsilon)_=^{-1}(S^*, F^\varepsilon(S^*, D^*)) = \emptyset$. Thus F^ε violates *disagreement simplicity* and by Lemma 2, is *nondecomposable*.

We will next show that $F^\varepsilon \in \text{Int}(\mathcal{F}_{\neq})$.

Step 2. There is $\gamma > 0$ such that for all $F^\gamma \in \mathcal{F}$ satisfying $\mu^{\mathcal{F}}(F^\varepsilon, F^\gamma) < \gamma$, we have $F^\gamma \in \mathcal{F}_{\neq}$.

Suppose that $\gamma = \frac{\|x^* - x^\varepsilon\|}{2 + \|x^* - x^\varepsilon\|}$ and $\eta = \frac{\gamma}{1-\gamma}$. Since $\mu^{\mathcal{F}}(F^\varepsilon, F^\gamma) < \gamma$, $\|F^\varepsilon(S, D) - F^\gamma(S, D)\| < \eta$ for each $(S, D) \in \mathcal{B}$. In particular,

$$\left\|F^\varepsilon(S^*, D^*) - F^\gamma(S^*, D^*)\right\| = \left\|x^* - F^\gamma(S^*, D^*)\right\| < \eta.$$

Now let $d \in \mathbb{R}^N$ be such that $(S^*, d) \in \mathcal{B}_=$. Since either $F^\varepsilon(S^*, d) = x^\varepsilon$ and, thus, $\|x^* - F^\varepsilon(S^*, d)\| = 2\eta$ or $\|x^* - F^\varepsilon(S^*, d)\| \geq \delta > 2\eta$, we have $\|x^* - F^\varepsilon(S^*, d)\| \geq 2\eta$. By the triangle inequality,

$$\begin{aligned} 2\eta &\leq \left\|x^* - F^\varepsilon(S^*, d)\right\| \\ &\leq \left\|x^* - F^\gamma(S^*, d)\right\| + \left\|F^\gamma(S^*, d) - F^\varepsilon(S^*, d)\right\|. \end{aligned}$$

But $\mu^{\mathcal{F}}(F^\gamma, F^\varepsilon) < \gamma$ implies $\|F^\gamma(S^*, d) - F^\varepsilon(S^*, d)\| < \eta$. Therefore,

$$\left\|x^* - F^\gamma(S^*, d)\right\| > \eta.$$

Combining the two displayed inequalities, we obtain $F^\gamma(S^*, D^*) \neq F^\gamma(S^*, d)$ for any $d \in \mathbb{R}^N$ such that $(S^*, d) \in \mathcal{B}_=$. Therefore, $\delta^{F^\gamma}(D^*) \subseteq (F^\gamma)_=^{-1}(S^*, F^\gamma(S^*, D^*)) = \emptyset$. Thus F^γ violates *disagreement simplicity* and by Lemma 2, is *nondecomposable*. This establishes that $F^\varepsilon \in \text{Int}(\mathcal{F}_{\neq})$.

Finally, we will show that any rule at the closure of $\mathcal{F}_=$ can be approximated by rules from $\text{Int}(\mathcal{F}_{\neq})$.

Step 3. For each $F \in \mathcal{F}_{\neq} \setminus \text{Int}(\mathcal{F}_{\neq})$ and $\varepsilon > 0$ there is $F' \in \text{Int}(\mathcal{F}_{\neq})$ such that $\mu^{\mathcal{F}}(F, F') < \varepsilon$.

Since $F \notin \text{Int}(\mathcal{F}_{\neq})$, there is $F'' \in \mathcal{F}_=$ such that $\mu^{\mathcal{F}}(F, F'') < \frac{\varepsilon}{2}$. Now since $F'' \in \mathcal{F}_=$, by steps 1 and 2, there is $F' \in \text{Int}(\mathcal{F}_{\neq})$ such that $\mu^{\mathcal{F}}(F', F'') < \frac{\varepsilon}{2}$. But then, by the triangle inequality, $\mu^{\mathcal{F}}(F, F') < \varepsilon$ proves the claim. □

While the class of *nondecomposable rules* contains an open and dense subset of $\mathcal{F}$, it is not open itself. Example 3 in the Appendix constructs a *nondecomposable rule* whose every neighborhood contains a *decomposable rule*.

Note that the relationship between problems with anonymous disagreement, $\mathcal{B}_=$, and problems with nonanonymous disagreement, $\mathcal{B}_{\neq}$, is quite similar to that between $\mathcal{F}_=$ and $\mathcal{F}_{\neq}$.

Remark 2. The class $\mathcal{B}_=$ is *nowhere dense* in $\mathcal{B}$.

While *decomposable rules* constitute a nowhere dense subset of all nonanonymous-disagreement rules, they are very central. In other extensions of the Nash model as well (such as Gupta and Livne, 1988; Livne, 1988; Chun and Thomson, 1992; Basu, 1996 and Smorodinsky, 2005), the axiomatic analyses always reveal rules that are decomposable into some "aggregator function" and a bargaining rule from the Nash domain. In what follows, we thus focus on such rules and, particularly, on the relationship between the properties satisfied by a *decomposable rule* $F = \phi \circ \alpha$, and its components.

Since $\phi = F|_{\mathcal{B}_=}$, any property satisfied by F passes on to ϕ. This is not the case with the *aggregator function* α. We however identify the exact conditions under which α retains the properties of F. We focus on *scale invariance* and *symmetry*. Other properties of F, such as *Pareto optimality, contraction independence*, and "monotonicity" are not related to changes in the disagreement point and thus do not have implications for the *aggregator function* α.

Scale invariance

If an *anonymous-disagreement rule* ϕ and an *aggregator function* α are both *scale invariant*, their composition $F = \phi \circ \alpha$ also satisfies the property. However, not all *scale invariant* F are created this way. As demonstrated next, composition of a *scale invariant* ϕ with a *non-scale invariant* α might also create a *scale invariant* F.

Example 1 (*A Scale Invariant $F = \phi \circ \alpha$ where α is Not Scale Invariant*). Let $D^* \in \mathbb{R}^{N \times N}$ be such that $\bar{d}(D^*) \neq \underline{d}(D^*)$. Then define ϕ and α as $\phi(S, d) = \arg\max_{x \in S} x_1$ *and*

$$\alpha(D) = \begin{cases} \bar{d}(D^*) & \text{if } D = D^*, \\ \underline{d}(D) & \text{otherwise.} \end{cases}$$

If, however, an *anonymous-disagreement rule* ϕ is sufficiently "sensitive" to changes in the disagreement point, its only *scale invariant* extensions are those that are obtained by compositions with *scale invariant* aggregator functions α. More surprisingly, the converse implication also holds. Formally, an *anonymous-disagreement rule* ϕ is *disagreement sensitive* if for each $d, d' \in \mathbb{R}^N$ such that $d \neq d'$, there is $S \subseteq \mathbb{R}^N$ such that $(S, d), (S, d') \in \mathcal{B}_=$ and $\phi(S, d) \neq \phi(S, d')$ (that is, any two distinct disagreement vectors can be combined with a feasible set so that the resulting two problems have distinct outcomes).[10] Note that both the Nash (1950) and the Kalai and Smorodinsky (1975) rules satisfy this property. For such rules, we have the following result.

Theorem 2. *An* anonymous-disagreement rule ϕ *is* disagreement sensitive *if and only if for any* scale invariant *and* decomposable rule $F = \phi \circ \alpha$, *the* aggregator function α *is also* scaleinvariant.

Proof. First, let ϕ be *disagreement sensitive* and let α be any *aggregator function*. Assume that $F = \phi \circ \alpha$ is *scale invariant*. Then, by Lemma 1, ϕ is *scale invariant*. Now suppose that α is not *scale invariant*. Then, there are $D \in \mathbb{R}^{N \times N}$ and $\lambda \in \Lambda$ such that $\lambda(\alpha(D)) \neq \alpha(\lambda(D))$. Since ϕ is *disagreement sensitive*, there is $S \subseteq \mathbb{R}^N$ such that $\phi(S, \lambda(\alpha(D))) \neq \phi(S, \alpha(\lambda(D)))$.

10 A similar (logically stronger) property can be imposed on nonanonymous-disagreement rules: F is *strongly disagreement sensitive* if for each $D, D' \in \mathbb{R}^{N \times N}$ such that $D \neq D'$, there is $S \subseteq \mathbb{R}^N$ such that $(S, D), (S, D') \in \mathcal{B}$ and $F(S, D) \neq F(S, D')$. However, there is no *decomposable* rule $F = \phi \circ \alpha$ that satisfies *strong disagreement sensitivity*. To see this, suppose for a contradiction that F is *strongly disagreement sensitive*. Let $D \in \mathbb{R}^{N \times N}$ be such that $D_i \neq D_j$ for some $i, j \in N$. Then for each $S \subseteq \mathbb{R}^N$ such that $(S, D) \in \mathcal{B}_{\neq}$, $F(S, D) = \phi(S, \alpha(D)) = F(S, \alpha(D), \ldots, \alpha(D))$. Since $D \neq (\alpha(D), \ldots, \alpha(D))$, by *strong disagreement sensitivity*, there is $S^* \subseteq \mathbb{R}^N$ such that $F(S^*, D) \neq F(S^*, \alpha(D), \ldots, \alpha(D))$. But then $F(S^*, D) \neq \phi(S^*, \alpha(D))$, a contradiction.

Let $T = \lambda^{-1}(S)$. Then, $\phi(\lambda(T), \lambda(\alpha(D))) \neq \phi(\lambda(T), \alpha(\lambda(D)))$. Since ϕ is *scale invariant* and $F = \phi \circ \alpha$, $\phi(\lambda(T), \lambda(\alpha(D))) = \lambda(\phi(T, \alpha(D))) = \lambda(F(T, D))$. Also, $F = \phi \circ \alpha$ implies $\phi(\lambda(T), \alpha(\lambda(D))) = F(\lambda(T), \lambda(D))$. But, then $\lambda(F(T, D)) \neq F(\lambda(T), \lambda(D))$, contradicting the *scale invariance* of F.

For the opposite direction, suppose that ϕ is *scale invariant* but not *disagreement sensitive*. We want to show that there is an *aggregator function* α that is not *scale invariant*, but $F = \phi \circ \alpha$ is *scale invariant*. Since ϕ is not *disagreement sensitive*, there exist $d', d'' \in \mathbb{R}^N$ such that $d' \neq d''$ and $\phi(., d') = \phi(., d'')$. Let $D^* \in \mathbb{R}^{N\times N}$ be such that $d', d'' \in \{x \in \mathbb{R}^N | \underline{d}(D^*) \leqq x \leqq \bar{d}(D^*)\}$. Then, define $\alpha : \mathbb{R}^{N\times N} \to \mathbb{R}^N$ as follows:

$$\alpha(D) = \begin{cases} d' & \text{if } D = D^*, \\ \lambda(d'') & \text{if } D = \lambda(D^*) \neq D^* \text{ for some } \lambda \in \Lambda, \\ \underline{d}(D) & \text{otherwise.} \end{cases}$$

Note that for any $\lambda \in \Lambda$ other than the identity function, $\lambda(\alpha(D^*)) = \lambda(d') \neq \lambda(d'') = \alpha(\lambda(D^*))$. Thus, α is not *scale invariant*. To show that $F = \phi \circ \alpha$ is *scale invariant*, suppose that $\lambda \in \Lambda$ and $(S, D) \in \mathcal{B}$. We want to show that $\lambda(F(S, D)) = F(\lambda(S), \lambda(D))$, that is $\lambda(\phi(S, \alpha(D))) = \phi(\lambda(S), \alpha(\lambda(D)))$. If λ is the identity function, the statement trivially holds. Thus, suppose that λ is different than the identity function.

Case 1. Suppose that $D = D^*$. Then, $\lambda(\phi(S, \alpha(D^*))) = \lambda(\phi(S, d'))$. Also, $\phi(\lambda(S), \alpha(\lambda(D^*))) = \phi(\lambda(S), \lambda(d''))$. Since ϕ is *scale invariant*, $\phi(\lambda(S), \lambda(d'')) = \lambda(\phi(S, d''))$. Also since $\phi(., d') = \phi(., d'')$, $\lambda(\phi(S, d'')) = \lambda(\phi(S, d'))$. Combining these equalities, $\lambda(\phi(S, \alpha(D^*))) = \phi(\lambda(S), \alpha(\lambda(D^*)))$.

Case 2. Suppose that $D = \lambda^*(D^*)$ for some $\lambda^* \in \Lambda$ such that $\lambda^*(D^*) \neq D^*$. Then, $\lambda(\phi(S, \alpha(D))) = \lambda(\phi(S, \lambda^*(d'')))$. First assume that $\lambda \neq (\lambda^*)^{-1}$. Then, $\phi(\lambda(S), \alpha(\lambda(D))) = \phi(\lambda(S), \alpha(\lambda(\lambda^*(D^*)))) = \phi(\lambda(S), \lambda(\lambda^*(d'')))$. Since ϕ is *scale invariant*, $\phi(\lambda(S), \lambda(\lambda^*(d''))) = \lambda(\phi(S, \lambda^*(d'')))$. Thus, $\lambda(\phi(S, \alpha(D))) = \phi(\lambda(S), \alpha(\lambda(D)))$. Secondly, assume that $\lambda = (\lambda^*)^{-1}$. Then, since ϕ is *scale invariant*, $\lambda(\phi(S, \alpha(D))) = \lambda(\phi(S, \lambda^*(d''))) = \phi(\lambda(S), \lambda(\lambda^*(d''))) = \phi(\lambda(S), d'')$. Also, $\phi(\lambda(S), \alpha(\lambda(D))) = \phi(\lambda(S), \alpha(\lambda(\lambda^*(D^*)))) = \phi(\lambda(S), \alpha(D^*)) = \phi(\lambda(S), d')$. But since $\phi(., d') = \phi(., d'')$, these imply that $\lambda(\phi(S, \alpha(D))) = \phi(\lambda(S), \alpha(\lambda(D)))$.

Case 3. Suppose that $D \neq \lambda^*(D^*)$ for any $\lambda^* \in \Lambda$. Then, $\lambda(D) \neq \lambda^*(D^*)$ for any $\lambda^* \in \Lambda$. Now first note that $\lambda(\phi(S, \alpha(D))) = \lambda(\phi(S, \underline{d}(D)))$. Next, $\phi(\lambda(S), \alpha(\lambda(D))) = \phi(\lambda(S), \underline{d}(\lambda(D)))$. Since ϕ and $\underline{d}(.)$ are both *scale invariant*, $\phi(\lambda(S), \underline{d}(\lambda(D))) = \lambda(\phi(S, \underline{d}(D)))$. Therefore, $\lambda(\phi(S, \alpha(D))) = \phi(\lambda(S), \alpha(\lambda(D)))$. □

Symmetry

If an *anonymous-disagreement rule* ϕ and an *aggregator function* α are both *symmetric*, their composition $F = \phi \circ \alpha$ also satisfies the property. However, not all *symmetric* F are created this way. As demonstrated next, composition of a *symmetric* ϕ with a *non-symmetric* α might also create a *symmetric* F, even when ϕ satisfies *disagreement sensitivity*.

Example 2 (*A* Symmetric $F = \phi \circ \alpha$ *where* α *is Not* Symmetric, *Even Though* ϕ *is* Disagreement Sensitive**)**. Suppose that $N = \{1, 2\}$. Let $D^* \in \mathbb{R}^{N\times N}$ be a symmetric matrix and let $S^* \subseteq \mathbb{R}^N$ be a symmetric set such that $(S^*, D^*) \in B_{\neq}$. Reminding the reader that γ stands for the egalitarian rule of Kalai (1977), we define ϕ and α as follows:

$$\alpha(D) = \begin{cases} D_1^* & \text{if } D = D^*, \\ \bar{d}(D) & \text{otherwise.} \end{cases}$$

Note that since $(S^*, D^*) \in B_{\neq}$, the vector D_1^* is not symmetric.

$$\phi(S, d) = \begin{cases} \gamma(S^*, \bar{d}(D^*)) & \text{if } S = S^* \text{ and } d = D_1^*, \\ d & \text{otherwise.} \end{cases}$$

If, however, an *anonymous-disagreement rule* ϕ is sufficiently "sensitive to asymmetry", its only *symmetric* extensions are those that are obtained by compositions with *symmetric aggregator functions* α. As was the case previously, the converse of this statement is also true. Formally, an *anonymous-disagreement rule* ϕ is *sensitive to asymmetry* if for each asymmetric $d \in \mathbb{R}^N$ and for each $\varepsilon \in \mathbb{R}_{++}$, there is a symmetric $S \subseteq \mathbb{R}^N$ such that (i) $N_\varepsilon(d) \subseteq S$ and (ii) $\phi(S, d)$ is asymmetric (that is, for any asymmetric disagreement vector, there are large enough feasible sets so that the resulting problems have asymmetric outcomes). Note that *symmetric anonymous-disagreement rules* such as the Nash (1950) and Kalai and Smorodinsky (1975), or the egalitarian (Kalai, 1977) rules all satisfy this property. On the other hand, the utilitarian rule (Thomson, 1981) does not. For such rules, we have the following result.

Theorem 3. *An* anonymous-disagreement rule ϕ *is* sensitive to asymmetry *if and only if for any* symmetric *and* decomposable *rule* $F = \phi \circ \alpha$, *the* aggregator function α *is also* symmetric.

Proof. First, let ϕ be *sensitive to asymmetry* and let α be any *aggregator function*. Assume that $F = \phi \circ \alpha$ is *symmetric*. Then, by Lemma 1, ϕ is *symmetric*. Now, suppose that α is not *symmetric*. Then, there is a symmetric $D \in \mathbb{R}^{N\times N}$ and $\pi \in \Pi$ such that $\pi(\alpha(D)) \neq \alpha(D)$. Let $\varepsilon \in \mathbb{R}_{++}$ be such that $D \in N_\varepsilon(\alpha(D))$. Then by *sensitivity to asymmetry*, there is a symmetric $S \subseteq \mathbb{R}^N$ such that (i) $N_\varepsilon(\alpha(D)) \subseteq S$ and (ii) $\phi(S, \alpha(D))$ is asymmetric. But then, $F(S, D)$ is asymmetric, while (S, D) is symmetric, a contradiction.

For the opposite direction, suppose that ϕ is *symmetric* but not *sensitive to asymmetry*. We want to show that there is an asymmetric *aggregator function* α such that $F = \phi \circ \alpha$ is *symmetric*. Since ϕ is not *sensitive to asymmetry*, there is an asymmetric $d^* \in \mathbb{R}^N$ and an $\varepsilon^* \in \mathbb{R}_{++}$ such that for any symmetric $S \subseteq \mathbb{R}^N$ with $N_{\varepsilon^*}(d^*) \subseteq S$, $\phi(S, d^*)$ is symmetric. Define $\alpha : \mathbb{R}^{N\times N} \to \mathbb{R}^N$ as follows:

$$\alpha(D) = \begin{cases} d^* & \text{if } N_{\varepsilon^*}(d^*) \subseteq \{x \in \mathbb{R}^N : \underline{d}(D) \leqq x \leqq \bar{d}(D)\}, \\ \underline{d}(D) & \text{otherwise.} \end{cases}$$

Now, let $D \in \mathbb{R}^{N\times N}$ be symmetric such that $N_{\varepsilon^*}(d^*) \subseteq \{x \in \mathbb{R}^N : \underline{d}(D) \leqq x \leqq \bar{d}(D)\}$. Then, by the definition of α, $\alpha(D) = d^*$. Since d^* is asymmetric, α is not *symmetric*. To see that $F = \phi \circ \alpha$ is *symmetric*, let $(S, D) \in \mathcal{B}$ be symmetric. If $N_{\varepsilon^*}(d^*) \subseteq \{x \in \mathbb{R}^N : \underline{d}(D) \leqq x \leqq \bar{d}(D)\}$, $F(S, D) = \phi(S, \alpha(D)) = \phi(S, d^*)$. Since $\phi(S, d^*)$ is symmetric, $F(S, D)$ is symmetric. Alternatively if $N_{\varepsilon^*}(d^*) \nsubseteq \{x \in \mathbb{R}^N : \underline{d}(D) \leqq x \leqq \bar{d}(D)\}$, $F(S, D) = \phi(S, \alpha(D)) = \phi(S, \underline{d}(D))$. Since $(S, \underline{d}(D))$ and ϕ are *symmetric*, $F(S, D)$ is symmetric. □

4. Extensions of the main bargaining rules

Kalai and Smorodinsky (1975) introduce the following *anonymous-disagreement rule* which picks the maximal feasible agreement at which the agents' payoff gains from disagreement are proportional to their maximal payoff gains: for each $(S, d) \in \mathcal{B}_=$,

$$\kappa(S, d) = \arg\max_{x\in S} \min_{i\in N} \frac{x_i - d_i}{m_i(S, d) - d_i}.$$

They show that, on the Nash domain, this rule uniquely satisfies *weak Pareto optimality, symmetry, scale invariance*, and "restricted monotonicity" (also see Roth, 1979 and Thomson, 1996).[11,12] We

[11] On anonymous disagreement problems, $\mathcal{B}_=$, the two restricted monotonicity properties of our paper coincide with that of Roth (1979).

[12] The results of Kalai and Smorodinsky (1975) and Roth (1979) are for two-agent problems. Thomson (1996) however notes that this statement holds for an arbitrary number of agents.

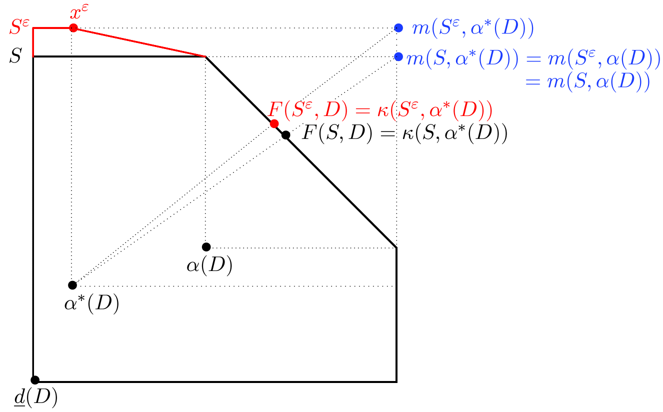

Fig. 2. Case 1 in the proof of Theorem 4.

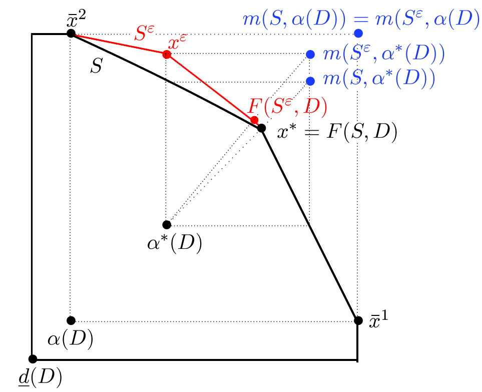

Fig. 3. Case 2 in the proof of Theorem 4.

next show that a similar result prevails among *decomposable* rules on our domain.[13]

Theorem 4. *For each* aggregator function α, *the* α*-extension of the Kalai–Smorodinsky rule* $(\kappa \circ \alpha)$ *uniquely satisfies* weak Pareto optimality, symmetry, scaleinvariance, restricted monotonicity with respect to α, *and* decomposability.

Proof. It is straightforward to show that $\kappa \circ \alpha$ satisfies the given properties. Conversely, let $F = \phi \circ \alpha^*$ be a *decomposable rule* (on $\mathcal{B}$) that satisfies *weak Pareto optimality, symmetry, scale invariance*, and *restricted monotonicity with respect to* α. Then F also satisfies them on $\mathcal{B}_=$. By Lemma 1 then, ϕ satisfies the given properties. Therefore by Kalai and Smorodinsky (1975), Roth (1979) and Thomson (1996), $\phi = \kappa$. Now suppose that $F \neq \kappa \circ \alpha$. Since $F = \kappa \circ \alpha^*$, we have $\alpha^* \neq \alpha$. Let $D \in \mathbb{R}^{N\times N}$ be such that $\alpha^*(D) \neq \alpha(D)$.

Case 1: (Fig. 2) Suppose that $\alpha^*(D) \ngeq \alpha(D)$. Let $k \in N$. Define $N' \subseteq N$ as $N' = \{i \in N : \alpha_i^*(D) \geqq \alpha_i(D)\}$. If $N' \neq \emptyset$, then let $k \in N'$. Let $t \in \mathbb{R}_{++}$ be such that $t > \sum_N (\bar{d}_i(D) - \alpha_i(D))$. Then define $S \subseteq \mathbb{R}^N$ as $S = \underline{d}(D)$-comp$\{$conv$\{\alpha(D) + te_i \mid i \in N\}\}$. Now for $\varepsilon \in \mathbb{R}_{++}$, let $x^\varepsilon \in \mathbb{R}^N$ be such that $x_k^\varepsilon = \alpha_k(D) + t + \varepsilon$ and for $i \in N \setminus \{k\}$, $x_i^\varepsilon = \alpha_i^*(D)$. Let $S^\varepsilon \subseteq \mathbb{R}^N$ be defined as $S^\varepsilon = \underline{d}(D)$-comp$\{$conv$\{x^\varepsilon, S\}\}$ and assume that $\varepsilon > 0$ is sufficiently small so that $PO(S) \subseteq PO(S^\varepsilon)$.

Now note that $m_k(S^\varepsilon, \alpha^*(D)) = m_k(S, \alpha^*(D)) + \varepsilon$ and for $i \in N \setminus \{k\}$, $m_i(S^\varepsilon, \alpha^*(D)) = m_i(S, \alpha^*(D))$. Thus by the definition of κ, $F_k(S^\varepsilon, D) = \kappa_k(S^\varepsilon, \alpha^*(D)) > \kappa_k(S, \alpha^*(D)) = F_k(S, D)$ and for $i \in N \setminus \{k\} F_i(S^\varepsilon, D) = \kappa_i(S^\varepsilon, \alpha^*(D)) < \kappa_i(S, \alpha^*(D)) = F_i(S, D)$. However, $S^\varepsilon \supset S$ and $m(S^\varepsilon, \alpha(D)) = m(S, \alpha(D))$, by *restricted monotonicity with respect to individual rationality*, imply $F(S^\varepsilon, D) \geqq F(S, D)$, a contradiction.

Case 2: (Fig. 3) Suppose that $\alpha^*(D) \geq \alpha(D)$. Let $k \in N$. Define $N' \subseteq N$ as $N' = \{i \in N : \alpha_i^*(D) \leqq \alpha_i(D)\}$. If $N' \neq \emptyset$, then suppose that $k \in N'$. Let $t, \tau \in \mathbb{R}_{++}$ be such that $t > \sum_N (\bar{d}_i(D) - \alpha_i^*(D))$ and $\tau < \frac{t}{n}$. Also, let $x^* = \alpha^*(D) + \left(\frac{t}{n} + \tau\right)\mathbf{1}$. Let $L_i = \{\alpha(D) + re_i \mid r \in \mathbb{R}\}$ be the line passing through $\alpha(D)$ in the direction of e_i. Let L_i' be the line passing through x^* and $\alpha^*(D) + te_i$. Since L_i and L_i' are not parallel, let $\bar{x}^i = L_i \cap L_i'$ and define $S \subseteq \mathbb{R}^N$ as $S = \underline{d}(D)$-comp$\{$conv$\{x^*, \bar{x}^1, \ldots, \bar{x}^n\}\}$. Now for $\varepsilon \in \mathbb{R}_{++}$, let $x^\varepsilon = \alpha^*(D) + (t + \varepsilon)e_k$. Let $S^\varepsilon \subseteq \mathbb{R}^N$ be defined as $S^\varepsilon = \underline{d}(D)$-comp$\{$conv$\{x^\varepsilon, S\}\}$ and assume that $\varepsilon > 0$ is sufficiently small so that $m_k(S^\varepsilon, \alpha(D)) = m_k(S, \alpha(D))$. Now note that $m_k(S^\varepsilon, \alpha^*(D)) = m_k(S, \alpha^*(D)) + \varepsilon$ and for $i \in N \setminus \{k\}$, $m_i(S^\varepsilon, \alpha^*(D)) = m_i(S, \alpha^*(D))$. Thus by definition of κ, $F_k(S^\varepsilon, D) = \kappa_k(S^\varepsilon, \alpha^*(D)) > \kappa_k(S, \alpha^*(D)) = F_k(S^\varepsilon, D)$ and for $i \in N \setminus \{k\} F_i(S^\varepsilon, D) = \kappa_i(S^\varepsilon, \alpha^*(D)) < \kappa_i(S, \alpha^*(D)) = F_i(S^\varepsilon, D)$. However, $S^\varepsilon \supset S$ and $m(S^\varepsilon, \alpha(D)) = m(S, \alpha(D))$, by *restricted monotonicity with respect to individual rationality*, imply $F(S^\varepsilon, D) \geqq F(S, D)$, a contradiction. □

It follows from Theorem 4 that the α^{diag}*-extension of* κ uniquely satisfies *weak Pareto optimality, symmetry, scale invariance, restricted monotonicity with respect to individual rationality*, and *decomposability*.

The uniqueness result of Theorem 4 turns out to be an exception. An infinite number of *decomposable rules* survive the Nash (1950), Kalai (1977) and Thomson (1981) properties even though, on problems with anonymous disagreement, each of these results characterizes a single rule. We next characterize the exact classes of *decomposable rules* that satisfy these properties.

Nash (1950) introduces the following *anonymous-disagreement rule* which maximizes the product of the agents' payoff gains from disagreement: for each $(S, d) \in \mathcal{B}_=$,

$$\nu(S, d) = \arg\max_{x \in S} \prod_{i \in N} (x_i - d_i).$$

He shows that, on the Nash domain, this rule uniquely satisfies *weak Pareto optimality, symmetry, scale invariance*, and *contraction independence*. On our domain, these properties characterize the following class of *decomposable* rules (see the Appendix for the proof).[14]

Proposition 3 (*Corollary to Nash, 1950*)**.** *A* decomposable *rule* F *on* $\mathcal{B}$ *satisfies* weak Pareto optimality, symmetry, scale invariance, *and* contraction independence *if and only if* F *is decomposable into the Nash rule* ν *and a* symmetric *and* scale invariant aggregator function α*; that is,* $F = \nu \circ \alpha$.

[13] On the other hand, an infinite number of *nondecomposable* rules satisfy the Kalai and Smorodinsky (1975) properties. The following class of two-agent rules is an example: suppose that $\rho, \beta_1, \beta_2 \in [0, 1]$ and for all $i \in N$ and for all $D \in \mathbb{R}^{N\times N}$, $x_i^*(D) = \rho\underline{d}_i(D) + (1-\rho)\bar{d}_i(D)$ and $y_i^*(D) = \beta_i\underline{d}_i(D) + (1-\beta_i)\bar{d}_i(D)$. Then, let

$$F^{\rho,\beta_1,\beta_2}(S, D) = \begin{cases} \arg\max\limits_{x \in S} \min\limits_{i \in N} \dfrac{x_i - x_i^*(D)}{m_i(S, \alpha^{\text{diag}}(D)) - x_i^*(D)} & \text{if } (S, D) \in \mathcal{B}^2_{\gg}, \\ \arg\max\limits_{x \in S} \min\limits_{i \in N} \dfrac{x_i - y_i^*(D)}{m_i(S, \alpha^{\text{diag}}(D)) - y_i^*(D)} & \text{if } (S, D) \notin \mathcal{B}^2_{\gg}. \end{cases}$$

[14] There also are infinitely many *nondecomposable* rules that satisfy the Nash (1950) properties. The following class of two-agent rules is an example: suppose that $\beta_1, \beta_2 \in [0, 1]$ and for any $D \in \mathbb{R}^{N\times N}$, $x_i^*(D) = \beta_i\underline{d}_i(D) + (1-\beta_i)\bar{d}(D)$. Define

$$F^{\beta_1,\beta_2}(S, D) = \begin{cases} WPO(S, D) \cap \{\underline{d}(D) + r(\bar{d}(D) - \underline{d}(D)) \mid r \in \mathbb{R}_+\} \\ \quad \text{if } (S, D) \in \mathcal{B}^2_{\gg}, \\ WPO(S, D) \cap \{\underline{d}(D) + r(x^*(D) - \underline{d}(D)) \mid r \in \mathbb{R}_+\} \\ \quad \text{if } (S, D) \in \mathcal{B}_{\neq} \setminus \mathcal{B}^2_{\gg}, \\ \nu(S, D) \quad \text{if } (S, D) \in \mathcal{B}^2_{=}. \end{cases}$$

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On the subclass $\mathcal{B}^2_{\gg}$ of two-agent problems, the *aggregator functions* characterized in Proposition 3 turn out to have a very particular structure (see the Appendix for the proof).

Proposition 4. *On* $\mathcal{B}^2_{\gg}$, *an* aggregator function α *is* symmetric *and* scale invariant *if and only if*

(i) *for each* $D \in \mathbb{R}^{N\times N}$, *there is* $r^D \in [0,1]$ *such that* $\alpha(D) = r^D \underline{d}(D) + (1-r^D)\bar{d}(D)$ *and*
(ii) *for each* $D \in \mathbb{R}^{N\times N}$ *and* $\lambda \in \Lambda$, $r^{\lambda(D)} = r^D$.

Outside $\mathcal{B}^2_{\gg}$ *symmetry* and *scale invariance* do not have similar implications. Two-agent problems where the agents agree on their strict ranking of the two disagreement alternatives are never *symmetric*. Thus on this class, *symmetry* has no bite. Alternatively for problems with more than two agents, *scale invariance* has much weaker implications.[15]

The following property is satisfied by a unique extension of the Nash rule. A rule F *respects the max lower bound* if for each $(S, D) \in \mathcal{B}$, $F(S, D) \in U(S, \alpha^{\max}(D))$.

Proposition 5. *The* $\alpha^{\max}$*- extension of the Nash rule* $(\nu\circ\alpha^{\max})$ *is the only extension of the Nash rule that* respects the max lower bound.

For the proof of Proposition 5, see the Appendix. This result is rather surprising since a large class of Nash extension rules always pick individually rational alternatives. Only when the two requirements coincide does "individual rationality" imply a unique extension of the Nash rule.

Kalai (1977) introduces the following *anonymous-disagreement rule*, called the *egalitarian rule*, which maximizes the minimum payoff gain from disagreement: for each $(S, d) \in \mathcal{B}_=$,

$$\gamma(S,d) = \arg\max_{x\in S}\min_{i\in N}(x_i - d_i)\,.$$

He shows that, on the Nash domain, this rule uniquely satisfies *weak Pareto optimality, symmetry*, and *strong monotonicity*. On our domain, these properties characterize the following class of *decomposable* rules (see the Appendix for the proof).

Proposition 6 (*Corollary to Kalai, 1977*)**.** *A* decomposable *rule F on* $\mathcal{B}$ *satisfies* weak Pareto optimality, symmetry, *and* strong monotonicity *if and only if there is a* symmetric *aggregator function* α *such that* $F = \gamma\circ\alpha$.

Among *nondecomposable* rules, a large class of monotone path rules satisfy the properties of Proposition 6. For more on this issue, see Kıbrıs and Tapkı(2010).

Thomson (1981) discusses the following *anonymous-disagreement rule*, called the *utilitarian rule*, which maximizes the sum of the agents' payoffs: for each $(S, d) \in \mathcal{B}^{sc}_=$,

$$\tau(S,d) = \arg\max_{x\in S}\sum_{i\in N} x_i.$$

He shows that, on the subset $\mathcal{B}^{sc}_=$ of the Nash domain, this rule uniquely satisfies *Pareto optimality, symmetry, translation invariance*, and *strong contraction independence*. The following is a simple corollary of his result and Lemma 1 and is presented without proof.

Proposition 7 (*Corollary to Thomson, 1981*)**.** *On* $\mathcal{B}^{sc}_=$, *a rule F satisfies* Pareto optimality, symmetry, translation invariance, strong contraction independence, *and* decomposability *if and only if there is an* aggregator function α *such that* $F = \tau\circ\alpha$.

15 With two agents, disagreement matrices on $\mathcal{B}^2_{\gg}$ are divided into six equivalence classes: two matrices in the same class are related by a positive affine transformation. In an equivalence class, it is sufficient to fix $\alpha(D)$ for a single D; *scale invariance* then defines the *aggregator function* for every other matrix in the same equivalence class. With three or more agents however, the number of equivalence classes becomes infinite. Thus, the class of *aggregator functions* that satisfy *symmetry* and *scale invariance* becomes much larger and lacks a similar structure.

5. Conclusion

Our domain extension creates an abundance of bargaining rules. Properties that characterize a unique bargaining rule on the Nash domain are now satisfied by a large class of rules. Exploration of their subclasses that satisfy additional desirable properties remains an open question.

Our model does not specify the outcome of a coalition jointly terminating the negotiations. Modeling coordinated disagreement by a coalition brings in questions about the bargaining process in that coalition and moves us further towards a *coalitional-form game* analysis. In this paper, we choose to remain in the bargaining framework and only consider individual deviations.

It is interesting to note that the equilibria of the noncooperative models mentioned at the end of Section 1 use only partial information on the implications of disagreement. For example, an agent's payoff from the opponent leaving has no effect on the equilibrium (except in extreme cases where the problem's individually rational region is empty). Thus, our conjecture is that all these noncooperative bargaining games implement *decomposable* rules. Gaining a better understanding of the relationship between noncooperative models and the concepts that we developed in this paper remains an important open question.

Appendix

Proof of Lemma 1. Let F be *decomposable* and let ϕ and α satisfy $F = \phi\circ\alpha$. Suppose that $(S, d) \in \mathcal{B}_=$. Then since $\alpha(d) = d$, we have $F(S, d) = \phi(S, \alpha(d)) = \phi(S, d)$, the desired conclusion. □

Proof of Lemma 2. Suppose first that F is *decomposable*. Suppose that $F = \phi\circ\alpha$. Take any $D \in \mathbb{R}^{N\times N}$. Then for each $S \subseteq \mathbb{R}^N$ such that $(S, D) \in \mathcal{B}$, $\phi(S, \alpha(D)) = F(S, D)$. By Lemma 1, $\phi(S, \alpha(D)) = F(S, \alpha(D))$. Thus, $\alpha(D) \in \delta^F(D)$. Since $\delta^F(D) \neq \emptyset$ for each $D \in \mathbb{R}^{N\times N}$, F is *disagreement-simple*.

Conversely assume that F is *disagreement-simple*. Then for each $D \in \mathbb{R}^{N\times N}$, $\delta^F(D) \neq \emptyset$. Define the *aggregator function* $\alpha : \mathbb{R}^{N\times N} \to \mathbb{R}^N$ as a selection from δ^F, that is, for each $D \in \mathbb{R}^{N\times N}$, $\alpha(D) \in \delta^F(D)$. Suppose that $\phi = F|_{\mathcal{B}_=}$. Then for each $(S, D) \in \mathcal{B}$, $F(S, D) = F(S, \alpha(D)) = \phi(S, \alpha(D))$. Thus $F = \phi\circ\alpha$, that is, F is *decomposable*. □

Proof of Proposition 3. It is straightforward to show that $\nu\circ\alpha$ satisfies the given properties. Conversely, let $F = \phi\circ\alpha$ be a *decomposable* rule on $\mathcal{B}$ that satisfies them. By Lemma 1 then, ϕ satisfies the given properties. Therefore by Nash (1950), $\phi = \nu$.

Next we show that the *symmetry* of F implies the *symmetry* of α (see Fig. 4, left). Let $D \in \mathbb{R}^{N\times N}$ be symmetric. Suppose that $\alpha(D)$ is not symmetric. Then for some $\pi \in \Pi$, $\pi(\alpha(D)) \neq \alpha(D)$. Let $H = \left\{x \in \mathbb{R}^N \mid \sum x_i = 1 + \sum \bar{d}_i(D)\right\}$ and that $S^H = \underline{d}(D)\text{-comp}\{H\}$. Then $(S^H, D) \in \mathcal{B}$ is symmetric. Since F is *symmetric*, $\pi(F(S^H, D)) = F(S^H, D)$. Thus, $\pi(\nu(S^H, \alpha(D))) = \pi\left(F\left(S^H, D\right)\right) = F\left(S^H, D\right) = \nu(S^H, \alpha(D))$. But by the definition of ν, $\nu(S^H, \alpha(D)) = \alpha(D) + t\mathbf{1}$ for some $t \in \mathbb{R}_+$ and by $\pi(\alpha(D)) \neq \alpha(D)$, we have $\pi(\nu(S^H, \alpha(D))) \neq \nu(S^H, \alpha(D))$, a contradiction.

We finally show that the *scale invariance* of F implies the *scale invariance* of α (see Fig. 4, right). Since F is *scale invariant*, for each $(S, D) \in \mathcal{B}$ and $\lambda \in \wedge$, we have $F(\lambda(S), \lambda(D)) = \lambda(F(S, D))$. Also, by *decomposability* of F and *scale invariance* of ν, $F(\lambda(S), \lambda(D)) = \nu(\lambda(S), \alpha(\lambda(D)))$ and $\lambda(F(S, D)) = \lambda(\nu(S, \alpha(D))) = \nu(\lambda(S), \lambda(\alpha(D)))$. This implies $\nu(T, \alpha(\lambda(D))) = \nu(T, \lambda(\alpha(D)))$ for $T = \lambda(S)$. Now suppose that there is $D \in \mathbb{R}^{N\times N}$ such that $\alpha(\lambda(D)) \neq \lambda(\alpha(D))$. For $x \in \mathbb{R}^N$, let $x^{-1} = \left(\frac{1}{x_i}\right)_{i\in N}$. Let $b \in \mathbb{R}^N_{++}$ be such that $b^{-1} \notin \{\alpha(\lambda(D)) - \lambda(\alpha(D)), \lambda(\alpha(D)) - \alpha(\lambda(D))\}$. Then let $H' = \{x \in \mathbb{R}^N \mid b\cdot x = 1 + b\cdot$

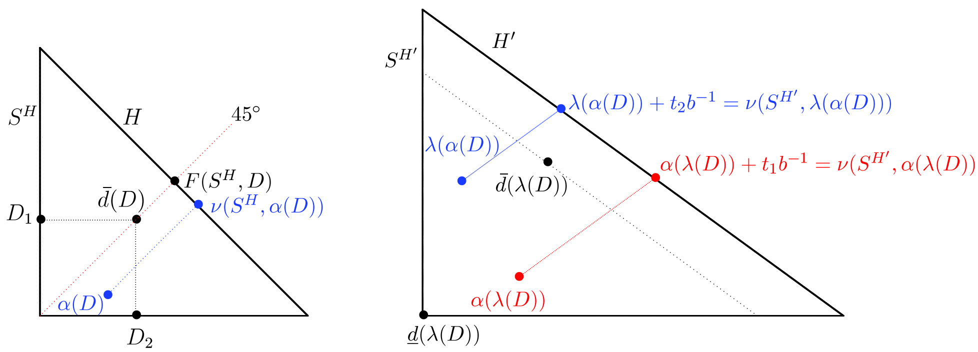

Fig. 4. Symmetry (on the left) and scale invariance in the proof of Proposition 3.

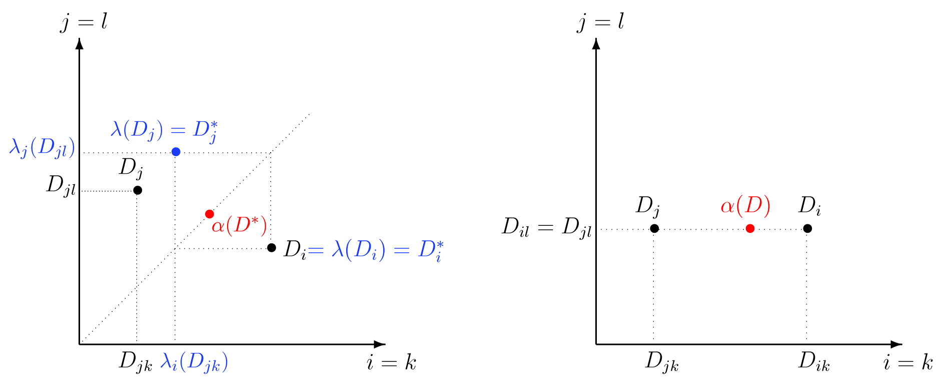

Fig. 5. Case 1 (on the left) and Case 2 in the proof of Proposition 4.

$\bar{d}(\lambda(D))\}$ and let $S^{H'} = \underline{d}(\lambda(D))$-comp$\{H'\}$. Then $\left(S^{H'}, \alpha(\lambda(D))\right) \in \mathcal{B}_=$ and $\left(S^{H'}, \lambda(\alpha(D))\right) \in \mathcal{B}_=$. But then for some $t_1, t_2 \in \mathbb{R}_{++}$, $\nu(S^{H'}, \alpha(\lambda(D))) = \alpha(\lambda(D)) + t_1 b^{-1}$ and $\nu(S^{H'}, \lambda(\alpha(D))) = \lambda(\alpha(D)) + t_2 b^{-1}$. Since $b^{-1} \notin \{\alpha(\lambda(D)) - \lambda(\alpha(D)), \lambda(\alpha(D)) - \alpha(\lambda(D))\}$, we have $\nu(S^{H'}, \alpha(\lambda(D))) \neq \nu(S^{H'}, \lambda(\alpha(D)))$, a contradiction. □

Proof of Proposition 4. First, let α satisfy conditions (i) and (ii). Since $\underline{d}(.)$ and $\bar{d}(.)$ are *symmetric* and *scale invariant*, α is *symmetric* and *scale invariant*.

For the opposite direction, let α be *symmetric* and *scale invariant*. Suppose first that condition (i) holds. To show that condition (ii) also holds, suppose that $D \in \mathbb{R}^{N \times N}$ and $\lambda \in \Lambda$. By condition (i), $\alpha(D) = r^D \underline{d}(D) + (1 - r^D)\bar{d}(D)$. Since α, $\underline{d}(.)$, and $\bar{d}(.)$ are scale invariant, $\alpha(\lambda(D)) = \lambda(\alpha(D)) = \lambda(r^D \underline{d}(D) + (1 - r^D)\bar{d}(D)) = r^D \underline{d}(\lambda(D)) + (1 - r^D)\bar{d}(\lambda(D))$. Also, by condition (i), we have $\alpha(\lambda(D)) = r^{\lambda(D)} \underline{d}(\lambda(D)) + (1 - r^{\lambda(D)})\bar{d}(\lambda(D))$. Thus, $r^{\lambda(D)} = r^D$. Now, to show that (i) holds, consider the following cases:

Case 1. (Fig. 5, left) Suppose that $D \in \{D \in \mathbb{R}^{N \times N} \mid$ there is $i, j \in \{1, 2\}$ such that $i \neq j$, $D_{i1} > D_{j1}$, and $D_{i2} < D_{j2}\}$. Then, consider the positive affine transformation λ such that $\lambda_i(D_{i1}) = D_{i1}$, $\lambda_i(D_{j1}) = D_{i2}$, $\lambda_j(D_{i2}) = D_{i2}$, and $\lambda_j(D_{j2}) = D_{i1}$. Note that $D^* = \lambda(D)$ is symmetric. Then, since α is *symmetric*, $\alpha(D^*)$ is symmetric. Since $\alpha(D^*) \in \{x \in \mathbb{R}^N \mid \underline{d}(D^*) \leqq x \leqq \bar{d}(D^*)\}$ and $\underline{d}(D^*)$ and $\bar{d}(D^*)$ are symmetric, there is $r^* \in [0, 1]$ such that $\alpha(D^*) = r^* \underline{d}(D^*) + (1 - r^*)\bar{d}(D^*)$. Since α, $\underline{d}(.)$, and $\bar{d}(.)$ are scale invariant, $\alpha(D) = \alpha(\lambda^{-1}(D^*)) = \lambda^{-1}(\alpha(D^*)) = \lambda^{-1}(r^* \underline{d}(D^*) + (1 - r^*)\bar{d}(D^*)) = r^* \underline{d}(\lambda^{-1}(D^*)) + (1 - r^*)\bar{d}(\lambda^{-1}(D^*)) = r^* \underline{d}(D) + (1 - r^*)\bar{d}(D)$. Since $D = \lambda^{-1}(D^*)$, by condition (ii), $r^D = r^*$. Thus, $\alpha(D) = r^D \underline{d}(D) + (1 - r^D)\bar{d}(D)$.

Case 2. (Fig. 5, right) Suppose that $D \in \{D \in \mathbb{R}^{N \times N} \mid$ there is $i, j \in \{1, 2\}$ such that $i \neq j$, $D_{i1} > D_{j1}$ and $D_{i2} = D_{j2}\}$. Since in this case conv$\{\underline{d}(D), \bar{d}(D)\} = \{x \in \mathbb{R}^N \mid \underline{d}(D) \leqq \alpha(D) \leqq \bar{d}(D)\}$, there is $r \in [0, 1]$ such that $\alpha(D) = r\underline{d}(D) + (1 - r)\bar{d}(D)$. Let $r^D = r$. □

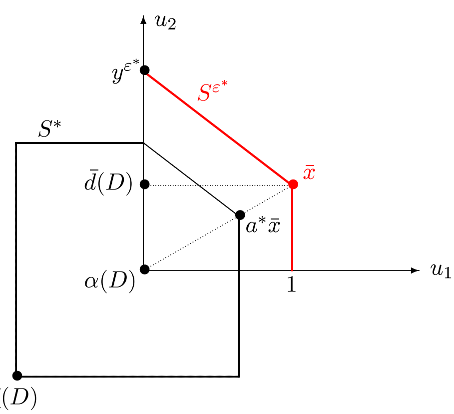

Fig. 6. The construction for $K = \{1\}$ in the proof of Proposition 5.

Proof of Proposition 5. It is straightforward to show that $\nu \circ \alpha^{\max}$ *respects the max lower bound*. To show that $\nu \circ \alpha^{\max}$ is unique, let $F = \nu \circ \alpha$ *respect the max lower bound* and suppose that there is $D \in \mathbb{R}^{N \times N}$ such that $\alpha(D) \leq \bar{d}(D)$. Since ν is *scale invariant*, without loss of generality, normalize $\alpha(D) = \mathbf{0}$. Let $K = \{i \in N \mid \bar{d}_i(D) = 0\}$, if this set is nonempty. Otherwise if $\bar{d}(D) > \mathbf{0}$, let $K = \{1\}$ (see Fig. 6).

Let $\bar{x} = \bar{d}(D) + \sum_{i \in K} e_i$ and let $\bar{S} = \mathbf{0}$-comp$\{\bar{x}\}$. Note that $(\bar{S}, \mathbf{0}) \in \mathcal{B}_=$. The normal vectors of the hyperplanes that support $\bar{S}$ at $\bar{x}$ constitute the set $E = $ conv$\{e_i \mid i \in N\}$. Also, the level curve for the Nash objective function at $\bar{x}$, $\prod \bar{x}_i$, has the normal vector $\bar{n} = \left(\frac{1}{\bar{x}_1}, \ldots, \frac{1}{\bar{x}_n}\right)$ at $\bar{x}$. Since there are $z \in E$ and $\gamma > 0$ such that $\bar{n} = \gamma z$, we have $\nu(\bar{S}, \mathbf{0}) = \bar{x}$.

Now for $\varepsilon > 0$, let $y^\varepsilon = \bar{d}(D) + \varepsilon \sum_{i \in N \setminus K} e_i$. Note that $\mathbf{0} = \alpha(D) \leq \bar{d}(D)$ implies $N \setminus K \neq \emptyset$. Thus for each $\varepsilon > 0$, $\bar{d}(D) \leq y^\varepsilon$. Next, define $S^\varepsilon = \mathbf{0}$-comp$\{$conv$\{\bar{x}, y^\varepsilon\}\}$ and note that (i) $(S^\varepsilon, \mathbf{0}) \in \mathcal{B}_=$ and (ii) $\bar{d}(D) \in$ Int(S^ε). The normal

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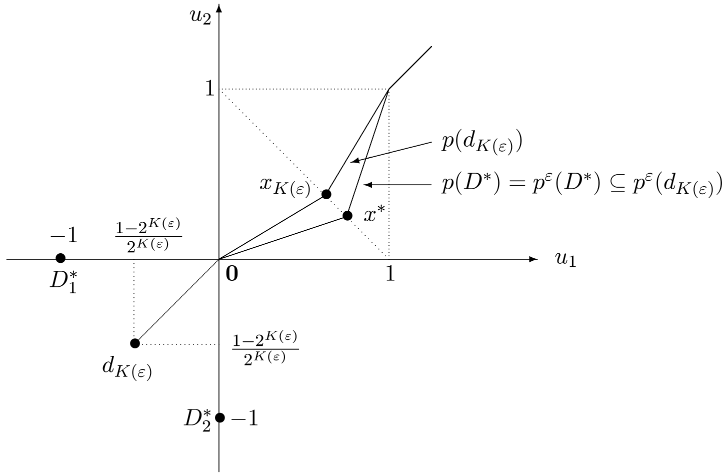

Fig. 7. In Example 3, for the *nondecomposable rule* F, the paths $p\,(D^*)$ and $p\,\left(d_{K(\varepsilon)}\right)$ are distinct. However, for the rule F^ε, the paths $p^\varepsilon\,(D^*)$ and $p^\varepsilon\,\left(d_{K(\varepsilon)}\right)$ overlap.

vectors of the hyperplanes that support S^ε at $\bar{x}$ constitute the set $E^\varepsilon = \text{conv}\left\{\{e_i \mid i \in K\} \cup \{e_j + \varepsilon e_i \mid i \in K \text{ and } j \in N \setminus K\}\right\}$. As $\lim_{\varepsilon \to 0} E^\varepsilon = E$, there is $\varepsilon^* > 0$ such that $\bar{n} \in E^{\varepsilon^*}$ and thus $\nu(S^{\varepsilon^*}, \mathbf{0}) = \bar{x}$.

Next, for $a \in (0, 1)$, let $\lambda^a \in \Lambda$ be such that for each $i \in N$, $\lambda_i^a(x) = ax_i$. Define $S^a = \lambda^a(S^{\varepsilon^*})$. Note that $(S^a, \mathbf{0}) \in \mathcal{B}_=$ and by scale invariance, $\nu(S^a, \mathbf{0}) = a\bar{x}$. Since $\lim_{a \to 1} S^a = S^{\varepsilon^*}$, there is $a^* \in (0, 1)$ such that $\bar{d}(D) \in \text{int}\, S^{a^*}$.

To conclude, let $S^* = \underline{d}(D)\text{-comp}\{S^{a^*}\}$ and note that since $\bar{d}(D) \in \text{Int}(S^{a^*})$, we have $(S^*, D) \in \mathcal{B}$. Also, since $I\left(S^{a^*}, 0\right) = I\left(S^*, 0\right)$ and $\nu\left(S^{a^*}, 0\right) = a^*\bar{x}$, we have $(\nu \circ \alpha)\,(S^*, D) = \nu(S^*, \mathbf{0}) = a^*\bar{x}$ and for each $i \in N \setminus K$, $a^*\bar{x}_i < \bar{d}_i(D)$. Thus, $\nu \circ \alpha$ violates the max lower bound at (S^*, D). □

Proof of Proposition 6. It is straightforward to show that the extensions $F = \gamma \circ \alpha$ of the egalitarian rule (γ), if the *aggregator function* α is *symmetric*, satisfy *weak Pareto optimality, symmetry*, and *strong monotonicity*. For uniqueness, let F be a *decomposable* rule on $\mathcal{B}$ that satisfies the given properties. By Lemma 1 then, ϕ satisfies the same properties. Therefore by Kalai (1977), $\phi = \gamma$.

Next we show that the *symmetry* of F implies the *symmetry* of α. Let $D \in \mathbb{R}^{N \times N}$ be symmetric. Suppose that for some $\pi \in \Pi, \pi(\alpha(D)) \neq \alpha(D)$. Let $H = \{x \in \mathbb{R}^N | \sum x_i = 1 + \sum \bar{d}_i(D)\}$ and that $S^H = \underline{d}(D)\text{-comp}\{H\}$. Then $(S^H, D) \in \mathcal{B}$ is symmetric. Since F is *symmetric*, $\pi(F(S^H, D)) = F(S^H, D)$. Thus, $\pi(\gamma(S^H, \alpha(D))) = \gamma(S^H, \alpha(D))$. Then, for all $i, j \in N$, $\gamma_i(S^H, \alpha(D)) = \gamma_j(S^H, \alpha(D))$. But by definition of the egalitarian rule, for all $i, j \in N$, $\gamma_i(S^H, \alpha(D)) - \alpha_i(D) = \gamma_j(S^H, \alpha(D)) - \alpha_j(D)$. Then, for all $i, j \in N$, $\alpha_i(D) = \alpha_j(D)$, contradicting $\pi(\alpha(D)) \neq \alpha(D)$. □

Example 3 (*A Nondecomposable Rule F Whose Every ε Neighborhood Contains a Decomposable Rule* F^ε**).** See Fig. 7. For $n = |N|$, suppose that $x^* = \frac{1}{n}\mathbf{1} + \frac{1}{2n}(e_1 - e_2)$. For $k \in N$, suppose that $x_k = \frac{1}{n}\mathbf{1} + \frac{k}{2n(k+1)}(e_1 - e_2)$ and note that $\lim_{k \to \infty} x_k = x^*$. Let $[a, b] \subseteq R^N$ represent the line segment that connects $a, b \in R^N$. For each $i \in N$, suppose that $D_i^* = -e_i$. For each $D \in R^{N \times N}$, let

$$p(D) = \begin{cases} [d_k, \mathbf{0}] \cup [\mathbf{0}, x_k] \cup [x_k, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} \\ \quad \text{if } D = d_k = \dfrac{1-2^k}{2^k}\mathbf{1} \text{ for } k \in \mathbb{N}, \\ [\mathbf{0}, x^*] \cup [x^*, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} \\ \quad \text{if } D = D^*, \\ \{\bar{d}(D) + r\mathbf{1} \mid r \geq 0\} & \text{otherwise.} \end{cases}$$

For each $(S, D) \in B$, let $F(S, D) = WPO(S, D) \cap p(D)$. Such rules are called monotone path rules (in reference to the monotone path $p(D)$). A monotone path rule F is *decomposable* if for each $D \in R^{N \times N}$ there is $d \in R^N$ such that $\underline{d}(D) \leqq d \leqq \bar{d}(D)$ and $p(D) \subseteq p(d)$. Since there is no $d \in R^N$ such that $p(D^*) \subseteq p(d)$, F is *nondecomposable*.

Given $\varepsilon > 0$, let $K(\varepsilon) \in N$ be such that $K(\varepsilon) > \frac{2(1-(n+1)\varepsilon)}{n\varepsilon}$. For each $D \in R^{N \times N}$, let

$$p^\varepsilon(D) = \begin{cases} [d_{K(\varepsilon)}, \mathbf{0}] \cup [\mathbf{0}, x^*] \cup [x^*, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} \\ \quad \text{if } D = d_{K(\varepsilon)} = \dfrac{1-2^{K(\varepsilon)}}{2^{K(\varepsilon)}}\mathbf{1}, \\ [d_k, \mathbf{0}] \cup [\mathbf{0}, x_k] \cup [x_k, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} \\ \quad \text{if } D = d_k = \dfrac{1-2^k}{2^k}\mathbf{1} \text{ for } k \in \mathbb{N} \setminus \{K(\varepsilon)\}, \\ [\mathbf{0}, x^*] \cup [x^*, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} \\ \quad \text{if } D = D^* \\ \{\bar{d}(D) + r\mathbf{1} \mid r \geq 0\} & \text{otherwise} \end{cases}$$

and for each $(S, D) \in B$, let $F^\varepsilon(S, D) = WPO(S, D) \cap p^\varepsilon(D)$. Now $p(D^*) \subseteq p\left(\frac{1-2^{K(\varepsilon)}}{2^{K(\varepsilon)}}\mathbf{1}\right)$ and thus, F^ε is decomposable. Furthermore, by the choice of $K(\varepsilon)$,

$$\begin{aligned} \mu^{\mathcal{F}}(F, F^\varepsilon) &= \max\left\{\frac{\|x-y\|}{1+\|x-y\|} \mid x \in p\left(\frac{1-2^{K(\varepsilon)}}{2^{K(\varepsilon)}}\mathbf{1}\right),\right. \\ &\qquad \left. y \in p^\varepsilon\left(\frac{1-2^{K(\varepsilon)}}{2^{K(\varepsilon)}}\mathbf{1}\right), x \ngeq y, y \ngeq x\right\} \\ &= \frac{\frac{2}{n(K(\varepsilon)+2)}}{1 + \frac{2}{n(K(\varepsilon)+2)}} < \varepsilon. \end{aligned}$$

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