Nash Bargaining in Ordinal Environments*

By Özgür Kıbrıs†

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Abstract

We analyze the implications of Nash's (1950) axioms in ordinal bargaining environments; there, the scale invariance axiom needs to be strenghtened to take into account all order-preserving transformations of the agents' utilities. This axiom, called ordinal invariance, is a very demanding one. For two-agents, it is violated by every strongly individually rational bargaining rule. In general, no ordinally invariant bargaining rule satisfies the other three axioms of Nash. Parallel to Roth (1977), we introduce a weaker independence of irrelevant alternatives axiom that we argue is better suited for ordinally invariant bargaining rules. We show that the three-agent Shapley-Shubik bargaining rule uniquely satisfies ordinal invariance, Pareto optimality, symmetry, and this weaker independence of irrelevant alternatives axiom. We also analyze the implications of other independence axioms.

Keywords: Bargaining, Shapley-Shubik rule, ordinal invariance, independence of irrelevant alternatives, brace.

JEL Classification: C78

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†Faculty of Arts and Social Sciences, Sabancı University, Orhanlı, Tuzla, Istanbul, 34956 Turkey. Fax: +90-216-483-9250. E-mail: ozgur@sabanciuniv.edu

1 Introduction

In his seminal work, Nash (1950) postulates four axioms that he argues solutions to bargaining problems do satisfy: **scale invariance**, **Pareto optimality**, **symmetry**, and **independence of irrelevant alternatives**. He proves that only one bargaining rule, now commonly known as the Nash bargaining rule, satisfies these axioms.

In this paper, we analyze the implications of Nash axioms in ordinal environments, that is, in bargaining situations where the agents' preferences are only restricted to be complete, transitive, and continuous. For ordinal environments, the *scale invariance* axiom of Nash is not sufficient to ensure the consistency of the bargaining solution (or equivalently, the invariance of the physical bargaining outcome) with respect to changes in the utility representation of the agents' underlying preferences. It needs to be replaced with a stronger axiom called **ordinal invariance**. Unfortunately, the Nash bargaining rule violates this axiom.

The previous literature on ordinal bargaining is quite small. Shapley (1969) shows that for two agents, no *strongly individually rational* bargaining rule satisfies *ordinal invariance*. However, it later turns out that this negative result is limited to two agents; an *ordinally invariant* and *strongly individually rational* bargaining rule for three agents appears in Shubik (1982). We will refer to it as the **Shapley-Shubik rule**.[1]

For Pareto surfaces with more than two agents, Sprumont (2000) proves the existence of an **ordinal basis**, that is, a subclass of surfaces which, through order-preserving transformations, generates all Pareto surfaces, and which is minimal. Furthermore, he constructs a "sufficiently symmetric" *ordinal basis* for three-agent surfaces. Kıbrıs (2004b) uses Sprumont's construction to describe a class of three-agent "ordinally normalized problems" which, through order-preserving transformations of the agents' utilities, generates all bargaining problems.[2] On this class, the *Shapley-Shubik rule* coincides with the *Egalitarian* (Kalai,

[1] There is no reference on the origin of this rule in Shubik (1982). However, Thomson [16] attributes it to Shapley. Furthermore, Roth (1979), in pages 72-73, mentions a three-agent ordinal bargaining rule proposed by Shapley and Shubik (1974, Rand Corporation, R-904/4) which, considering the scarcity of ordinal rules in the literature at the time, is most probably the same bargaining rule.

[2] The class of ordinally normalized problems can be interpreted as the ordinal counterpart of the class of 0-1 normalized problems (that is, agents' disagreement payoffs are normalized to 0 and aspiration payoffs to

1977) and the *Kalai-Smorodinsky* (1975) rules since it always chooses the maximizer of a Leontief type order. Kıbrıs also shows that the *Shapley-Shubik* rule is the only *symmetric* member of a class of (ordinal) monotone path rules and that these are the only *ordinally invariant* rules that are *Pareto optimal* and "monotonic".

For problems with more than three agents, our knowledge is quite restricted. It follows from Sprumont (2000) that for such problems *ordinally invariant*, *Pareto optimal*, and *strictly individually rational* bargaining rules exist. Furthermore, Safra and Samet (2004, 2005) propose two generalizations of the *Shapley-Shubik* formula to this case. However, since these formulas (and in general, formulas defining non-dictatorial ordinal rules) are quite complicated,[3] construction of a class of ordinally normalized problems (over which the rule under question has a simpler definition) significantly helps to facilitate an axiomatic analysis.[4] Such a construction does not yet exist for more than three agents.[5] It is for this reason that the analysis of this paper is restricted to the three-agent case.

Three-agent bargaining problems are also of independent interest since they can be used to model a set of real life applications where a third party with individual interests on the resulting agreement is involved in the negotiations between two parties. As an example, consider an international conflict between two countries where a third country or the United Nations (either of which might conceivably have different preferences than either of the two parties) are also involved in the negotiations.

Ordinally invariant rules do not satisfy all of the other three axioms of Nash (1950). Among them we consider *Pareto optimality* and *symmetry* to be more basic. We therefore look for *ordinally invariant, Pareto optimal*, and *symmetric* rules that satisfy a weaker form

1) for cardinal bargaining rules. Both classes have the property that any physical bargaining problem has a utility image in this subclass (and therefore, any problem outside the class is equivalent to a member of this class).

[3] The Shapley-Shubik and the Safra-Samet solutions to arbitrary bargaining problems are defined as the limit of a sequence constructed on the problem's Pareto surface.

[4] For example, Kıbrıs (2004b) utilizes the fact that on the class of ordinally normalized problems, the Shapley-Shubik rule coincides with the Egalitarian rule.

[5] Note that the issue is not the existence of a normalized class but that of constructing one that is desirable in the aforementioned sense. In fact, Sprumont presents a highly asymmetric construction for more than three agents and notes that "it may be of little use to define attractive solutions".

of **independence of irrelevant alternatives (IIA)**: the original axiom of Nash (1950) requires that the solution to a bargaining problem should not change as some of the alternatives (other than the original solution) cease to be feasible, but the problem's disagreement point remains unchanged. *IIA* has been frequently criticized on the basis that it requires the bargaining rule to be too insensitive to changes in the set of feasible utility profiles (*e.g.* see Luce and Raiffa, 1957). Based on this criticism, Roth (1977) proposes a *weakening* of the axiom appropriate for cardinal bargaining environments; this axiom, called **independence of irrelevant alternatives other than the aspiration points (IIA-aspiration)**, requires that the problem's aspiration points (additional to the disagreement point) remain unchanged as the feasible set contracts. Since an affine transformation with two fixed points can only be the identity mapping, *IIA-aspiration* effectively rules out the possibility of comparing two alternative (cardinal) utility representations of the same physical problem. It is also useful to note that *IIA-aspiration* is the restriction of *IIA* to a "cardinal basis" of bargaining problems, namely the class of 0-1 normalized problems.

For *Pareto optimal* rules, the *scale invariance* and *IIA* properties do not make contradictory statements about how solutions to problem pairs should be related.[6] This however is not the case when *scale invariance* is replaced with *ordinal invariance* (*e.g.* see Example 4). Due to the nonlinear transformations allowed, it becomes essential to separate the jurisdictions of *ordinal invariance* from a compatible IIA property. Thus we repeat the exercise of Roth (1977) for ordinal rules and observe that restricting the comparison to problems which have a common set of Pareto optimal points, called the "extended brace", rules out the possibility of comparing two alternative (ordinal) utility representations of the same physical problem. Motivated by this observation, we propose **independence of irrelevant alternatives other than the extended brace (IIA-extended brace)** which only considers contractions of the feasible set in which the extended brace of the original problem (additional to the original solution and the disagreement point) remain feasible. As in IIA-

[6]Without Pareto optimality, the two requirements can contradict. Consider for example $d = (0,0)$, $S = conv\{(0,0),(1,0),(0,1)\}$ and $S' = conv\{(0,0),(\frac{1}{2},0),(0,\frac{1}{2})\}$. The problems (S,d) and (S',d) are related to each other through both an affine transformation of the agents utilities and by a contraction of S to S'. For example if $F(S,d) = (\frac{1}{4},\frac{1}{4})$, scale invariance requires $F(S',d) = (\frac{1}{8},\frac{1}{8})$ and IIA requires $F(S',d) = (\frac{1}{4},\frac{1}{4})$.

aspiration, this axiom is nothing but a restriction of Nash's IIA property to an "ordinal basis" of bargaining problems that will be introduced in Section 2. *IIA-extended brace* is weaker than both *IIA-aspiration* and *IIA*.

In Section 2, we demonstrate that the extended brace of a problem is constructed via a set of Pareto optimal points which we call the "brace". The aspiration points of a problem summarize what each agent can achieve if he dictates the decision taken in the bargaining process. The brace points have a similar interpretation; they generalize the idea of dictatorship from individuals to coalitions (for further discussion, see *Subsection 2.1*). We thus also propose a *stronger version of the above property* called **independence of irrelevant alternatives other than the brace (IIA-brace)**. It requires the brace of the original problem (additional to the original solution and the disagreement point) remain feasible in a contraction of the feasible set. As will be discussed below, the relation between these two properties resemble that between *IIA* and *IIA-aspiration*.

Our results are as follows. We first show that a class of ordinal rules, including the Shapley-Shubik rule satisfies *IIA-extended brace* and all of these rules coincide on a large class of bargaining problems. We next show that the *Shapley-Shubik* bargaining rule uniquely satisfies *ordinal invariance*, *Pareto optimality*, *symmetry*, and *IIA-brace*. These results are of interest for three reasons. First, they suggests that similar axioms lead to two different practices in cardinal and ordinal approaches to bargaining: while the product of the agents' utility gains is maximized in the former, the "utility" of the worst-off agent is maximized in the latter. Second, when analyzed in relation to the findings of Kıbrıs (2004b), these results suggest that the Shapley-Shubik rule is the ordinal counterpart of both the Nash and the Kalai-Smorodinsky bargaining rules which, in the cardinal approach seem to be based on different principles. Finally, this observation might provide some intuition on explaining why equal division is the most prominent outcome in bargaining experiments: this is what the Shapley-Shubik rule proposes for any three-agent bargaining problem on the allocation of a single divisible good (such as money).

Finally we show that *IIA-brace* is the strongest axiom of its kind which is satisfied by *ordinally invariant*, *Pareto optimal*, and *symmetric* bargaining rules. Furthermore, such rules all violate the *IIA-aspiration* axiom of Roth (1977).

2 Model

Let $N = \{1, 2, 3\}$ be the set of **agents**. Vector inequalities are written as $\leqq$, $\leq$, and $<$. For each $i \in N$, $e(i)$ stands for the vector in $\mathbb{R}^N_+$ whose ith coordinate is 1 and all other coordinates are 0. Let Π be the set of all permutations π of N. For each $\pi \in \Pi$, $x \in \mathbb{R}^N$, and $S \subset \mathbb{R}^N$, let $\pi(x) = (x_{\pi(i)})_{i \in N}$ and $\pi(S) = \{\pi(y) : y \in S\}$. For each $X \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$, $conv\{X\}$ is the convex hull of X and $comp\{X \mid x\} = \{y \in \mathbb{R}^N : y \geqq x \text{ and } y \leqq z \text{ for some } z \in X\}$ is the comprehensive hull of X down to x.

A pair $(S, d) \in 2^{\mathbb{R}^N} \times \mathbb{R}^N$ is a **bargaining problem** if (i) S is compact, (ii) $d \in S$, and (iii) there is $x \in S$ with $x > d$. A bargaining problem (S, d) is **strictly d-comprehensive** if for each $x \in S$ and $y \in \mathbb{R}^N$ such that $d \leq y \leq x$, $y \in S$ and there is $z \in S$ such that $z > y$. This axiom has two implications: first, that utility is disposable down to the disagreement point; second, that any individually rational and weakly Pareto optimal point is also Pareto optimal. Let $\mathcal{B}$ denote the set of all strictly d-comprehensive bargaining problems.

For each $(S, d) \in \mathcal{B}$ and $i \in N$, agent i's aspiration payoff is $a_i(S, d) = \max\{s_i : (s_i, d_{-i}) \in S\}$ and his **aspiration point** is $asp_i(S, d) = (a_i, d_{-i})$. Let $P(S, d)$ denote the set of **Pareto optimal profiles in S**: $x \in P(S, d)$ if and only if $x \in S$ and for each $y \geq x$, $y \notin S$. Let $I(S, d)$ denote the set of **individually rational profiles in S**: $x \in I(S, d)$ if and only if $x \in S$ and $x \geqq d$. Let $IP(S, d)$ denote the set of **Pareto optimal and individually rational profiles in S**. Let $\mathcal{B}_I = \{(S, d) \in \mathcal{B} \mid S = I(S, d)\}$.

For each $i \in N$, let f_i be an increasing continuous function on $\mathbb{R}$ and define $f = (f_i)_{i \in N} : \mathbb{R}^N \to \mathbb{R}^N$ by $f(x) = (f_i(x_i))_{i \in N}$ for each $x \in \mathbb{R}^N$. Let $\mathcal{F}$ denote the set of such functions. Two problems (S, d), $(S', d') \in \mathcal{B}$ are **ordinally equivalent** if there is $f \in \mathcal{F}$ such that $f(S) = S'$ and $f(d) = d'$. Otherwise, they are called **ordinally distinct**. A subclass $\mathcal{B}' \subseteq \mathcal{B}$ **ordinally spans** $\mathcal{B}$ if for any problem $(S, d) \in \mathcal{B}$, there is a problem $(S', d') \in \mathcal{B}'$ which is *ordinally equivalent* to it. The subclass $\mathcal{B}'$ is an **ordinal basis** of $\mathcal{B}$ if it *ordinally spans* $\mathcal{B}$ and all problems in $\mathcal{B}'$ are *ordinally distinct*.

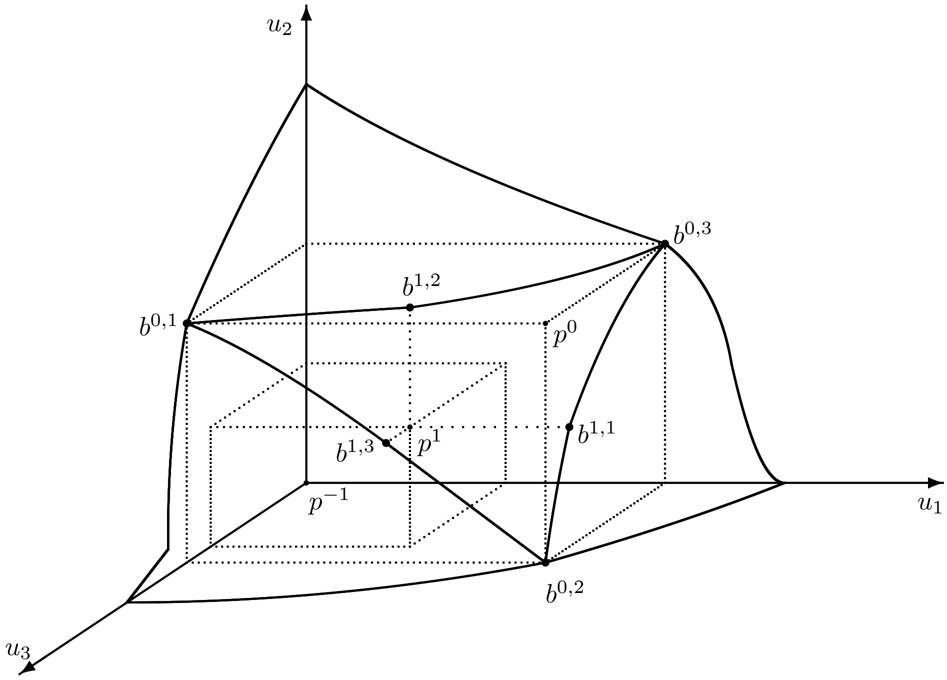

Figure 1: Constructing the brace of an arbitrary problem.

2.1 Brace and ordinally normalized problems

Let $(S,d) \in \mathcal{B}$. Define $p^{-1}(S,d) = d$ and for each $n \in \mathbb{N}$ define $p^n(S,d) \in \mathbb{R}^N$ to be such that

$$\begin{aligned}
b^{n,1}(S,d) \quad &: \quad = (p_1^{n-1}(S,d), p_2^n(S,d), p_3^n(S,d)) \in P(S,d),\\
b^{n,2}(S,d) \quad &: \quad = (p_1^{n}(S,d), p_2^{n-1}(S,d), p_3^n(S,d)) \in P(S,d), \text{ and}\\
b^{n,3}(S,d) \quad &: \quad = (p_1^{n}(S,d), p_2^n(S,d), p_3^{n-1}(S,d)) \in P(S,d).
\end{aligned}$$

The sequence $\{p^n(S,d)\}_{n\in\mathbb{N}}$ is uniquely defined and it is convergent. Also note that for each $i \in N$, $\lim_{n\to\infty} p^n(S,d) = \lim_{n\to\infty} b^{n,i}(S,d)$.The **brace** of (S,d) is a subset of $IP(S,d)$ and it is defined as

$$br(S,d) = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in N} \{b^{n,i}(S,d)\}. \qquad \text{(see } \textit{Figure 1}\text{)}$$

As seen in *Figure 1*, the payoff vectors $p^n(S,d)$ are either "above" or "below" $IP(S,d)$ and they are related to each other through their projections on this set. These projections on $IP(S,d)$ constitute the brace. For example, projecting $p^0(S,d)$ on the Pareto set of S by *decreasing* agent i's payoff results in the brace point $b^{0,i}(S,d)$ in which agent i receives exactly his payoff in $p^{-1}(S,d)$. Similarly, projecting $p^1(S,d)$ on the Pareto set of S by *increasing* agent

i's payoff results in the brace point $b^{1,i}(S,d)$ in which agent i receives exactly his payoff in $p^0(S,d)$.

The sequences $\{p^n(S,d)\}_{n\in\mathbb{N}}$ and $\{b^{n,i}(S,d)\}_{n\in\mathbb{N}}$ also have an intuitive interpretation. Each brace point $b^{n,i}(S,d)$ represents a "coalitional ideal payoff" for a two-agent coalition. To see this, consider a problem in which decisive two-agent coalitions can form and the agents compete to be a member of the decisive coalition.[7] If formation of such a coalition is part of the bargaining process, in any negotiating pair, the agents' conjectures on what they can achieve from negotiating with the third agent serves as a threat point. This class of problems, called **multicoalitional bargaining problems,** is introduced by Bennett (1997) who also introduced and characterized a solution rule, the **Bennett rule**. Bennett solutions to a problem also correspond to the stationary subgame perfect equilibria of a coalition formation game. Kıbrıs (2004a) shows that each $(S,d) \in \mathcal{B}$ generates a unique sequence of Bennett solutions which is equal to the $\{p^n(S,d)\}_{n\in\mathbb{N}}$ sequence introduced above.

Intuitively, given a (standard) bargaining problem $(S,d) \in \mathcal{B}$, the unique *Bennett solution* of the multicoalitional bargaining problem associated with it is $p^0(S,d)$. For each $i \in N$, the *brace* point $b^{0,i}(S,d) = (d_i, p^0_{-i}(S,d))$ summarizes the (feasible) payoff distribution if the two-agent coalition excluding i forms. The profile $p^0(S,d)$, being infeasible, generates a reduced multicoalitional bargaining problem in which each two-agent coalition negotiates on how to allocate the remaining payoffs in case the third agent, say i, receives $p^0_i(S,d)$.[8] The unique *Bennett solution* to this new problem is the profile $p^1(S,d)$ which for each $i \in N$, leads to the *brace* point $b^{1,i}(S,d) = (p^0_i(S,d), p^1_{-i}(S,d))$ and generates a third multicoalitional problem. Continuing this way generates the whole sequence.

We use the *brace* to define a subclass of $\mathcal{B}$. Let $(S,d) \in \mathcal{B}$ and define $b^{-1,1}(S,d) := (asp_1(S,d), d_2, d_3)$, $b^{-1,2}(S,d) := (d_1, asp_2(S,d), d_3)$, and $b^{-1,3}(S,d) := (d_1, d_2, asp_3(S,d))$. Note that $\{b^{-1,i}(S,d)\}_{i\in N}$ are not brace points but aspiration points. For each $n \in \mathbb{N}$ and

[7] Mathematically, this new problem will be a non-transferable utility game where the feasible set of a two-agent coalition is the projection of the grand coalition's feasible set on their utility subspace. Note that any three-party negotiation in which the outcome is determined through majority voting will be of this form.

[8] Note that in this case, the first round of negotiations lead to two extreme divisions for these two agents. Therefore, this second round can be interpreted as an attempt of the agents to insure themselves against the outcome in which they receive nothing.

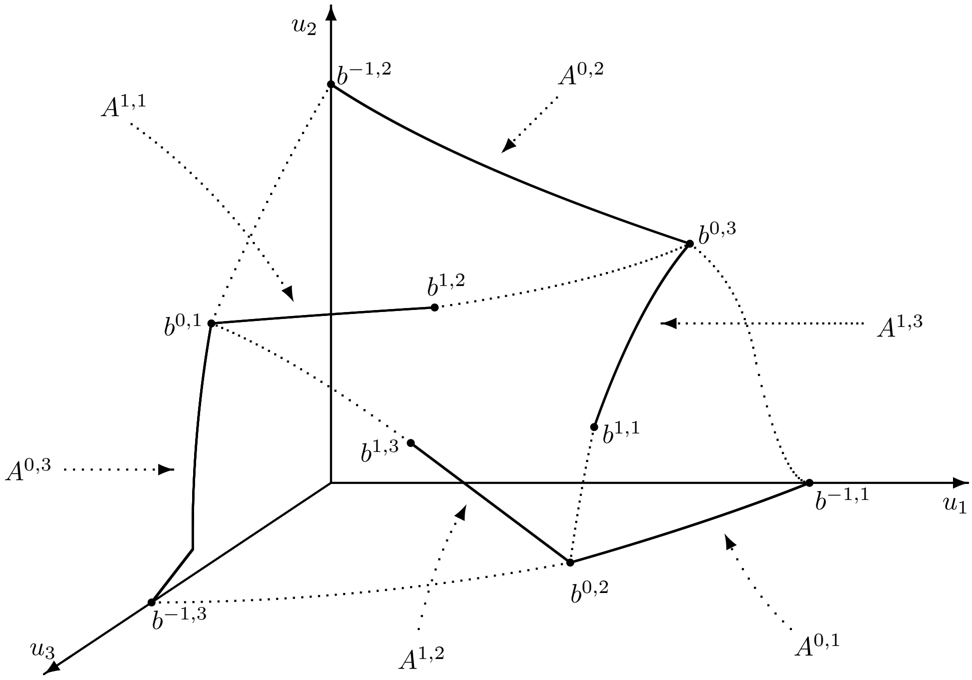

Figure 2: Constructing the extended brace of an arbitrary problem.

$i \in N$, let $A^{n,i}(S,d)$ be the Pareto optimal curve that connects $b^{n-1,i}(S,d)$ and $b^{n,i+1}(S,d)$ (with the convention that for $i=3$, $i+1=1$) as follows:

$$A^{n,i}(S,d) = \left\{ \begin{array}{c} x \in P(S,d) : \text{for each } j \in N, \\ \min\{b_j^{n-1,i}(S,d), b_j^{n,i+1}(S,d)\} \leqq x_j \leqq \max\{b_j^{n-1,i}(S,d), b_j^{n,i+1}(S,d)\} \end{array} \right\}$$

The **extended brace** of (S,d) is (see *Figure 2*)

$$A(S,d) = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in N} A^{n,i}(S,d).$$

As seen in *Figure 2*, the extended brace "extends" the brace by adding to it Pareto optimal payoff vectors that connect two brace points in which the payoff of an agent is constant (such as the part connecting $b^{0,3}(S,d)$ and $b^{1,1}(S,d)$ in *Figure 2*) and these connections are chosen in such a way that the resulting set is a union of three helices, each originating from one of the three aspiration points of (S,d).

For bargaining problems whose set of individually rational and Pareto optimal points coincide with the unit simplex, the brace and the extended brace are trivially defined. Let $d^* = 0$ and $S^* = comp\{conv\{e(1), e(2), e(3)\} \mid d^*\}$. Then, $p^0(S^*,d^*) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $p^1(S^*,d^*) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, and for $n \geq 2$, $p^n(S^*,d^*) = \frac{1}{2}(p^{n-1}(S,d) + p^{n-2}(S,d))$. This implies that for each $i \in$

N, $b^{-1,i}(S^*, d^*) = e(i)$ and for each $n \in \mathbb{N}$, $b^{n,i}(S^*, d^*) = \frac{1}{2}(b^{n-1,i-1}(S^*, d^*) + b^{n-1,i+1}(S^*, d^*))$. That is, each brace point of this problem is obtained as the midpoint average of two other brace points and the averaging starts with the unit vectors. The **normalized extended brace**, A^*, is defined as the extended brace of this problem: $A^* = A(S^*, d^*)$. A bargaining problem $(S, d) \in \mathcal{B}$ is **ordinally normalized** (see *Figure 3*) if (i) $d = 0$, (ii) for each $i \in N$, $asp_i(S, d) = e(i)$, and (iii) $A(S, d) = A^*$. Let $\mathcal{B}_{ord}$ denote the set of all such problems in $\mathcal{B}$.

The following result states that for each $(S, d) \in \mathcal{B}$ there are order-preserving transformations of the agents utilities $(f_i)_{i \in N} = f \in \mathcal{F}$ such that the transformed problem $(f(S), f(d))$ is in $\mathcal{B}_{ord}$. Furthermore, the transformation is unique on the individually rational part $I(S, d)$ of (S, d).

Proposition. (Kıbrıs, 2004b) *The subclass B_{ord} ordinally spans B. Moreover, $B_I \cap B_{ord}$ is an ordinal basis of B_I.*

This result implies that to define an *individually rational* and *ordinally invariant* bargaining rule, it suffices to describe the rule on the class of ordinally normalized problems, $\mathcal{B}_{ord}$. The ordinal invariance axiom then determines the solution to an arbitrary problem in relation to the rule's solution to an ordinally equivalent problem in $\mathcal{B}_{ord}$.

2.2 Bargaining rules

A **bargaining rule** $F : \mathcal{B} \to \mathbb{R}^N_+$ assigns each bargaining problem $(S, d) \in \mathcal{B}$ to a feasible payoff profile $F(S, d) \in S$. The following bargaining rule plays a central role in our analysis. For each $(S, d) \in \mathcal{B}$, the **Shapley-Shubik bargaining rule,** $Sh : \mathcal{B} \to \mathbb{R}^N$ selects the limit of the *brace* points (equivalently the limit of the sequence $\{p^n(S, d)\}_{n \in \mathbb{N}}$) as the solution:

$$Sh(S, d) := \lim_{n \to \infty} p^n(S, d).$$

A bargaining rule F is **Pareto optimal** if it assigns each bargaining problem $(S, d) \in \mathcal{B}$ to a Pareto optimal payoff profile, $F(S, d) \in P(S, d)$. It is **symmetric** if given $(S, d) \in \mathcal{B}$, for each $\pi \in \Pi$, $\pi(S) = S$ and $\pi(d) = d$ implies that for each $i, j \in N$ $F_i(S, d) = F_j(S, d)$. A bargaining rule F is **ordinally invariant** if for each $(S, d) \in \mathcal{B}$ and $f \in \mathcal{F}$, $F(f(S), f(d)) =$

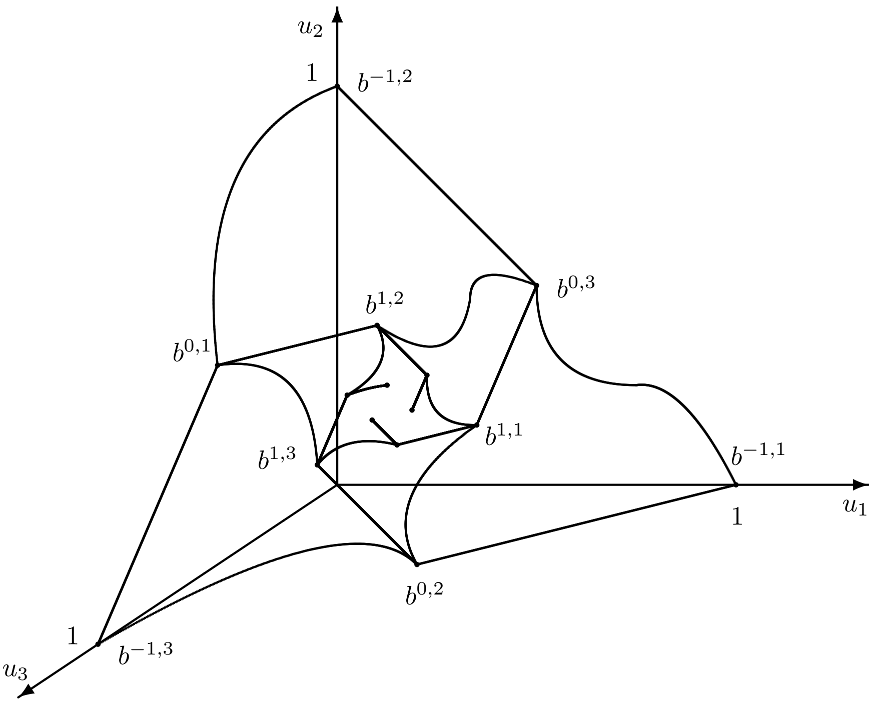

Figure 3: A typical ordinally normalized problem.

$f(F(S,d))$. The **scale invariance** axiom of Nash is a weaker version which only allows positive affine transformations.

A bargaining rule F is **independent of irrelevant alternatives (IIA)** if for each (S,d), $(S',d) \in \mathcal{B}$, $F(S,d) \in S' \subseteq S$ implies $F(S,d) = F(S',d)$. It is **independent of irrelevant alternatives other than the aspiration points (IIA-aspiration)** if for each (S,d), $(S',d) \in \mathcal{B}$, $F(S,d) \in S' \subseteq S$ and for each $i \in N$ $asp_i(S,d) \in S'$ imply $F(S,d) = F(S',d)$. It is **independent of irrelevant alternatives other than the brace (IIA-brace)** if for each (S,d), $(S',d) \in \mathcal{B}$, $F(S,d) \in S' \subseteq S$ and $br(S,d) \subset S'$ imply $F(S,d) = F(S',d)$. It is **independent of irrelevant alternatives other than the extended brace (IIA-extended brace)** if for each (S,d), $(S',d) \in \mathcal{B}$, $F(S,d) \in S' \subseteq S$ and $A(S,d) \subset S'$ imply $F(S,d) = F(S',d)$.

3 Results

In the introduction, we stated that there is no *ordinally invariant* bargaining rule that satisfies the other three axioms of Nash (1950). Weakening the *IIA* axiom in this list to

IIA- extended brace (that is, restricting *IIA* to only apply to ordinally normalized problems) however, changes this result. The Shapley-Shubik rule satisfies the property. Moreover, there are many *ordinally invariant, Pareto optimal,* and *symmetric* bargaining rules that satisfy *IIA-extended brace.* The following is the example of such a rule.

Example 1 *Let* $G^1 = \left[(0,0,0), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\right]$ *and* $G^2 = \left((\frac{3}{4}, \frac{1}{4}, 0), (1, \frac{1}{2}, 0)\right)$ *be two monotone paths. Let* F^* *be a bargaining rule which, for each* $(S,d) \in \mathcal{B}_{ord}$*, is defined as:*

$$F^*(S,d) = \begin{cases} P(S,d) \cap G^2 & \text{if } P(S,d) \cap G^2 \neq \emptyset, \\ P(S,d) \cap G^1 & \text{otherwise.} \end{cases}$$

To find the solution of F^* *to each* $(S,d) \in \mathcal{B}$*, (i) normalize* (S,d) *to an* $(S',d') \in \mathcal{B}_{ord}$*, (ii) find* $F^*(S',d')$ *by the above method, and (iii) transform* $F^*(S',d')$ *back to* (S,d) *by using the inverse of the normalization functions.*

By definition F^* *is ordinally invariant, Pareto optimal, and satisfies IIA-extended brace. We will next prove that this rule is indeed symmetric. First note that if* $(S,d) \in \mathcal{B}_{ord}$ *is a symmetric problem,* $P(S,d) \cap G^2 = \emptyset$*. Next, let* $(S,d) \in \mathcal{B} \setminus \mathcal{B}_{ord}$ *be a symmetric problem. Let* $f \in \mathcal{F}$ *be such that* $(f(S), f(d)) \in \mathcal{B}_{ord}$*. Note that* f *is uniquely defined on* $I(S,d)$ *and by symmetry of* (S,d)*, satisfies* $f_1 = f_2 = f_3$*. Therefore,* $I(f(S), f(d))$ *is symmetric. This implies that* $P(f(S), f(d)) \cap G^2 = \emptyset$*. So in both cases, the solution is determined via* G^1*. Then, since* G^1 *is a symmetric path,* F^* *satisfies symmetry for any* $(S,d) \in \mathcal{B}$*.*

However, the rule F^* as well as any rule that satisfies the above properties must coincide with the Shapley-Shubik rule on a large subclass of $\mathcal{B}_{ord}$ which can be defined as follows: Let $d^* = 0$ and $S^* = comp\{conv\{e(1), e(2), e(3)\} \mid d^*\}$. Let the construct C^* be defined as

$$C^* = \bigcup_{n=-1}^{\infty} \bigcup_{i=1}^{3} [b^{n,i}(S^*, d^*), b^{n,i+1}(S^*, d^*)]$$

with the convention that for $i = 3$, $i + 1 = 1$ (See *Figure 4*). Note that the construct C^* is the minimal symmetric extension of the *normalized extended brace* A^*. A bargaining problem $(S,d) \in \mathcal{B}$ is **restricted and ordinally normalized** if

1. $(S,d) \in \mathcal{B}_{ord}$
   ($i.e.$ (S,d) is ordinally normalized) and

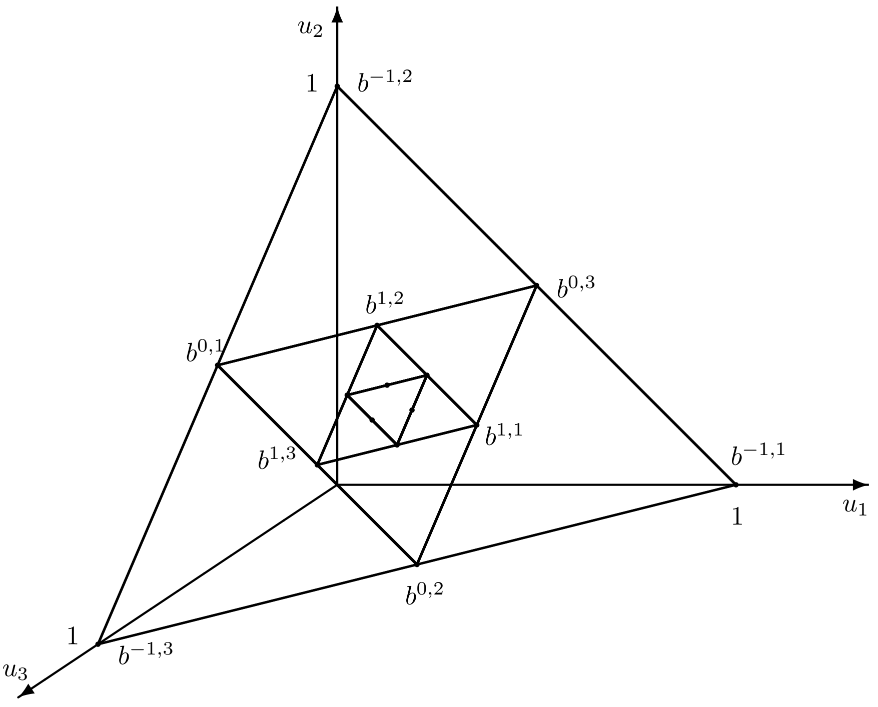

Figure 4: Constructing C^*.

2. $x \in C^*$ and $y > x$ imply $y \notin S$
   ($i.e.$ S does not contain profiles that Pareto dominate C^*).

Let $\mathcal{B}_{rest\text{-}ord}$ denote the class of all *restricted and ordinally normalized problems.* Note that $\mathcal{B}_{rest\text{-}ord} \subset \mathcal{B}_{ord}$. A bargaining problem $(S, d) \in \mathcal{B}$ is **restricted** if it is *ordinally equivalent* to a *restricted and ordinally normalized problem* $(S', d') \in \mathcal{B}_{rest\text{-}ord}$. Let $\mathcal{B}_{rest}$ denote the class of all *restricted problems.*

Proposition 2 *Let F be a bargaining rule that satisfies* ordinal invariance, Pareto optimality, symmetry, *and* IIA-extended brace. *Then F coincides with the Shapley-Shubik rule on the class of restricted bargaining problems $\mathcal{B}_{rest}$.*

Proof. Let F be a bargaining rule satisfying these four axioms. Let $(S, d) \in \mathcal{B}_{rest}$. We will show that $F(S, d) = Sh(S, d)$. By *ordinal invariance* of the two rules, we can assume without loss of generality that $(S, d) \in \mathcal{B}_{rest\text{-}ord}$. Let $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Since $\mathcal{B}_{rest\text{-}ord} \subset \mathcal{B}_{ord}$, $A(S, d) = A^*$ and $Sh(S, d) = x^*$.

Let $T = \cup_{\pi \in \Pi} \pi(S)$. Note that T is a symmetric bargaining problem and that $x^* \in P(T, d)$. Then by *Pareto optimality* and *symmetry* of F, $F(T, d) = x^*$ and therefore,

$F(T,d) \in S$. Now note that $(T,d) \in \mathcal{B}_{rest\text{-}ord}$. This implies that $A(T,d) = A^*$ and therefore, $A(T,d) \subset S$. This observations together show that the pair (S,d), (T,d) satisfy the conditions of *IIA-extended brace.* Then, by applying this axiom, we get $F(S,d) = F(T,d) = Sh(S,d)$. ■

Note that the class $\mathcal{B}_{rest\text{-}ord}$ is not dense in $\mathcal{B}_{ord}$. Therefore, there is a continuum of bargaining rules other than the Shapley-Shubik rule that satisfy *ordinal invariance, Pareto optimality*, *symmetry*, and *IIA-extended brace.* We next show that if *IIA-extended brace* is strenghtened to *IIA-brace*, a unique rule survives.

Theorem 3 *The Shapley-Shubik rule uniquely satisfies* ordinal invariance*,* Pareto optimality*,* symmetry*, and* IIA-brace*.*

Proof. It is straightforward to show that Sh satisfies these four axioms. Let F be a bargaining rule that also satisfies them. Let $(S,d) \in \mathcal{B}$. We will show that $F(S,d) = Sh(S,d)$. By *ordinal invariance* of the two rules, we can assume without loss of generality that $(S,d) \in \mathcal{B}_{ord}$. Let $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and note that $Sh(S,d) = x^*$.

Let $T = \cup_{\pi \in \Pi} \pi(S)$. Note that T is a symmetric bargaining problem and that $x^* \in P(T,d)$. Then by *Pareto optimality* and *symmetry* of F, $F(T,d) = x^*$ and therefore, $F(T,d) \in S$. Also note that for each $\pi \in \Pi$, $\pi(d) = d$ and for each $n \in \mathbb{N}$ and $i \in N$, $\pi(b^{n,i}(S,d)) = b^{n,\pi(i)}(\pi(S), \pi(d))$. Thus for each $\pi \in \Pi$, $br(S,d) = br(\pi(S), d)$. This implies that $br(T,d) = br(S,d) \subset S$. These observations together show that the pair (S,d), (T,d) satisfy the conditions of *IIA-brace.* Then, by applying this axiom, we get $F(S,d) = F(T,d) = Sh(S,d)$. ■

IIA-brace is the strongest axiom of its kind that *ordinally invariant, Pareto optimal*, and *symmetric* bargaining rules satisfy. To see this, we will introduce the following axiom which is *slightly stronger* than *IIA-brace*: it drops the feasibility requirement for one arbitrary point in the brace. A bargaining rule F is **independent of irrelevant alternatives other than brace minus 1 (IIA-brace-1)** if for each (S,d), $(S',d) \in \mathcal{B}$, $F(S,d) \in S' \subseteq S$ and $br(S,d) \setminus \{b^{n,i}(S,d)\} \subset S'$ for some $i \in N$ and $n \in \mathbb{N}$ imply $F(S,d) = F(S',d)$. The next example demonstrates a case in which the Shapley-Shubik rule violates this stronger axiom.[9]

[9] In this example, the feasible set contracts in a way that all left out profiles are the ones that assign Agent 1 a smaller payoff compared to the other two agents. Therefore, it is only intuitive that Agent 1 be

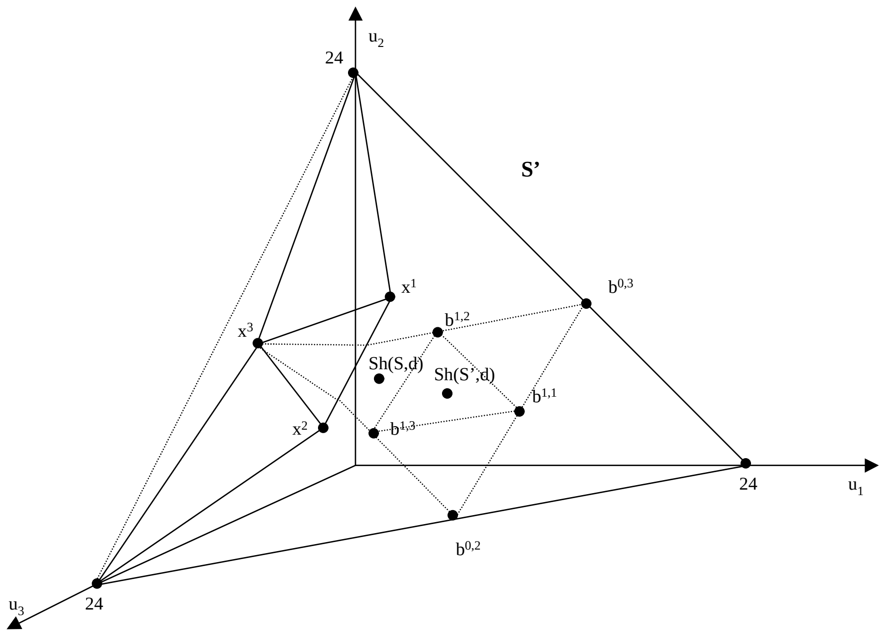

Figure 5: The Shapley-Shubik solutions to (S, d) and (S', d) are different.

Example 4 *Let $y(i) = 24e(i)$ for each $i \in N$. Let $d = 0$ and $S = comp\{conv\{y(1), y(2), y(3)\} \mid d\}$. Note that $Sh(S, d) = (8, 8, 8)$. S' is constructed as demonstrated in* Figure 5*: let $x^1 = (6, 12, 6)$, $x^2 = (6, 6, 12)$, and $x^3 = (0, 10, 10)$. Let $Q^1 = conv\{y(1), y(2), x^1\}$, $Q^2 = conv\{y(1), x^1, x^2\}$, $Q^3 = conv\{y(1), y(3), x^2\}$, $Q^4 = conv\{y(3), x^2, x^3\}$, $Q^5 = conv\{x^1, x^2, x^3\}$, and $Q^6 = conv\{y(2), x^1, x^3\}$. Let $S' = comp\{\cup_{k=1}^{6} Q^k \mid d\}$. Note that $S' \subset S$ and $asp(S', d) = asp(S, d)$. Note that all brace points of the original problem (S, d) except $b^{0,1}(S, d) = (0, 12, 12)$ remain feasible in the smaller problem. Therefore, IIA-brace-*1 *requires that the solution to (S', d) be $(8, 8, 8)$.*
It is straightforward to check that the first six brace points of (S', d) are $b^{0,1} = (0, 10, 10)$, $b^{0,2} = (14, 0, 10)$, $b^{0,3} = (14, 10, 0)$, $b^{1,1} = (14, 5, 5)$, $b^{1,2} = (9, 10, 5)$, and $b^{1,3} = (9, 5, 10)$. Then, it follows from the definition of $Sh(S', d)$ that $b_1^{1,2} = 9 < Sh_1(S', d)$. Since $Sh_1(S, d) =$ 8*, $Sh(S', d) \neq Sh(S, d)$.*

The following proposition follows from this example and Theorem 3.

better-off as result of such a contraction.

Proposition 5 *No bargaining rule simultaneously satisfies* ordinal invariance, Pareto optimality, symmetry*, and* IIA-brace-1*.*

Proof. Any bargaining rule that satisfies *IIA-brace-1* also satisfies *IIA-brace*. Therefore, by Theorem 3, the only bargaining rule that can satisfy the above list of axioms is the Shapley-Shubik rule. However, by Example 1, this rule violates *IIA-brace*-1. ■

Also note that, in Example 1, the aspiration points of (S, d) remain feasible in the smaller problem (S', d). This shows that the Shapley-Shubik rule violates the *IIA-aspiration* axiom of Roth (1977).[10] The following proposition follows from this observation and Theorem 3.

Proposition 6 *No bargaining rule simultaneously satisfies* ordinal invariance, Pareto optimality, symmetry*, and* IIA-aspiration*.*

Proof. Define the following axiom which combines the requirements of both IIA-aspiration and IIA-brace: a bargaining rule F is **independent of irrelevant alternatives other than the aspiration and the brace (IIA-aspiration-brace)** if for each (S, d), $(S', d) \in \mathcal{B}$, $F(S, d) \in S' \subseteq S$, $br(S, d) \subset S'$, and for each $i \in N$, $asp_i(S, d) \in S'$ imply $F(S, d) = F(S', d)$. Note that any bargaining rule that satisfies IIA-aspiration also satisfies this axiom. Since for each $(S, d) \in \mathcal{B}_{ord}$ and $i \in N$ $asp_i(S, d) = e(i)$, it is straightforward to modify the proof of Theorem 3 to show that the Shapley-Shubik rule uniquely satisfies *ordinal invariance, Pareto optimality, symmetry*, and *IIA-aspiration-brace*. Therefore, the Shapley-Shubik rule is also the only bargaining rule that can satisfy the above list of axioms. However, by Example 1, it violates *IIA-aspiration*. ■

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