On Recursive Solutions to Simple Allocation Problems*

Özgür Kıbrıs†

Sabancı University

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Abstract

We propose and axiomatically analyze a class of *rational* solutions to *simple allocation problems* where a policy maker allocates an *endowment* E among n agents described by a *characteristic vector* c. We propose a class of **recursive rules** which mimic a decision process where the policy maker initially starts with a *reference allocation* of E in mind and then uses the data of the problem to *recursively adjust* his previous allocation decisions. We show that recursive rules uniquely satisfy *rationality, c-continuity,* and *other-c monotonicity.* We also show that a well-known member of this class, the **Equal Gains rule**, uniquely satisfies *rationality, c-continuity* and *equal treatment of equals.*

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†Faculty of Arts and Social Sciences, Sabancı University, 34956, Istanbul, Turkey. E-mail: ozgur@sabanciuniv.edu Tel: +90-216-483-9267 Fax: +90-216-483-9250

1 Introduction

Revealed preference theory studies conditions under which by observing the choice behavior of an agent, one can discover the underlying preferences that govern it. Choice rules for which this is possible are called *rational.* Most of the earlier work on rationality analyzes consumers' demand choices from budget sets (e.g. see Samuelson (1938, 1948)). The underlying premise that choices reveal information about preferences, however, is applicable to a wide range of choice situations. For example, applications of the theory to bargaining games (Nash (1950)) characterize bargaining rules which can be "rationalized" as maximizing the underlying preferences of an impartial arbitrator (Peters and Wakker, 1991; Bossert, 1994; Ok and Zhou, 2000; Sánchez, 2000).

In this paper, we propose and axiomatically analyze a class of *rational* solutions to *simple allocation problems.* A simple allocation problem for a society N is an $|N|+1$ dimensional nonnegative real vector $\left(c_1, ..., c_{|N|}, E\right)$ satisfying $\sum_N c_i \geqq E$ where E, the **endowment** has to be allocated among agents in N who are characterized by c, the **characteristic vector**. Simple allocation problems have a wide range of applications some of which are discussed at the end of this section.

We interpret an allocation rule on simple allocation problems as data on the choices of a policy-maker. As is standard in revealed preference theory, we say that a policy maker's choices are (i) *rational* (ii) *transitive-rational*, and (iii) *representable* if they coincide with maximization of a (i) binary relation, (ii) transitive binary relation, and (iii) numerical function on the allocation space.

We propose a class of **recursive rules**. This is a large class of rules which mimic a recursive decision process where the policy maker initially starts with a *reference allocation* of E in mind and then uses the data of the problem and his previous allocation decisions to *recursively adjust* his allocation choice.

Our main result, Theorem 1, is a characterization of recursive rules. It can be divided into the following two statements. First, recursive rules all satisfy three axioms: **rationality**, **other-c monotonicity**, and **c-continuity**. As noted above, *rationality* means that a recursive rule's choices are consistent with the maximization of a binary relation. Given that the definition of a recursive rule makes no reference to such a binary relation, this is a surprising observation. The second axiom, *other-c monotonicity*, is a standard property which is satisfied by most allocation rules in the literature (Barberà, Jackson, and Neme, 1997; Kıbrıs 2011; Thomson, 2003, 2007). It means that in a recursive rule, a change in an agent i's characteristic value c_i affects the rest of the society in the same way, that is, no two other agents' shares are affected in opposite directions. The third axiom, *c-continuity*, means that a recursive rule is continuous with respect to changes in the characteristic vector.

The second, and more surprising statement of Theorem 1 is that recursive rules are the only rules to satisfy *rationality, other-c monotonicity*, and *c-continuity.*

This paper is related to Kıbrıs (2011) which, for simple allocation problems, analyzes the logical relationships among the three central notions of revealed preference theory (rationality, transitive-rationality, representability) and other well-known axioms in the literature. A combination of Kıbrıs (2011) and Theorem 1 leads to two interesting observations. First, every *recursive rule* is *transitive-rational.* That is, recursive rules never exhibit cyclic choice behavior. Second, every recursive rule which is continuous with respect to E is additionally *representable* by a numerical function. For more on this discussion, please see Section 3.

Our second result, Theorem 2, is a characterization of an **Equal Gains rule.** This rule is an important member of the family of recursive rules. It is called the *Uniform rule* in the single-peaked allocation literature, the *Constrained Equal Awards rule* in the bankruptcy literature, and the *Leveling Tax* in the taxation literature. The *Equal Gains rule* allocates the endowment in each problem equally, subject to no agent receiving more than his characteristic value.

Being a recursive rule, the Equal Gains rule satisfies *rationality, other-c monotonicity*, and *c-continuity.* Theorem 2 shows that it additionally satisfies a well-known anonymity axiom called **equal treatment of equals**. This means that the Equal Gains rule always assigns identical shares to two agents with identical characteristics.

Theorem 2 also shows that the Equal Gains rule is the only rule to satisfy *rationality, c-continuity*, and *equal treatment of equals.* Note that this statement does not include *other-c monotonicity.* The addition of *equal treatment of equals* makes it redundant.

We conclude this section with some applications of simple allocation problems: (i) **Tax-**

ation: A public authority is to collect an amount E of *tax* from a society N. Each agent i has *income* c_i. (E.g., Edgeworth, 1898; Young, 1987) (*ii*) **Bankruptcy:** A bankruptcy judge is to allocate the remaining *assets* E of a bankrupt firm among its creditors, N. Each agent i has *credited* c_i to the bankrupt firm.[1] (E.g., O'Neill, 1982; Aumann and Maschler, 1985; Thomson, 2003 and 2007) (*iii*) **Permit Allocation**: The Environmental Protection Agency is to allocate an amount E of *pollution permits* among firms in N. Each firm i, depending on its location, is imposed by the local authority an *emission constraint* c_i on its pollution level. (E.g., Kıbrıs, 2003) (*iv*) **Single-peaked or Saturated Preferences**: A social planner is to allocate E units of a perfectly divisible commodity among members of N. Each agent i is known to have preferences with *peak (saturation point)* c_i.[2] (E.g., Sprumont, 1991) (*v*) **Demand Rationing**: A supplier is to allocate its *production* E among demanders in N. Each $i \in N$ *demands* c_i units. (E.g. Cachon and Lariviere, 1999) (*vi*) **Bargaining with Quasilinear Preferences and Claims:** An arbitrator is to allocate E units of a *numeriare good* among agents who have quasilinear preferences with respect to it. Each agent holds a *claim* c_i on what he should receive. (E.g. Chun and Thomson, 1992; Moulin, 1985) (*vii*) **Consumer Choice under fixed prices and rationing:** A consumer has to allocate his *income* E among a set N of commodities. The prices are fixed and the consumer faces a "rationing constraint" c_i on his consumption of each commodity i. (E.g. Bénassy, 1993; Kıbrıs and Küçükşenel, 2008)

2 Model

Let $N = \{1, ..., n\}$ be the set of agents (or commodities as in the consumer choice application mentioned above). For $i \in N$, let e_i be the i^{th} unit vector in $\mathbb{R}^N_+$. Let $e = \sum_N e_i$. We use the vector inequalities $\leqq, \leq, <$.[3] For $c \in \mathbb{R}^N_+$, $\alpha \in \mathbb{R}_+$, and $S \subseteq N$, with an abuse of notation, we write $(c_S, \alpha_{N \setminus S})$ to denote the vector which coincides with c on S and which chooses α

[1] The dual of this problem, called surplus sharing, constitutes another application (e.g. Moulin, 1987).

[2] The rest of the preference information is disregarded as typical in several well-known solutions to this problem, such as the Uniform rule or the Proportional rule.

[3] That is, $x \leqq y$ if and only if $x_i \leqq y_i$ for each $i \in N$; $x \leq y$ if and only if $x \leqq y$ and $x \neq y$; $x < y$ if and only if $x_i < y_i$ for each $i \in N$.

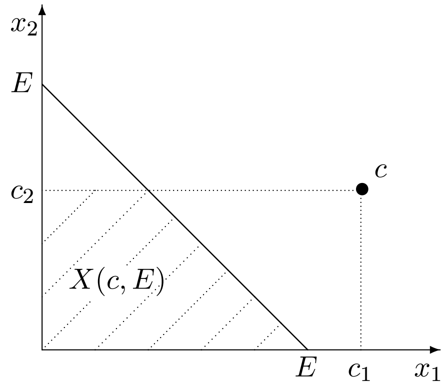

Figure 1: A typical simple allocation problem.

for every coordinate in $N \setminus S$.

A **simple allocation problem** for N is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ such that $\sum_N c_i \geqq E$ (please see *Figure 1*). We call E the **endowment** and c the **characteristic vector**. As discussed at the end of Section 1, depending on the application, E can be an asset or a liability and c can be a vector of incomes, claims, demands, preference peaks, or consumption constraints. Let $\mathcal{C}$ be the set of all simple allocation problems for N. Given a simple allocation problem $(c, E) \in \mathcal{C}$, let $X(c, E) = \left\{ x \in \mathbb{R}_+^N \mid x \leqq c \text{ and } \sum_N x_i \leqq E \right\}$ be the **choice set of** (c, E).

An allocation **rule** $F : \mathcal{C} \rightarrow \mathbb{R}_+^N$ assigns each simple allocation problem (c, E) to an allocation $F(c, E) \in X(c, E)$ such that $\sum_N F_i(c, E) = E$. Each rule F satisfies $F(c, E) \leqq c$ which, depending on the application, might be interpreted as a consumption constraint (as in permit allocation) or an efficiency requirement (as in single-peaked preferences). Also, $\sum_N F_i(c, E) = E$ can be interpreted as an *efficiency* property (as in permit allocation) or a feasibility requirement (as in taxation). In consumer choice, this condition is equivalent to the Walras law.

A rule F is continuous in characteristics (**c-continuous**) if for each $E \in \mathbb{R}_+$, $F(., E)$ is a continuous function. This is a standard regularity property which eliminates rules that can change the proposed allocation radically in response to very small changes in the characteristic vector.

A rule F is monotonic in others' characteristics (**other-c monotonic**) if a change in agent i's characteristic value affects other agents in the same way: for each $(c, E) \in \mathcal{C}$, $i \in N$, and $c_i' \in \mathbb{R}_+$ with $(c_i', c_{-i}, E) \in \mathcal{C}$, either $F_{-i}(c, E) \geqq F_{-i}(c_i', c_{-i}, E)$ or $F_{-i}(c, E) \leqq F_{-i}(c_i', c_{-i}, E)$. *Other-c monotonicity* is a weak property; it does not specify how much the share of each agent will change or how these changes will be related to the agents' characteristics (so for instance, among two agents with identical characteristics, one's share may remain the same while the other's share increases). As a result, it is satisfied by almost all of the well-known solutions to simple allocation problems.

A rule F is **rational** if its choices always coincide with the maximization of a binary relation on the allocation space $\mathbb{R}^N_+$, that is, if there is a binary relation $B \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$ such that for each $(c, E) \in \mathcal{C}$, $F(c, E) = \{x \in X(c, E) \mid \text{for each } y \in X(c, E), \ xBy\}$. *Rationality* has been a central concept in economics since the seminal work of Samuelson (1938). It has been used to analyze a wide variety of choice behavior, including consumption, arbitration, and voting. For simple allocation problems, Kıbrıs (2011) discusses this axiom in detail and additionally, shows that a rule F is *rational* if and only if it satisfies a well-known axiom: F is **contraction independent** if for each $(c, E), (c', E) \in \mathcal{C}$, $F(c, E) \leqq c' \leqq c$ implies $F(c', E) = F(c, E)$. *Contraction independence* states that a decrease in characteristic values does not change the initially chosen allocation as long as it remains feasible. This axiom is also known as *independence of irrelevant alternatives* in cooperative bargaining theory (Nash, 1950) and *Sen's property* α in revealed preference theory (Sen, 1971).

Remark 1 *(Kıbrıs, 2011)* A rule F is *rational* if and only if it is *contraction independent.*

3 Recursive Rules

In this section, we present and characterize a class of **recursive rules**. This is a large class of rules which mimic a recursive decision process where the policy maker initially starts with a *reference allocation* of E in mind and then uses the data of the problem and his previous allocation decisions to *recursively adjust* his allocation choice.

The well-known **Equal Gains rule** is a member of this class.[4] This rule allocates the endowment in each problem equally, subject to no agent receiving more than his characteristic value: for each $i \in N$, $EG_i(c, E) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum_N \min\{c_i, \lambda\} = E$.

[4] In the single-peaked allocation literature, the Equal Gains rule is called the *Uniform rule,* in the bankruptcy literature it is called the *Constrained Equal Awards rule*, and in the taxation literature, it is called the *Leveling Tax.*

The *Equal Gains rule* can equally be defined as choosing the outcome of the following recursive decision algorithm: let $(c, E) \in \mathcal{C}$,

Step 1. Determine the set of agents whose characteristic value, c_i, is less than his share from equal division, $\frac{E}{|N|}$. If no such agent exists, pick equal division and terminate the algorithm. Otherwise, assign each such agent his characteristic value and move to the next step.

Step 2. Determine the remaining agents (say N') to be allotted and the remaining endowment to be allotted (say E'). Repeat Step 1 by replacing N with N' and E with E'.

The class of *recursive rules* generalize this "Equal Gains algorithm" in several ways.[5] First, the starting allocation of the algorithm does not have to be equal division. Instead, it is given by a **reference function** $r : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N$ that maps each endowment level E to an initial allocation $r(E) \in \mathbb{R}_+^N$ such that $\sum_N r_i(E) = E$. Second, the proposed allocation can be updated in a variety of ways. At each step of the algorithm, this updating will be done by a general function g according to the previous allocation x and the problem's parameters c and E. That is, g assigns each problem $(c, E) \in C$ and previous allocation $x \in \mathbb{R}_+^N$ (with $\sum_N x_i = E$) to an adjusted allocation $g(x, c, E) = x' \in \mathbb{R}_+^N$ with $\sum_N x_i' = E$. For each $t \in \{1, ..., n\}$, let $g^t(x, c, E) = g\left(g^{t-1}(x, c, E), c, E\right)$ with the convention that $g^0(x, c, E) = x$. (That is, g^t represents g composed itself with t times.) The function g is a **recursive adjustment function with respect to r** if g^n is *continuous in* c and the following are true for any $t \in \{1, ..., n\}$ and $x^t = g\left(x^{t-1}, c, E\right) = g^t(r(E), c, E)$:[6]

1. $x_i^{t-1} \geqq c_i$ implies $x_i^t = c_i$.

2. $x_i^{t-1} < c_i$ implies $x_i^t \geqq x_i^{t-1}$.

3. Let $x^{t-1} = g^{t-1}(r(E), c_i, c_{-i}, E) = g^{t-1}(r(E), \widetilde{c}_i, c_{-i}, E)$. Then, $x_i^{t-1} < \widetilde{c}_i < c_i$ implies $g\left(x^{t-1}, c_i, c_{-i}, E\right) = g\left(x^{t-1}, \widetilde{c}_i, c_{-i}, E\right)$.

4. Let $x = g^n(r(E), c_i, c_{-i}, E)$ and $\widetilde{x} = g^n(r(E), \widetilde{c}_i, c_{-i}, E)$. Then, $c_i < \widetilde{c}_i$ implies $x_j \geqq \widetilde{x}_j$

[5] Recursive rules are closely related to a family of rules introduced and analyzed by Barberà, Jackson and Neme (1997) on the domain of allocation problems with single-peaked preferences.

[6] Similar to Barberà, Jackson and Neme (1997), we only require these four properties to be satisfied at an allocation obtained at some step of the recursive adjustment process. (And this is why g is defined in reference to r.) The function g could be arbitrary on other parts of the domain and induce a perfectly well-behaved allocation rule.

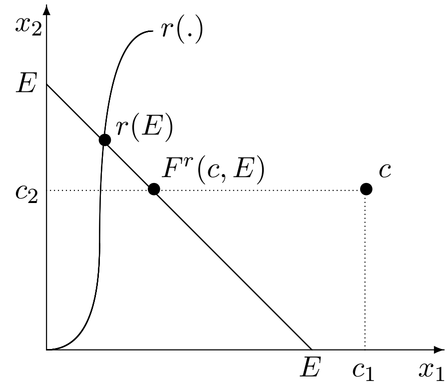

Figure 2: The reference function r completely defines a two-agent algorithmic rule. In the figure, since $r_2(E) > c_2$, $F_2^r(c,E) = c_2$ and $F_1^r(c,E) = E - c_2$.

for each $j \neq i$.

Property 1 requires that if at any stage, an agent's proposed share is greater than or equal to his characteristic value, it is adjusted in the next step to be equal to his characteristic value. This property is intimately linked to *c-continuity* (see *Lemma 1*). It also guarantees that once an agent is served his characteristic value, his share is fixed in the rest of the algorithm. *Property* 2 requires that, if the proposed share of an agent is smaller than his characteristic value, the adjustment can not be to decrease it further. This property guarantees that, as some additional endowment is freed up in a step (due to some agents receiving their c_i and leaving the algorithm), none of the remaining agents are worse off. *Property* 3 requires that a change in c_i that does not affect the feasibility of agent i's proposed share does not affect the adjustment made to it by the recursive algorithm. This property is intimately linked to *rationality* (which, by *contraction independence* makes a similar requirement). *Property* 4 requires that a decrease in c_i does not decrease the final share of any other agent. This property is intimately linked to *other-c monotonicity.* To see this, note that by *rationality*, decreasing c_i does not increase x_i. So the remaining agents consume at least as much in total. Therefore, if some agent j's share decreased in response to a decrease in c_i, there would be another agent k whose share increased. This would violate *other-c monotonicity.*

Given a *reference function* r and a *recursive adjustment function* g *with respect to* r, a **recursive rule with respect to g and r**, $F^{g,r}$, is defined for each $(c,E) \in \mathcal{C}$ as $F^{g,r}(c,E) = g^n(r(E), c, E)$. That is $F^{g,r}(c,E)$ is the allocation obtained at the end of n steps of the recursive adjustment algorithm.[7]

[7] The recursive adjustment algorithm in fact obtains the final allocation in at most $(n-1)$ steps. Thus

Two-agent recursive rules can be completely defined via the reference function r (please see *Figure 2*). If $r(E)$ is feasible at a problem (c, E), it is chosen as the final allocation. Otherwise, there is at most one agent i such that $c_i < r_i(E)$. The adjusted (and final) allocation is then $x_i = c_i$ and $x_j = E - c_i$.

Recursive rules all satisfy *c-continuity* and *other-c monotonicity.* The main result of this section, presented next, states that they are also the only *rational* rules to satisfy these properties.

Theorem 1 A rule F is *rational, c-continuous* and *other-c monotonic* if and only if F is a *recursive rule*, that is, there is a *reference function* r and a *recursive adjustment function* g *with respect to* r *such that* $F = F^{g,r}$.

Theorem 1 has interesting implications regarding *transitive-rationality* and *representability* of recursive rules. For this discussion, we need to mention two findings by Kıbrıs (2011): (i) every *rational* and *other-c monotonic* rule is *transitive-rational* and (ii) every *rational*, *other-c monotonic,* and *continuous* rule is *representable* by a numerical function.[8] A combination of Theorem 1 and finding (i) implies that every recursive rule is *transitive-rational*, that is, recursive rules can be rationalized by transitive binary relations. This means that recursive rules never exhibit cyclic choice behavior (unlike the rule we present at the end of this section). Secondly, Theorem 1 and finding (ii) of Kıbrıs (2011) together imply that every *E-continuous* recursive rule is additionally *representable* by a numerical function. This is a large subclass of recursive rules $F^{g,r}$ for which the functions g and r are continuous with respect to E.

The above characterization is tight. Without *rationality*, *Proportional rule* becomes admissible. The following example presents a rule that violates *other-c monotonicity* but satisfies the other properties. Finally, *Example 2*, at the end of the next section, presents a rule that violates *c-continuity* but satisfies the other properties.

$g^n = g^{n-1}$. We use g^n since it slightly simplifies the notation as well as the argument in Step 3 of the proof of *Theorem 1.*

[8] Kıbrıs (2011) uses the term *continuity* as a combination of *c-continuity,* defined in Section 2, and *E-continuity* (*i.e.* the rule being continuous with respect to E).

Example 1 *Let* $N = \{1, 2, 3\}$. *Let*

$$F(c, E) = \begin{cases} \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) & if \quad \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) \leqq c \\ (c_1, c_1, E - 2c_1) & else\ if \quad c_1 < \frac{E}{3} \text{ and } (c_1, c_1, E - 2c_1) \leqq c \\ (E - 2c_2, c_2, c_2) & else\ if \quad c_2 < \frac{E}{3} \text{ and } (E - 2c_2, c_2, c_2) \leqq c \\ (c_3, E - 2c_3, c_3) & else\ if \quad c_3 < \frac{E}{3} \text{ and } (c_3, E - 2c_3, c_3) \leqq c \\ (c_1, c_2, E - c_1 - c_2) & else\ if \quad E - c_1 - c_2 > c_2 \text{ and } c_1 > c_2 \\ (c_1, E - c_1 - c_3, c_3) & else\ if \quad E - c_1 - c_3 > c_1 \text{ and } c_3 > c_1 \\ (E - c_2 - c_3, c_2, c_3) & else\ if \quad E - c_2 - c_3 > c_3 \text{ and } c_2 > c_3. \end{cases}$$

Using Remark 1, it is straightforward to check that this rule is rational*. It is also* c-continuous*. It is not* other-c monotonic *since, for* $x = (c_1, c_1, E - 2c_1)$, x_2 *is increasing in* c_1 *while* x_3 *is decreasing. We will next demonstrate that this rule is not recursive. Suppose otherwise. Let* $E = 9$, $c = (9, 2, 1)$. *The initial reference allocation should then be* $r(E) = F(E_N, E) = (3, 3, 3)$. *Since agents* 2 *and* 3 *have smaller characteristic values than* 3*, they must receive their characteristic values:* $x_2 = 2$ *and* $x_3 = 1$. *Thus* $x_1 = 6$. *However,* $F(c, E) = (7, 1, 1)$, *a contradiction.*

4 Equal Gains Rule

Being a recursive rule, the Equal Gains rule satisfies *rationality, other-c monotonicity*, and *c-continuity*. Given the very symmetric structure of its algorithm, it is not surprising that it additionally satisfies a well-known anonymity axiom called *equal treatment of equals*. A rule F satisfies **equal treatment of equals** if two agents with identical characteristics are awarded equal shares: for each $(c, E) \in \mathcal{C}$ and $i, j \in N$, $c_i = c_j$ implies $F_i(c, E) = F_j(c, E)$. A large class of rules also satisfy this rather weak axiom (*e.g.* see Young, 1987; Thomson, 2003, 2007). But, as demonstrated in the following theorem, the *Equal Gains rule* is the only *rational* and *c-continuous* rule to satisfy it.

Theorem 2 A rule F satisfies *rationality, c-continuity,* and *equal treatment of equals* if and only if it is the *Equal Gains rule.*

Theorem 2 implies that the Equal Gains rule is the only *recursive rule* to satisfy *equal treatment of equals*. This theorem, however, does not make use of the *c-monotonicity* property of recursive rules (as *Theorem 1* did). This is because the three axioms in *Theorem 2* imply *c-monotonicity.*

The above characterization is tight. Without *rationality*, *Proportional rule* becomes admissible. Without *equal treatment of equals*, other recursive rules become admissible. Finally, the following example presents a rule that violates *c-continuity* but satisfies the other properties.

Example 2 *Let $N = \{1, 2\}$. Let F be defined as*

$$F(c, E) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } c_1 \geqq \frac{E}{2} \text{ and } c_2 \geqq \frac{E}{2}, \\ (E, 0) & \text{if } c_1 \geqq E \text{ and } c_2 < \frac{E}{2}, \\ (c_1, E - c_1) & \text{if } E - c_2 < c_1 < E \text{ and } c_2 < \frac{E}{2}, \\ (0, E) & \text{if } c_1 < \frac{E}{2} \text{ and } c_2 \geqq E, \\ (E - c_2, c_2) & \text{if } c_1 < \frac{E}{2} \text{ and } E - c_1 < c_2 < E. \end{cases}$$

Note that F satisfies rationality *and* equal treatment of equals*. It is not* c-continuous *since for each $\varepsilon \in (0, \frac{E}{2}]$, $F\left(E_1, \left(\frac{E}{2} - \varepsilon\right)_2, E\right) = (E, 0)$ but $F\left(E_1, \left(\frac{E}{2}\right)_2, E\right) = \left(\frac{E}{2}, \frac{E}{2}\right)$.*

5 Concluding Comments

Theorem 1 is related to Barberà, Jackson, and Neme (1997) who, for allocation problems with single peaked preferences, analyze the implications of strategy proofness. These authors show that a similar class of recursive rules uniquely satisfy strategy proofness, efficiency, and a monotonicity property similar to ours. Given this similarity as well as the fact that both strategy proofness and rationality demands a certain insensitivity of the allocation to changes in the c vector (e.g. see Lemma 2 in Ching 1994), analyzing the logical relationship between strategy proofness and rationality remains an interesting open question.

Theorem 2 is related to the literature on the *Equal Gains rule (EG)* as follows. Dagan (1996) shows that *EG* uniquely satisfies *equal treatment of equals, truncation invariance*, and *composition up*.[9] Schummer and Thomson (1997) show that EG minimizes (i) the difference between the largest and the smallest share and (ii) the variance of the shares. In a related result, Bosmans and Lauwers (2007) show that EG Lorenz dominates every other "order-preserving" rule. Herrero and Villar (2002) and Yeh (2004) show that EG uniquely satisfies *conditional full compensation* and *composition down*.[10] Finally, Yeh (2006) shows that EG uniquely satisfies *conditional full compensation* and "own-claim monotonicity". Our characterization is logically independent from these previous results. Furthermore, the main principles employed in these characterizations (such as "composition", *full compensation*, or *Lorenz domination*) are quite different than *rationality*. Also, with the exception of Schummer and Thomson (1997) and Bosmans and Lauwers (2007), the above characterizations use properties that relate the rule's behavior at different social endowment levels. This is not the case for *Theorem 2*.

6 Appendix

The following lemmata are useful in the proof of *Theorem 1*. The first lemma shows under *rationality* and *c-continuity* that, if c_i decreases below agent i's current share, agent i's updated share should be his new characteristic value.

Lemma 1 Assume that F is *rational* and *c-continuous*. Let $(c, E), (c', E) \in \mathcal{C}$, and $i \in N$ be such that $c_{-i} = c'_{-i}$, $c_i > c'_i$, *and* $F_i(c, E) > c'_i$. Then $F_i(c', E) = c'_i$.

Proof. Suppose $F_i(c', E) < c'_i$. By *c-continuity*, there is $c''_i \in \mathbb{R}_+$ such that $c'_i < c''_i < c_i$ and $F_i(c''_i, c_{-i}, E) = F_i\left(c''_i, c'_{-i}, E\right) = c'_i$. Then, $F\left(c''_i, c'_{-i}, E\right) \leqq c' \leqq \left(c''_i, c'_{-i}\right)$, by *contraction*

[9] *Composition up* requires that dividing the social endowment in two, first allocating one part, revising the characteristic vector accordingly, and then allocating the rest produces the same final allocation as allocating all the social endowment at once. *Truncation invariance* says that the excess of c_i over E does not affect the allocation choice.

[10] *Conditional full compensation* roughly requires agents with sufficiently small characteristic values to receive their characteristic values. *Composition down* deals with the following scenario: after the social endowment is allocated, we discover that the actual social endowment is smaller; then, it requires that using the original characteristic vector or the initially chosen allocation should produce the same final outcome.

independence, implies $F(c', E) = F\left(c_i'', c_{-i}', E\right)$, a contradiction. ■

The following lemma states that if c_i decreases, the share of agent i can not increase and the shares of other agents can not decrease in response.

Lemma 2 Assume that F is *rational*, *c-continuous*, and *other-c monotonic.* Let $(c, E), (c', E) \in \mathcal{C}$, and $i \in N$ be such that $c_{-i} = c'_{-i}$ and $c_i > c'_i$. Then $F_i(c, E) \geqq F_i(c', E)$ and for each $j \in N \setminus \{i\}$, $F_j(c', E) \geqq F_j(c, E)$.

Proof. If $F_i(c, E) \leqq c'_i$, by *contraction independence,* we have $F(c', E) = F(c, E)$. Thus the result trivially holds. Alternatively assume $F_i(c, E) > c'_i$. Then by *Lemma 1*, $F_i(c', E) = c'_i < F_i(c, E)$. Thus, $\sum_{N\setminus\{i\}} F_j(c, E) < \sum_{N\setminus\{i\}} F_j(c', E)$. Therefore, there is $k \in N \setminus \{i\}$ such that $F_k(c, E) < F_k(c', E)$. By *other-c monotonicity*, this implies for each $j \in N \setminus \{i\}$, $F_j(c, E) \leqq F_j(c', E)$. ■

We next present the proofs of *theorems 1* and *2*.

Proof. (Theorem 1)

($\Leftarrow$) Let $F^{g,r}$ be a recursive rule. Since g^n is *continuous in* c, $F^{g,r}$ is *c-continuous*. Let $(c, E), (c', E) \in \mathcal{C}$ and $i \in N$ be such that $c_{-i} = c'_{-i}$.

Claim 1: *If* $F_i^{g,r}(c, E) \leqq c'_i \leqq c_i$ *or* $F_i^{g,r}(c, E) < c_i \leqq c'_i$, *we have* $F^{g,r}(c, E) = F^{g,r}(c', E)$. If $c_i = c'_i$, the claim trivially holds. If $F_i^{g,r}(c, E) < c'_i < c_i$ or $F_i^{g,r}(c, E) < c_i < c'_i$, the claim follows from Property 3 of g. Finally if $F_i^{g,r}(c, E) = c'_i < c_i$, the claim follows from the previous case and *c-continuity* of $F^{g,r}$.

Claim 2: $F^{g,r}$ *is rational.* Applying *Claim 1* to each $i \in N$ shows that it is *contraction independent*. Then, by *Remark 1, F* is *rational.*

Claim 3: $F^{g,r}$ *is other-c monotonic.* Assume $c_i \neq c'_i$. Then by Property 4, either [for each $j \in N \setminus \{i\}$, $F_j^{g,r}(c, E) \geqq F_j^{g,r}(c', E)$] or [for each $j \in N \setminus \{i\}$, $F_j^{g,r}(c, E) \leqq F_j^{g,r}(c', E)$].

($\Rightarrow$) Let F satisfy the given properties.

Step 1: *defining* g *and* r. For each $E \in \mathbb{R}_+$, let $r(E) = F(E_N, E)$. For each $(c, E) \in \mathcal{C}$ and $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = E$, let $g(x, c, E) = F\left(c_{M(x,c)}, E_{N\setminus M(x,c)}, E\right)$ where $M(x, c) = \{i \in N \mid c_i \leqq x_i\}$.

For the following steps, we will introduce some notation. Let $x^0 = r(E)$ and for $t \in \{1, ..., n\}$, let $x^t = g(x^{t-1}, c, E) = g^t(r(E), c, E)$. Let $M^{-1} = \emptyset$ and for each $t \in \{0, ..., n\}$, let $M^t = M(x^t, c)$.

Step 2: *if* $t \in \{0, ..., n\}$ *and* $i \in M^{t-1}$, *then* $i \in M^t$ *and* $x_i^t = c_i$. The proof is by induction. For $t = 0$, $M^{-1} = \emptyset$ implies the desired conclusion. Now assume $M^{-1} \subseteq ... \subseteq M^{t-1}$. Let $i \in M^{t-1}$. Then $x_i^{t-1} \geqq c_i$. Let $K = M^{t-2} \cup \{i\}$ and note that $K \subseteq M^{t-1}$. If $x_i^{t-1} = c_i$, by *contraction independence,* $x^{t-1} = F\left(c_K, E_{N \backslash K}, E\right)$. Thus, $F_i\left(c_K, E_{N \backslash K}, E\right) = c_i$. Alternatively, if $x_i^{t-1} > c_i$, by *Lemma 1,* $F_i\left(c_K, E_{N \backslash K}, E\right) = c_i$. If $K = M^{t-1}$, by construction of g in Step 1, we have $i \in M^t$ and $x_i^t = c_i$. Otherwise, $F_i\left(c_K, E_{N \backslash K}, E\right) = c_i$ and *Lemma 2* imply $x_i^t \geqq c_i$. Thus $i \in M^t$. Since $i \in M^{t-1}$, by definition, $x_i^t \leqq c_i$. Thus overall, $x_i^t = c_i$.

Step 3: $x^n \leqq c$. First assume $M^{t-1} = M^t$ for some $t \in \{0, ..., n-1\}$. Then by definition, $x^t = x^{t+1}$ and thus, $M^t = M^{t+1}$. Iterating, $x^t = x^n$. Also, by Step 2, for each $i \in M^t$, $x_i^t = c_i$. Thus, $x^t = x^n \leqq c$. Alternatively, assume $M^{t-1} \neq M^t$ for each $t \in \{0, ..., n-1\}$. By Step 2, $M^{n-1} = N$. Thus $x^n = F(c, E) \leqq c$.

Step 4: $F = F^{g,r}$. Let $(c, E) \in \mathcal{C}$. By *Remark 1,* assume $c \leqq E_N$.[11] Note that $F^{g,r}(c, E) = x^n = F\left(c_{M^{n-1}}, E_{N \backslash M^{n-1}}, E\right)$. By Step 3, $x^n \leqq c \leqq \left(c_{M^{n-1}}, E_{N \backslash M^{n-1}}\right)$. Then, by *contraction independence*, $x^n = F(c, E)$.

Step 5: *g is a recursive adjustment function.* Since F is *c-continuous*, g^n is continuous in c. Also, Step 2 above proves Property 1. Now let $i \in N$ and $t \in \{1, ..., n\}$.

For Property 2, assume $x_i^{t-1} < c_i$. Then $i \notin M^{t-1}$ implies $i \notin M^{t-2}$. If $M^{t-2} = M^{t-1}$, by definition, $x^t = x^{t-1}$ and thus, $x_i^t \geqq x_i^{t-1}$. Otherwise, by *Lemma 2,* $x_i^t \geqq x_i^{t-1}$.

For Property 3, assume $x_i^{t-1} < \widetilde{c}_i < c_i$. Let $\widetilde{c} = (\widetilde{c}_i, c_{-i})$. Then $i \notin M(x^{t-1}, c) = M(x^{t-1}, \widetilde{c})$. Thus, by definition of g, we have $g(x^{t-1}, c, E) = g(x^{t-1}, \widetilde{c}, E)$.

For Property 4, assume $c_i < \widetilde{c}_i$. Let $\widetilde{c} = (\widetilde{c}_i, c_{-i})$, $x = g^n(r(E), c, E)$ and $\widetilde{x} = g^n(r(E), \widetilde{c}, E)$. Then, $x = F(c, E)$ and $\widetilde{x} = F(\widetilde{c}, E)$. By *Lemma 2,* $x_i \leqq \widetilde{x}_i$ and for each $j \in N \backslash \{i\}$, $x_j \geqq \widetilde{x}_j$.

■

Proof. (Theorem 2)

[11] If $c_i > E$, $\min\{c_i, E\} = E$, and by *Remark 1,* both $F(c, E) = F\left(c_{N \backslash i}, E_i, E\right)$ and $F^{g,r}(c, E) = F^{g,r}\left(c_{N \backslash i}, E_i, E\right)$. Then $F^{g,r}\left(c_{N \backslash i}, E_i, E\right) = F\left(c_{N \backslash i}, E_i, E\right)$ implies $F^{g,r}(c, E) = F(c, E)$.

It is straightforward to show that EG satisfies the given properties. Conversely, let F be a rule that satisfies them. We next show $F = EG$. Let $(c, E) \in \mathcal{C}$. By *Remark 1,* assume $c \leqq E_N$ (see *Footnote 12*). Without loss of generality, assume $c_1 \leqq ... \leqq c_n$. Let $c^0 = E_N$, $c^n = c$, and for each $k \in \{1, ..., n-1\}$, let $c^k = \left(c_{\{1,...,k\}}, E_{\{k+1,...,n\}}\right)$.

We inductively show that for each $k \in \{0, ..., n\}$, we have $F\left(c^k, E\right) = EG\left(c^k, E\right)$. For $k = n$, this will imply the desired conclusion. Initially, let $k = 0$. By *equal treatment of equals*, $F\left(c^0, E\right) = EG\left(c^0, E\right)$. Now let $k \in \{1, ..., n\}$ and assume that the statement holds for each $l < k$.

Case 1: There is $l < k$ such that $F\left(c^k, E\right) = F\left(c^l, E\right)$. Then, by our assumption, $F\left(c^l, E\right) = EG\left(c^l, E\right)$. Thus, $EG\left(c^l, E\right) \leqq c^k \leqq c^l$. This, by *contraction independence*, implies $EG\left(c^l, E\right) = EG\left(c^k, E\right)$. Combining the equalities, we then have $F\left(c^k, E\right) = EG\left(c^k, E\right)$.

Case 2: For each $l < k$, $F\left(c^k, E\right) \neq F\left(c^l, E\right)$. Thus, $F\left(c^k, E\right)$ is first obtained at c^k.

We first show that $F_k\left(c^k, E\right) = c_k$. For this, note that $F\left(c^k, E\right) \neq F\left(c^{k-1}, E\right)$ implies, by *contraction independence*, $F_k\left(c^{k-1}, E\right) > c_k$. Thus, by *Lemma 1*, $F_k\left(c^k, E\right) = c_k$.

We next show that for each $l < k$, $F_l\left(c^k, E\right) = c_l$. Let $\bar{c} = c^k + (c_k - c_l)\, e_l$ and note that $\bar{c} \geqq c^k$. First assume $F(\bar{c}, E) \neq F\left(c^k, E\right)$. Then $c_l < c_k$. Thus, by *contraction independence*, $F_l(\bar{c}, E) > c_l$. This, by *Lemma 1*, implies $F_l\left(c^k, E\right) = c_l$. Next, assume $F(\bar{c}, E) = F\left(c^k, E\right)$. By *equal treatment of equals*, $F_l(\bar{c}, E) = F_k(\bar{c}, E)$. Then, $F_l\left(c^k, E\right) = F_k\left(c^k, E\right)$. This implies $F_l\left(c^k, E\right) = c_k \geqq c_l$. Thus, $F_l\left(c^k, E\right) = c_l$

Overall, for each $l \in \{1, ..., k\}$, $F_l\left(c^k, E\right) = c_l$. This, by *equal treatment of equals*, implies for each $i \in \{k+1, ..., n\}$, $F_i\left(c^k, E\right) = \frac{E - \sum_{j=1}^{k} c_j}{n-k}$. Applying the same arguments to EG shows that it picks the same allocation. Thus, $F\left(c^k, E\right) = EG\left(c^k, E\right)$. ■

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