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Bargaining with nonanonymous disagreement: Monotonic rules ☆

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ABSTRACT

We analyze bargaining situations where the agents' payoffs from disagreement depend on who among them breaks down the negotiations. We model such problems as a superset of the standard domain of Nash [Nash, J.F., 1950. The bargaining problem. Econometrica 18, 155–162]. On our extended domain, we analyze the implications of two central properties which, on the Nash domain, are known to be incompatible: *strong monotonicity* [Kalai, E., 1977. Proportional solutions to bargaining situations: Interpersonal utility comparisons. Econometrica 45, 1623–1630] and *scale invariance* [Nash, J.F., 1950. The bargaining problem. Econometrica 18, 155–162]. We first show that a class of monotone path rules uniquely satisfy *strong monotonicity*, *scale invariance*, *weak Pareto optimality*, and "continuity". We also show that dropping *scale invariance* from this list characterizes the whole class of monotone path rules. We then introduce a *symmetric* monotone path rule that we call the *Cardinal Egalitarian rule* and show that it is *weakly Pareto optimal, strongly monotonic, scale invariant*, *symmetric* and that it is the only rule to satisfy these properties on a class of two-agent problems.

1. Introduction

A typical bargaining problem, as modeled by Nash (1950) and the vast literature that follows, is made up of two elements. The first is a set of alternative agreements on which the agents negotiate. The second element is an alternative realized in case of disagreement. This "disagreement outcome" does not however contain detailed information about the nature of disagreement. Particularly, it is assumed in the existing literature that the realized disagreement alternative is independent of who among the agents disagree(s).

In real life examples of bargaining, however, the identity of the agent who terminates the negotiations turns out to have a significant effect on the agents' "disagreement payoffs." The 2004 reunion negotiations between the northern and the southern parts of Cyprus constitute a good example. Due to a very strong support from the international community towards the island's reunion, neither party preferred to be the one to disagree. Also, each preferred the other's disagreement to some agreements which they in turn preferred to leaving the negotiation table themselves.[1] Wage negotiations between firms and labor unions constitute another example to the dependence of the disagreement payoffs on the identity of the

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[1] There is a vast number of articles that discuss the issue. For example, see the *Economist* articles dated April 17, 2004 (volume 371, issue 8371), *Cyprus: A Greek Wrecker* (page 11) and *Cyprus: A Derailment Coming* (page 25); also see *Greece's Election: Sprinting Start?* dated March 13, 2004 (volume 370, issue

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disagreer. There, the disagreement action of the union, a strike, and that of the firm, a lockout, can be significantly different in terms of their payoff implications.[2]

Note that, neither of these examples can be fully represented in the confines of Nash's (1950) standard model. We therefore extend this model to a nonanonymous-disagreement model of bargaining by allowing the agents' payoffs from disagreement to depend on who among them disagrees. For this, we replace the disagreement *payoff vector* in the Nash (1950) model with a disagreement *payoff matrix*. The ith row of this matrix is the payoff vector that results from agent i terminating the negotiations. The standard (anonymous-disagreement) domain of Nash (1950) is a "measure-zero" subset of ours where all rows of the disagreement matrix are identical.

Our domain extension significantly increases the amount of admissible rules. Every bargaining rule on the Nash domain has counterparts on our domain (we call such rules decomposable since they are a composition of a rule from the Nash domain and a function that transforms disagreement matrices to disagreement vectors). But our domain also offers an abundance of rules that are nondecomposable (that is, they are not counterparts of rules from the Nash domain).

On our extended domain, we analyze the implications of two central properties which, on the Nash domain, are known to be incompatible (Thomson, in press). The first property, called strong monotonicity (Kalai, 1977) states that an expansion of the set of possible agreements should not make any agent worse-off. Kalai (1977) motivates it as both a normative and a positive property and argues that "if additional options were made available to the individuals in a given situation, then no one of them should lose utility because of the availability of these new options." The second property, called scale invariance (Nash, 1950) ensures the invariance of the physical bargaining outcome with respect to utility-representation changes that leave the underlying von Neumann–Morgenstern (1944) preferences intact. Scale invariant rules use information only about the agents' preferences (and not their utility representation) to determine the bargaining outcome.

Our first result establishes the existence of nonanonymous-disagreement bargaining rules that are both *strongly monotonic* and *scale invariant*. More specifically, in Subsection 3.1, we first present a class of monotone path rules which assign each disagreement matrix to a monotone increasing path in the payoff space and for a given problem, picks the maximal feasible point of this monotone path as the solution.[3] In Theorem 1, we show that *strong monotonicity*, *scale invariance*, *weak Pareto optimality*, and "continuity" characterize the whole class of monotone path rules. Next, we show in Theorem 2 that adding *scale invariance* to this list characterizes a class of monotone path rules.

In this subsection, we also analyze two-agent problems. We show in Proposition 4 that a *scale invariant* monotone path rule for two-agent problems can be fully defined by the specification of at most eight monotone paths.

Finally, in Subsection 3.2, we introduce a *symmetric* monotone path rule that we call the Cardinal Egalitarian rule. This is a nondecomposable rule and it is a *scale invariant* version of the well-known Egalitarian rule (Kalai, 1977). (The Egalitarian rule violates scale invariance since it makes interpersonal utility comparisons.) The Cardinal Egalitarian rule coincides with the Egalitarian rule on a class of normalized problems and solves every other problem by using scale invariance and this normalized class. In Theorem 5, we show that the Cardinal Egalitarian rule is *weakly Pareto optimal, strongly monotonic, scale invariant*, *symmetric* and that it is the only rule to satisfy these properties on a class of two-agent problems where the agents disagree on their strict ranking of the disagreement alternatives (as, for example, was the case for the 2004 Cyprus negotiations).

In a companion paper (Kıbrıs and Tapkı, 2007), we show that the class of *decomposable* rules is a *nowhere dense* subset of all bargaining rules. This class, however, contains the (uncountably many) extensions of each rule that has been analyzed in the literature until now. Thus, we then enquire if the counterparts of some standard results on the Nash domain continue to hold for decomposable rules on our extended domain. We first show that an extension of the Kalai–Smorodinsky bargaining rule uniquely satisfies the Kalai–Smorodinsky (1975) properties. This uniqueness result, however, turns out to be an exception. An infinite number of decomposable rules survive the Nash (1950), Kalai (1977), Perles–Maschler (1981), and Thomson (1981) properties even though, on the Nash domain each of these results characterizes a single rule. In that paper, we also observe that extensions to our domain of a standard independence property (by Peters, 1986) imply *decomposability*.

Gupta and Livne (1988) analyze bargaining problems with an additional reference point (in the feasible set), interpreted as a past agreement. Chun and Thomson (1992) analyze an alternative model where the reference point is not feasible (and is interpreted as a vector of "incompatible" claims). Both studies characterize rules that allocate gains proportionally to the reference point. Neither of these two papers focus on disagreement. Livne (1988) and Smorodinsky (2005), on the other hand, analyze cases where the implications of disagreement are uncertain. They thus extend the Nash (1950) model to allow probabilistic disagreement points. They characterize alternative extensions of the Nash rule to their domain. Finally, Basu (1996) analyzes cases where disagreement leads to a noncooperative game with multiple equilibria and to model them, he extends the Nash model to allow for a set of disagreement points over which the players do not have probability distributions. He characterizes an extension of the Kalai–Smorodinsky (1975) rule to this domain.

8366, page 31). Also see the special issue on Cyprus of *International Debates* (2005, 3:3). Finally, an interview (in Turkish) with a former Turkish minister of external affairs, which appeared in the daily newspaper Radikal on February 16, 2004, presents a detailed discussion of the implications of disagreement.

[2] A similar case may arise between two countries negotiating at the brink of a war. Among the two possible disagreement outcomes, each country might prefer the one where it leaves the negotiation table first and makes an (unexpected) "preemptive strike" against the other.

[3] This monotone path can be interpreted as an agenda in which the agents jointly improve their payoffs until doing so is no more feasible. On the Nash domain, monotone path rules are introduced by Thomson and Myerson (1980) and further discussed by Peters and Tijs (1984) (also see Thomson, in press).

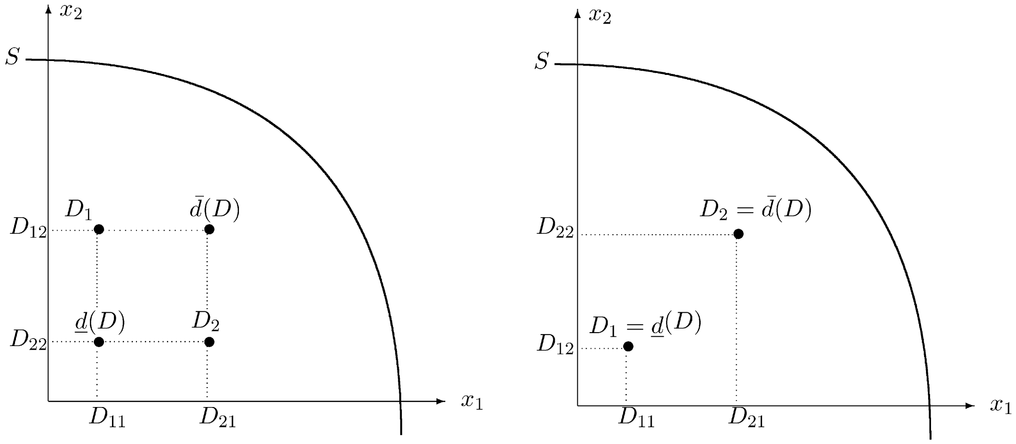

Fig. 1. A typical bargaining problem with nonanonymous disagreement.

Chun and Thomson (1990a, 1990b) and Peters and van Damme (1991) remain in the Nash (1950) model but they introduce axioms to represent cases where the agents are not certain about the implications of disagreement. Chun and Thomson (1990a) show that a basic set of properties characterize the weighted Egalitarian rules. Chun and Thomson (1990b) and Peters and van Damme (1991) show that the Nash rule uniquely satisfies alternative sets of properties. Some other papers that discuss disagreement-related properties on the Nash (1950) model are Dagan et al. (2002), Livne (1986), and Thomson (1987).

The common feature of all of the above papers (and the current cooperative bargaining literature for that matter) is that the implications of disagreement are independent of the identity of the agent who causes it. On the other hand, there are noncooperative bargaining models in which agents are allowed to leave and take an outside option. Shaked and Sutton (1984) present one of the first examples. Ponsatí and Sákovics (1998) analyze a model where both agents can leave at each period (but the resulting payoffs are independent of who leaves) and Corominas-Bosch (2000) analyzes a model where the disagreement payoffs depend on who the last agent to reject an offer was (but the agents are not allowed to leave, disagreement is randomly determined by nature). Our model can be seen as to provide a cooperative counterpart to these noncooperative models.

2. Model

Let $N = \{1, \dots, n\}$ be the set of agents. For each $i \in N$, let $e_i \in \mathbb{R}^n$ be the vector whose ith coordinate is 1 and every other coordinate is 0. Let $\mathbf{1} \in \mathbb{R}^n$ (respectively, $\mathbf{0}$) be the vector whose every coordinate is 1 (respectively, 0). For vectors in $\mathbb{R}^n$, inequalities are defined as: $x \leqq y$ if and only if $x_i \leqq y_i$ for each $i \in N$; $x \leqslant y$ if and only if $x \leqq y$ and $x \neq y$; $x < y$ if and only if $x_i < y_i$ for each $i \in N$. For each $S \subseteq \mathbb{R}^n$, $Int(S)$ denotes the interior of S and $Cl(S)$ denotes the closure of S. For each $S \subseteq \mathbb{R}^n$ and $s \in S$, $conv\{S\}$ denotes the convex hull of S and $s\text{-}comp\{S\} = \{x \in \mathbb{R}^n \mid s \leqq x \leqq y \text{ for some } y \in S\}$ denotes the s-comprehensive hull of S. The set S is *s-comprehensive* if $s\text{-}comp\{S\} \subseteq S$. The set S is *strictly s-comprehensive* if it is *s-comprehensive* and for each $x, y \in S$ such that $x \geqslant y \geqslant s$, there is $z \in S$ such that $z > y$.

Let the Euclidean metric be defined as $\|x - y\| = \sqrt{\sum (x_i - y_i)^2}$ for $x, y \in \mathbb{R}^n$ and let the Hausdorff metric be defined as $\mu^H(S^1, S^2) = \max_{i \in \{1,2\}} \max_{x \in S^i} \min_{y \in S^j} \|x - y\|$ for compact sets $S^1, S^2 \subseteq \mathbb{R}^n$. Let

$$D = \begin{bmatrix} D_{11} & \cdots & D_{1n} \\ \vdots & \ddots & \vdots \\ D_{n1} & \cdots & D_{nn} \end{bmatrix} = \begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

be a matrix in $\mathbb{R}^{n \times n}$. The ith row vector $D_i = (D_{i1}, \dots, D_{in}) \in \mathbb{R}^n$ represents the disagreement payoff profile that arises from agent i terminating the negotiations. For each $i \in N$, let $\bar{d}_i(D) = \max\{D_{ji} \mid j \in N\}$ be the maximum payoff agent i can get from disagreement and let $\underline{d}_i(D) = \min\{D_{ji} \mid j \in N\}$ be the minimal payoff. Let $\bar{d}(D) = (\bar{d}_i(D))_{i \in N}$ and $\underline{d}(D) = (\underline{d}_i(D))_{i \in N}$. Let the metric $\mu^{\mathbf{M}}$ on $\mathbb{R}^{n \times n}$ be defined as $\mu^M(D, D') = \max_{i \in N} \|D_i - D'_i\|$ for $D, D' \in \mathbb{R}^{n \times n}$.

Let Π be the set of all permutations π on N. A function $f : \mathbb{R} \to \mathbb{R}$ is *positive affine* if there is $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $f(x) = ax + b$ for each $x \in \mathbb{R}$. Let Λ be the set of all $\lambda = (\lambda_1, \dots, \lambda_n)$ where each $\lambda_i : \mathbb{R} \to \mathbb{R}$ is a positive affine function.

For $\pi \in \Pi$, $S \subseteq \mathbb{R}^n$, and $D \in \mathbb{R}^{n\times n}$, let $\pi(S) = \{y \in \mathbb{R}^n \mid y = (x_{\pi(i)})_{i\in N}$ for some $x \in S\}$ and $\pi(D) = (D_{\pi(i)\pi(j)})_{i,j\in N}$. The set S (respectively, the matrix D) is symmetric if for every permutation $\pi \in \Pi$, $\pi(S) = S$ (respectively, $\pi(D) = D$). For $\lambda \in \Lambda$, let $\lambda(S) = \{(\lambda_1(x_1), \ldots, \lambda_n(x_n)) \mid x \in S\}$ and

$$\lambda(D) = \begin{bmatrix} \lambda_1(D_{11}) & \cdots & \lambda_n(D_{1n}) \\ \vdots & \ddots & \vdots \\ \lambda_1(D_{n1}) & \cdots & \lambda_n(D_{nn}) \end{bmatrix} = \begin{bmatrix} \lambda(D_1) \\ \vdots \\ \lambda(D_n) \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

A (nonanonymous-disagreement bargaining) problem for N is a pair (S, D) where $S \subseteq \mathbb{R}^n$ and $D \in \mathbb{R}^{n\times n}$ satisfy (see Fig 1): (i) for each $i \in N$, $D_i \in S$; (ii) S is compact, convex, and $\underline{d}(D)$-comprehensive; (iii) there is $x \in S$ such that $x > \bar{d}(D)$. Assumptions (i), (ii) and a counterpart of (iii) are standard in the literature.[4] They essentially come out of problems where the agents have expected utility functions on a bounded set of lotteries.

Let $\mathcal{B}$ be the class of all problems for agents in N. Let $\mathcal{B}_= = \{(S, D) \in \mathcal{B} \mid D_1 = D_2 = \cdots = D_n\}$ be the subclass of problems with anonymous disagreement. Let $\mathcal{B}_{\neq} = \mathcal{B} \setminus \mathcal{B}_=$ be the subclass of problems with nonanonymous disagreement.

Let $\mathcal{B}^2_{\neq}$ be the class of two-agent problems with nonanonymous disagreement. Problems in $\mathcal{B}^2_{\neq}$ can be grouped into three distinct classes. In the first class of problems, the disagreement of one agent is strictly preferred by both agents to the disagreement of the other:

$$\mathcal{B}^2_{>>} = \big\{(S, D) \in \mathcal{B}^2_{\neq} \;\big|\; \text{there are } i, j \in \{1, 2\} \text{ such that } i \neq j \text{ and for all } k \in \{1, 2\},\, D_{ik} > D_{jk}\big\}.$$

The second class of problems represents cases where one agent is indifferent between the two disagreement alternatives and the other has strict preferences:

$$\mathcal{B}^2_{>=} = \big\{(S, D) \in \mathcal{B}^2_{\neq} \;\big|\; \text{there are } i, j, k, l \in \{1, 2\} \text{ such that } i \neq j, k \neq l,\ D_{ik} > D_{jk} \text{ and } D_{il} = D_{jl}\big\}.$$

In the third class of problems, the agents disagree on their (strict) ranking of the two disagreement alternatives:

$$\mathcal{B}^2_{><} = \big\{(S, D) \in \mathcal{B}^2_{\neq} \;\big|\; \text{there are } i, j \in \{1, 2\} \text{ such that } i \neq j,\ D_{i1} > D_{j1}, \text{ and } D_{i2} < D_{j2}\big\}.$$

Let the metric $\mu^{\mathcal{B}}$ on $\mathcal{B}$ be defined as $\mu^{\mathcal{B}}((S, D), (S', D')) = \max\{\mu^H(S, S'), \mu^M(D, D')\}$ for $(S, D), (S', D') \in \mathcal{B}$. Given $(S, D) \in \mathcal{B}$, the set of Pareto optimal alternatives is $PO(S, D) = \{x \in S \mid y \geqslant x$ implies $y \notin S\}$ and the set of weakly Pareto optimal alternatives is $WPO(S, D) = \{x \in S \mid y > x$ implies $y \notin S\}$.

A (nonanonymous-disagreement bargaining) rule $F : \mathcal{B} \to \mathbb{R}^n$ is a function that satisfies $F(S, D) \in S$ for each $(S, D) \in \mathcal{B}$. Let $\mathcal{F}$ be the class of all rules. A rule F is Pareto optimal if for each $(S, D) \in \mathcal{B}$, $F(S, D) \in PO(S, D)$. It is weakly Pareto optimal if for each $(S, D) \in \mathcal{B}$, $F(S, D) \in WPO(S, D)$. It is symmetric if for each $(S, D) \in \mathcal{B}$ with *symmetric* S and D, $F(S, D)$ is also *symmetric*, that is, $F_1(S, D) = \cdots = F_n(S, D)$.

The following property requires small changes in the data of a problem not to have a big effect on the agreement. A rule F is set-continuous if for every $D \in \mathbb{R}^{n\times n}$ and for every sequence $\{(S^m, D)\}_{m\in\mathbb{N}} \subseteq \mathcal{B}$ that converges with respect to $\mu^{\mathcal{B}}$ to some $(S, D) \in \mathcal{B}$, we have $\lim_{m\to\infty} F(S^m, D) = F(S, D)$.

Next, we present two central properties in bargaining theory. The first one requires the physical bargaining outcome to be invariant under utility-representation changes as long as the underlying von Neumann–Morgenstern (1944) preference information is unchanged. A rule F is scale invariant if for each $(S, D) \in \mathcal{B}$ and each $\lambda \in \Lambda$, $F(\lambda(S), \lambda(D)) = \lambda(F(S, D))$. The second property requires that an expansion of the set of possible agreements make no agent worse-off. A rule F is strongly monotonic (Kalai, 1977) if for each $(S, D), (T, D) \in \mathcal{B}$, $T \subseteq S$ implies $F(T, D) \leqq F(S, D)$.

3. Results

3.1. Monotone path rules

On our domain, a very large class of rules simultaneously satisfy three properties which, on the Nash domain, $\mathcal{B}_=$, are incompatible: *weak Pareto optimality, strong monotonicity,* and *scale invariance* (see Thomson, in press, for a discussion). We introduce them next.

A monotone path on $\mathbb{R}^n$ is the image $G \subseteq \mathbb{R}^n$ of a function $g : \mathbb{R}_+ \to \mathbb{R}^n$ which is such that for all $i \in N$, $g_i : \mathbb{R}_+ \to \mathbb{R}$ is continuous and nondecreasing and for some $j \in N$, $g_j(\mathbb{R}_+) = [g_j(0), \infty)$. Let $\mathbb{G}$ be the set of all monotone paths.

Let $p : \mathbb{R}^{n\times n} \to \mathbb{G}$ be a path generator (function) that maps each disagreement matrix D to a monotone path $p(D)$ such that (i) $x = \min p(D)$ is the unique member of $p(D)$ that satisfies $\underline{d}(D) \leqq x \leqq \bar{d}(D)$ and $x_i = \bar{d}_i(D)$ for some $i \in N$, and (ii) there are no $x, y \in p(D)$ such that $x \neq y$ and $x_i = y_i > \bar{d}_i(D)$ for some $i \in N$. Condition (i) requires the path $p(D)$ to start from a point x that is at the weak Pareto boundary of the rectangular set in-between $\underline{d}(D)$ and $\bar{d}(D)$. Condition (ii)

[4] When Assumption 3 is violated, the agents are not guaranteed to reach an agreement. Particularly, for each alternative x, there will be an agent who receives higher payoff from someone (including himself) leaving the negotiation table. It will be in the interest of this agent then to follow strategies that induce disagreement rather than to accept x.

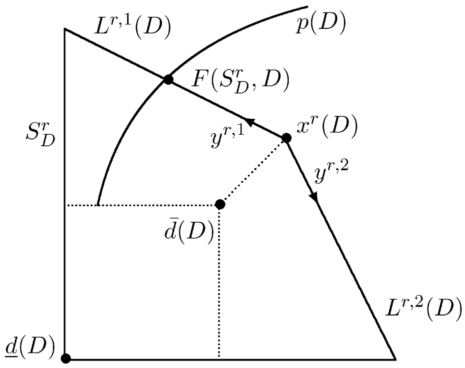

Fig. 2. The construction of Step 1 in the proof of Theorem 1.

strengthens the monotonicity requirement on the path $p(D)$. For example, in $\mathbb{R}^2$ it requires the path not to be vertical (respectively, horizontal) on the half-space of vectors whose first (respectively, second) coordinate is greater than that of $\bar{d}(D)$. The path generator p is scale invariant if for each $D \in \mathbb{R}^{n\times n}$ and $\lambda \in \Lambda$, $p(\lambda(D)) = \lambda(p(D))$.

The monotone path rule $\mathbf{F^p} : \mathcal{B} \to \mathbb{R}^n$ with respect to the path generator $\mathbf{p}$ maps each $(S, D) \in \mathcal{B}$ to the maximal point of S along $p(D)$, that is, $F^p(S, D) = WPO(S, D) \cap p(D)$.[5]

The following theorem shows that monotone path rules uniquely satisfy three basic properties.

Theorem 1. *A rule $F : \mathcal{B} \to \mathbb{R}^n$ is weakly Pareto optimal, strongly monotonic, and set-continuous if and only if it is a monotone path rule F^p.*

Proof. Monotone path rules by definition are *weakly Pareto optimal*. *Strong monotonicity* follows from the monotonicity of the paths $p(D)$ and *set-continuity* follows from Condition (ii) on the path generator p.

For the uniqueness part of the first statement, let $F : \mathcal{B} \to \mathbb{R}^n$ be a rule that is *weakly Pareto optimal, strongly monotonic*, and *set-continuous*.

Step 1 *(define p).* Fix arbitrary $D \in \mathbb{R}^{n\times n}$ and for each $r \in \mathbb{R}_+$, let $x^r(D) = \bar{d}(D) + r\mathbf{1}$ (see Fig. 2). Let $f : \mathbb{R}_+ \to [0, 1]$ be an increasing continuous function such that $f(0) = 0$ and $\lim_{r\to\infty} f(r) = 1$. For each $i \in N$, let $y^{r,i} \in \mathbb{R}^n$ be such that $y^{r,i}_i = -1$ and for $j \neq i$, $y^{r,i}_j = f(r)$. Note that $y^{r,i}$ is a vector whose ith coordinate is -1 and whose jth coordinate is $f(r)$. It is used to construct the following line segment: let $L^{r,i}(D) = \{x^r(D) + ly^{r,i} \mid 0 \leqq l \leqq x^r_i(D) - \underline{d}_i(D)\}$. Now let $S^r_D = \underline{d}(D)\text{-}comp\{L^{r,1}(D), \ldots, L^{r,n}(D)\}$ and note that for all $r \in \mathbb{R}_{++}$, (i) S^r_D is *strictly $\underline{d}(D)$-comprehensive* and (ii) $F(S^r_D, D) \in PO(S^r_D, D)$. Finally, define $p(D) = Cl\{F(S^r_D, D) \mid r \in \mathbb{R}_{++}\}$. Note that, for each $D \in \mathbb{R}^{n\times n}$, $p(D)$ is ordered with respect to $\geqslant$ by *strong monotonicity* of F. Let $F^p : \mathcal{B} \to \mathbb{R}^n$ be defined as follows: for each $(S, D) \in \mathcal{B}$, $F^p(S, D) = \max(p(D) \cap WPO(S, D))$.

Step 2 *($F = F^p$).* Now let $(S, D) \in \mathcal{B}$ and let $x^* = F^p(S, D)$ (see Fig. 3). Then by definition of p, there is $r \in \mathbb{R}_{++}$ such that $x^* = F^p(S^r_D, D) = F(S^r_D, D)$. Let $T = S \cap S^r_D$ and note that $x^* \in T$. Since $x^* \in PO(S^r_D, D)$, we also have $x^* \in PO(T, D)$. Now, by *strong monotonicity* of F, $F(T, D) \leqq x^*$. First assume that $y \leqslant x^*$ implies $y \notin WPO(T, D)$. Then *weak Pareto optimality* of F implies $F(T, D) = x^*$. Alternatively if there is $y \leqslant x^*$ such that $y \in WPO(T, D)$, let $\{T^m\}_{m\in\mathbb{N}} \to T$ be such that for each $m \in \mathbb{N}$, $T^m \subseteq S^r_D$ is *strictly $\underline{d}(D)$-comprehensive* (this is possible since S^r_D is strictly $\underline{d}(D)$-comprehensive) and $x^* \in PO(T^m, D)$ (this is possible since $x^* \in PO(S^r_D, D)$). Then by the previous case, $F(T^m, D) = x^*$ for each $m \in \mathbb{N}$ and by *set-continuity* of F, $F(T, D) = x^*$. Finally, $T \subseteq S$, by *strong monotonicity* of F implies $x^* \leqq F(S, D)$. If $x^* \in PO(S, D)$, this implies $x^* = F(S, D)$. Alternatively if $x^* \in WPO(S, D) \setminus PO(S, D)$, let $\{S^m\}_{m\in\mathbb{N}} \to S$ be such that for each $m \in \mathbb{N}$, $T \subseteq S^m$ and $x^* \in PO(S^m, D)$ (this is possible since $x^* \in PO(T, D)$). Then by the previous case, $F(S^m, D) = x^*$ and by *set-continuity* of F, $F(S, D) = x^*$.

Step 3 *($F = F^p$ is a monotone path rule).* We show that for each $(S, D) \in \mathcal{B}$, the set $p(D) \cap WPO(S, D)$ is a singleton and thus, $F^p(S, D) = p(D) \cap WPO(S, D)$. Suppose $(S, D) \in \mathcal{B}$ is such that there is $y \leqslant F^p(S, D)$ satisfying $y \in p(D) \cap WPO(S, D)$. Let $\{S^m\}_{m\in\mathbb{N}} \to S$ be such that for each $m \in \mathbb{N}$, $p(D) \cap WPO(S^m, D) = \{y\}$. Then for each $m \in \mathbb{N}$, $F^p(S^m, D) = y$. This, by *set-continuity* of F^p implies $F^p(S, D) = y$, contradicting $y \neq F^p(S, D)$. □

The following theorem characterizes *scale invariant* monotone path rules. Note that the domain reduces from $\mathcal{B}$ to $\mathcal{B}_{\neq}$ as we introduce *scale invariance*. This is because the stated properties (of Theorem 2) are not compatible on $\mathcal{B}_{=}$.

[5] Continuity and monotonicity of the path $p(D)$ guarantee that this intersection is nonempty while Condition (ii) on the generator function p guarantees that it is a singleton.

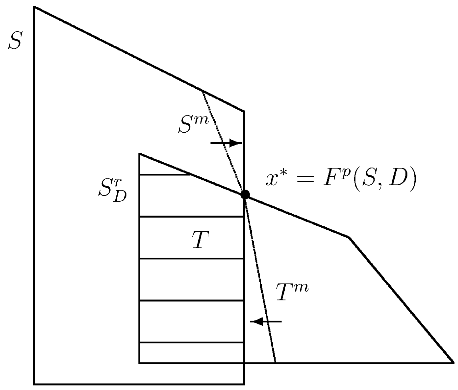

Fig. 3. The construction of Step 2 in the proof of Theorem 1.

Theorem 2. *A rule on $\mathcal{B}_{\neq}$, $F : \mathcal{B}_{\neq} \to \mathbb{R}^n$ is weakly Pareto optimal, strongly monotonic, set-continuous, and scale invariant if and only if it is a monotone path rule F^p where p is scale invariant.*

Proof. The proof of Theorem 1 does not rely on the existence of a problem in $\mathcal{B}_{=}$. Therefore, its statement also holds on the subclass $\mathcal{B}_{\neq}$ of $\mathcal{B}$. That is, a rule $F : \mathcal{B}_{\neq} \to \mathbb{R}^n$ is *weakly Pareto optimal, strongly monotonic*, and *set-continuous* if and only if it is a monotone path rule F^p on $\mathcal{B}_{\neq}$. Therefore, to prove Theorem 2 it suffices to show that $F = F^p$ is *scale invariant* if and only if p is *scale invariant*. For this, let $(S, D) \in \mathcal{B}_{\neq}$ and $\lambda \in \Lambda$.

First note that $F^p(\lambda(S), \lambda(D)) = WPO(\lambda(S), \lambda(D)) \cap p(\lambda(D))$ and $WPO(\lambda(S), \lambda(D)) = \lambda(WPO(S, D))$. Then, $\lambda(F^p(S, D)) = \lambda(WPO(S, D) \cap p(D)) = \lambda(WPO(S, D)) \cap \lambda(p(D)) = WPO(\lambda(S), \lambda(D)) \cap \lambda(p(D))$. Therefore, $p(\lambda(D)) = \lambda(p(D))$ implies $F^p(\lambda(S), \lambda(D)) = \lambda(F^p(S, D))$ (that is, *scale invariance* of p implies *scale invariance* of F^p).

For the other direction, suppose $p(\lambda(D)) \neq \lambda(p(D))$ for some $D \in \mathbb{R}^{n\times n}$ and $\lambda \in \Lambda$. For each $\omega \in \mathbb{R}^n_{++}$ satisfying $\sum \omega_i = 1$ and $r > \sum \omega_i \bar{d}_i(\lambda(D))$, let $T^{r,\omega} = \{x \in \mathbb{R}^n \mid \sum \omega_i x_i \leqq r \text{ and } x \geqq \underline{d}(\lambda(D))\}$ and note that $(T^{r,\omega}, \lambda(D)) \in \mathcal{B}_{\neq}$. Now, by $p(\lambda(D)) \neq \lambda(p(D))$ and the fact that both $p(\lambda(D))$ and $\lambda(p(D))$ are images of continuous functions, there is $\omega^* \in \mathbb{R}^n_{++}$ satisfying $\sum \omega^*_i = 1$ and $r^* > \sum \omega^*_i \bar{d}_i(\lambda(D))$ such that $WPO(T^{r^*,\omega^*}, \lambda(D)) \cap p(\lambda(D)) \neq WPO(T^{r^*,\omega^*}, \lambda(D)) \cap \lambda(p(D))$. But the expression on the left is $F^p(T^{r^*,\omega^*}, \lambda(D))$ and the expression on the right is $\lambda(F^p(\lambda^{-1}(T^{r^*,\omega^*}), D))$. This contradicts *scale invariance* of F^p (therefore, *scale invariance* of F^p implies *scale invariance* of p). □

Remark 3. On $\mathcal{B}_{=}$, the Nash (1950) bargaining rule uniquely satisfies *weak Pareto optimality, symmetry, scale invariance*, and an "independence of irrelevant alternatives" property. On $\mathcal{B}$, a large class of monotone path rules satisfy all of these properties. They are characterized by *scale invariant* and "symmetric" path generators.

Two-agent problems are central in bargaining theory. We next show that for this case, *scale invariant* monotone path rules have a very simple form (for its proof, please see the Appendix A).

Proposition 4. *On $\mathcal{B}^2_{\neq}$, a* scale invariant *monotone path rule can be completely characterized by at most eight distinct paths. On $\mathcal{B}^2_{>=}$, these paths are either vertical or horizontal.*

With more than two agents, Proposition 4 no more holds: constructing a monotone path rule potentially involves the specification of an infinite number of paths.[6]

3.2. Cardinal Egalitarian rule

In this subsection, we analyze the implications of *symmetry* together with *strong monotonicity* and *scale invariance*. *Symmetry* is a weakening of "anonymity" which requires that agents with identical payoff functions receive the same payoff. It thus concerns negotiations where the agents have equal "bargaining power." Implications of *symmetry* has been analyzed by many authors including Nash (1950), Kalai and Smorodinsky (1975), and Kalai (1977).

[6] This has got to do with the fact that with two agents, all disagreement matrices are divided into eight equivalence classes: two matrices in the same class are related by a positive affine transformation. In an equivalence class, it is sufficient to specify a monotone path for one matrix; scale invariance then defines the paths of the other matrices. With more agents however, the number of equivalence classes becomes infinite.

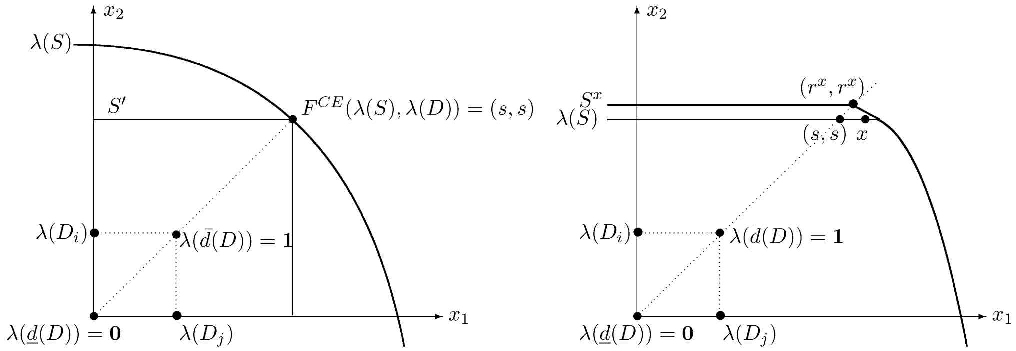

Fig. 4. Constructing S' (on the left) and S^x (on the right) in the proof of Theorem 5.

We first present a *symmetric* monotone path rule. The Cardinal Egalitarian rule, F^{CE} picks the maximizer, for each $D \in \mathbb{R}^{n\times n}$, of the linear monotone path that passes through $\underline{d}(D)$ and $\bar{d}(D)$: $p^{CE}(D) = \{\bar{d}(D) + r(\bar{d}(D) - \underline{d}(D)) \mid r \in \mathbb{R}_+\}$; that is, $F^{CE} = F^{p^{CE}}$.[7]

The Cardinal Egalitarian rule is well-defined for all nonanonymous-disagreement problems, $\mathcal{B}_{\neq}$, independent of the number of agents. Additional to the properties stated in the next theorem, it is *set-continuous*. Also, it is a "nondecomposable" rule. That is, it can not be written as a composition of a rule from the Nash domain and a function that transforms disagreement matrices to disagreement vectors (for more on decomposability, see Kıbrıs and Tapkı (2007)).

The Cardinal Egalitarian solution to a problem utilizes, for each agent, the difference between his maximum and minimum disagreement payoffs. Agents for whom this difference is higher receive a higher share of the surplus (over $\bar{d}(D)$) than others. As a result of this feature, the Cardinal Egalitarian rule violates "disagreement payoff monotonicity"; that is, an increase in an agent's disagreement payoff can make him worse-off (because, it can decrease the aforementioned difference). Also note that the Cardinal Egalitarian solution can be very sensitive to small changes in D. This sensitivity increases as $\underline{d}(D)$ and $\bar{d}(D)$ get closer to each other.

The following result analyzes the properties of the Cardinal Egalitarian rule.

Theorem 5. *The Cardinal Egalitarian rule, F^{CE}, is* weakly Pareto optimal, strongly monotonic, scale invariant, *and* symmetric *on $\mathcal{B}_{\neq}$. Furthermore on $\mathcal{B}^2_{><}$, it is the unique rule that satisfies these properties.*

Proof. It is straightforward to show that F^{CE} satisfies these properties. Conversely let F be any rule on $\mathcal{B}^2_{><}$ that satisfies them. Take any $(S, D) \in \mathcal{B}^2_{><}$. We want to show that $F(S, D) = F^{CE}(S, D)$.

Consider the positive affine transformation $\lambda \in \Lambda$ such that $\lambda_i(x) = \frac{x_i - \underline{d}_i(D)}{\bar{d}_i(D) - \underline{d}_i(D)}$ for $i \in N$. Note that $\lambda(\bar{d}(D)) = \mathbf{1}$ and $\lambda(\underline{d}(D)) = \mathbf{0}$. Then, by definition $F^{CE}(\lambda(S), \lambda(D)) = (s, s)$, for some $s > 1$. Consider $S' = 0\text{-}comp\{(s, s)\}$. Note that $\lambda(D) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ or $\lambda(D) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Since S' is symmetric, $(S', \lambda(D))$ is a symmetric problem. Then, by *symmetry* and *weak Pareto optimality* of F, $F(S', \lambda(D)) = (s, s)$. Since $\lambda(S) \supseteq S'$, *strong monotonicity* of F implies $F(\lambda(S), \lambda(D)) \geqq F(S', \lambda(D))$.

Now if $(s, s) \in PO(\lambda(S), \lambda(D))$ (as in Fig. 4, left), then $F(\lambda(S), \lambda(D)) = (s, s) = F^{CE}(\lambda(S), \lambda(D))$. Alternatively, assume that $(s, s) \in WPO(\lambda(S), \lambda(D))$ (as in Fig. 4, right). Suppose $F(\lambda(S), \lambda(D)) = x \geqslant (s, s)$. Let $r^x \in \mathbb{R}$ be such that $s < r^x < \max\{x_1, x_2\}$. Let $S^x \subseteq \mathbb{R}^n$ be such that $S^x = conv\{0\text{-}comp\{r^x, r^x\}, \lambda(S)\}$. Note that $(S^x, \lambda(D)) \in \mathcal{B}_{\neq}$ and $F^{CE}(S^x, \lambda(D)) = (r^x, r^x) \in PO(S^x, \lambda(D))$. So by the previous argument, $F(S^x, \lambda(D)) = (r^x, r^x)$. Also since $s < r^x$, $S^x \supseteq \lambda(S)$. Thus by *strong monotonicity* of F, $F(S^x, \lambda(D)) = (r^x, r^x) \geqq x = F(\lambda(S), \lambda(D))$, contradicting $r^x < \max\{x_1, x_2\}$. Therefore, $F(\lambda(S), \lambda(D)) = (s, s)$.

Finally, by *scale invariance* of F and F^{CE}, $F(S, D) = \lambda^{-1}(F(\lambda(S), \lambda(D))) = \lambda^{-1}(F^{CE}(\lambda(S), \lambda(D))) = F^{CE}(S, D)$. □

Since there are no symmetric problems in $\mathcal{B}^2_{>>} \cup \mathcal{B}^2_{>=}$, any rule is *symmetric* on those classes of problems. Therefore, the properties of Theorem 5 do not pinpoint a single rule on $\mathcal{B}^2_{>>} \cup \mathcal{B}^2_{>=}$. Also, we do not state uniqueness for more than two agents. The following is an example of a rule that satisfies all the above properties and that is different from the Cardinal Egalitarian rule for problems with more than two agents. Let $\xi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined as

[7] We call this rule the Cardinal Egalitarian rule because it is a version of the Egalitarian rule (Kalai, 1977) that is covariant under *cardinal* (i.e. positive affine) transformations. To see this, let (S, D) be called a normalized problem when D is any one of the matrices $\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]\}$. For such problems, the Cardinal Egalitarian rule coincides with the Egalitarian rule. Any other problem is a cardinal transformation of a normalized problem and its Cardinal Egalitarian solution is the same cardinal transformation of the Egalitarian solution to the associated normalized problem.

$$\big[\xi(D)\big]_{ji} = \begin{cases} D_{ji} & \text{if } D_{ji} = \underline{d}_i(D), \\ \min\{D_{ji} \mid D_{ji} \neq \underline{d}_i(D)\} & \text{otherwise.} \end{cases}$$

Then, let $F^{\xi} \in \mathcal{F}_{\neq}$ be defined as $F^{\xi}(S, D) = F^{CE}(S, \xi(D))$ for each $(S, D) \in \mathcal{B}_{\neq}$.

Remark 6. The properties *weak Pareto optimality, strong monotonicity, set-continuity, scale invariance*, and *symmetry* are logically independent. For this, note that the rule F^1 *defined as* $F^1(S, D) = \bar{d}(D)$ satisfies all properties except *weak Pareto optimality*. The rule F^2 defined as $F^2(S, D) = (\max\{x_1 \mid x \in S \text{ and } x_2 = \underline{d}_2(D)\}, \underline{d}_2(D))$ satisfies all properties except *symmetry*. The rule F^3 defined as $F^3(S, D) = \arg\max_{x \in S} \min_{i \in N} x_i - \underline{d}_i(D)$ satisfies all properties except scale invariance. Finally, let $m_i^* = \max\{x_i \mid x \in S \text{ and } x \geqq \underline{d}(D)\}$ and define F^4 as $F^4(S, D) = \arg\max_{x \in S} \min_{i \in N} \frac{x_i - \bar{d}_i(D)}{m_i^* - \bar{d}_i(D)}$. This rule satisfies all properties except *strong monotonicity*. Finally, let F^5 coincide with F^{CE} everywhere except $\mathcal{B}^2_{>>}$. There, let F^5 be as explained in Claim 1 of the proof of Proposition 4 where $G_i = [\mathbf{1}, \mathbf{1} + 1e_i] \cup \{\mathbf{1} + 1e_i + re_j \mid r \in \mathbb{R}_+\}$ for $i \neq j$. Since each G_i violates Condition (ii) in the definition of the path generators, F^5 violates *set-continuity* but it satisfies all the other properties above.

4. Conclusion

In this paper, we analyze bargaining processes where the disagreement outcome depends on who terminates the negotiations. We present a cooperative bargaining model that captures this feature and we carry out an axiomatic analysis of the implications of *strong monotonicity, scale invariance*, and *symmetry* on *weakly Pareto optimal* bargaining rules.

One restriction of our model is that it does not specify the outcome of a coalition of agents jointly terminating the negotiations. Modeling coordinated disagreement by a coalition would bring in questions about the bargaining process in that coalition and move us further towards a non-transferable utility game analysis. In this paper, we remain in the bargaining framework and only consider individual deviations.

In our opinion, it is essential to complement our analysis with a noncooperative approach. Studies such as Shaked and Sutton (1984), Ponsatí and Sákovics (1998), and Corominas-Bosch (2000) present an excellent starting point. The equilibria of these models, however, use only partial information on the implications of disagreement. For example, an agent's payoff from his opponent leaving has no effect on the equilibrium (except in extreme cases where the problem's individually rational region is empty).[8] Therefore, the design and analysis of noncooperative bargaining games which, in equilibrium, use full disagreement information remains an important open question.

Appendix A

Proof of Proposition 4. Let F^p be a *scale invariant* monotone path rule. Note that $\mathcal{B}^2_{\neq} = \mathcal{B}^2_{>>} \cup \mathcal{B}^2_{>=} \cup \mathcal{B}^2_{><}$. Let $G_1 = p(\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right])$, $G_2 = p(\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right])$, $G_3 = p(\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right])$, $G_4 = p(\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right])$, $G_{11} = p(\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right])$, $G_{12} = p(\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right])$, $G_{21} = p(\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right])$, and $G_{22} = p(\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right])$.

Claim 1. On $\mathcal{B}^2_{>>}$, G_1 and G_2 suffice to describe F^p. To see this, let $(S, D) \in B^2_{>>}$ and assume $D_i > D_j$. Let $\lambda_1(x) = \frac{x - D_{j1}}{D_{i1} - D_{j1}}$ and $\lambda_2(x) = \frac{x - D_{j2}}{D_{i2} - D_{j2}}$. Note that $\lambda(D_i) = \mathbf{1}$ and $\lambda(D_j) = \mathbf{0}$. Thus $p(\lambda(D)) = G_i$. Then, by *scale invariance*, $F^p(S, D) = \lambda^{-1}(F^p(\lambda(S), \lambda(D))) = \lambda^{-1}(WPO(\lambda(S), \lambda(D)) \cap G_i)$ only uses the path G_i.

Claim 2. On $\mathcal{B}^2_{><}$, G_3 and G_4 suffice to describe F^p. The proof is similar to Claim 1.

Claim 3. On $\mathcal{B}^2_{>=}$, G_{11}, G_{12}, G_{21}, and G_{22} suffice to describe F^p. To see this, let $(S, D) \in B^2_{>=}$ and assume $D_{ik} > D_{jk}$ and for $l \neq k$, $D_{il} = D_{jl}$. Let $\lambda_k(x) = \frac{x - D_{jk}}{D_{ik} - D_{jk}}$ and for $l \neq k$, let $\lambda_l(x) = x - D_{il}$. Note that $\lambda(D_i) = e_k$ and $\lambda(D_j) = \mathbf{0}$. Therefore, $\lambda(p(D)) = G_{ik}$. Then by *scale invariance*, $F^p(S, D) = \lambda^{-1}(F^p(\lambda(S), \lambda(D))) = \lambda^{-1}(WPO(\lambda(S), \lambda(D)) \cap G_{ik})$ only uses the path G_{ik}.

By Claims 1, 2 and 3, F^p can be completely characterized by at most eight distinct paths (two for $\mathcal{B}^2_{>>}$, two for $\mathcal{B}^2_{><}$, and four for $\mathcal{B}^2_{>=}$).

Claim 4. The paths G_{ik} on $\mathcal{B}^2_{>=}$ are either vertical or horizontal (see Fig. 5). To see this, let D be as in Fig. 5 (that is, $D_i = e_k$ and $D_j = \mathbf{0}$). Thus $p(D) = G_{ik}$. Note that, for each $r \in \mathbb{R}_+$ and $\lambda^r \in \Lambda$ defined as $\lambda^r_k(x) = x$ and $\lambda^r_l(x) = rx$, we have $\lambda^r(D) = D$. Thus $p(\lambda^r(D)) = G_{ik}$. Now, let $y \in G_{ik}$ be such that $y_l > 0$. (If there is no such y, G_{ik} is a vertical line on the k-axis and we are done.) Then $\lambda^r(y) \in \lambda^r(p(D)) = p(\lambda^r(D)) = G_{ik}$. Since $\lambda^r(y) \in G_{ik}$ holds for all r, G_{ik} is horizontal. $\square$

[8] Thus, our conjecture is that these noncooperative games implement *decomposable* rules.

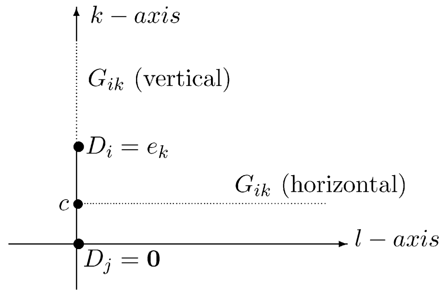

Fig. 5. The configuration of the monotone paths in Proposition 4.

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