Available online at www.sciencedirect.com
SCIENCE DIRECT•

mathematical social sciences

Mathematical Social Sciences 45 (2003) 313–331
www.elsevier.com/locate/econbase

Consistency, converse consistency, and aspirations in TU-games

Toru Hokari[a,*], Özgür Kıbrıs[b]

[a]Institute of Social Sciences, University of Tsukuba, 1-1-1, Ten'no-dai, Tsukuba, Ibaraki 306-8571, Japan
[b]Faculty of Arts and Social Sciences, Sabancı University, Orhanlı, 81474 Tuzla, Istanbul, Turkey

Received 1 January 2001; received in revised form 1 May 2002; accepted 1 October 2002

Abstract

In problems of choosing 'aspirations' for TU-games, we study two axioms, 'MW-consistency' and 'converse MW-consistency.' In particular, we study which subsolutions of the aspiration correspondence satisfy MW-consistency and/or converse MW-consistency. We also provide axiomatic characterizations of the aspiration kernel and the aspiration nucleolus.

Keywords: Aspirations; Consistency; Converse consistency; Reduced games; TU-games

JEL classification: C71

1. Introduction

A transferable utility coalitional game (TU-game, for short) associates with each coalition of agents a real number representing what the coalition can achieve on its own, its 'worth'. Given a class of TU-games, a 'solution' associates with each game in the class a non-empty set of payoff vectors. The analysis of TU-games proposes solutions that answer two basic questions:

(i) Which coalitions form?
(ii) What is the payoff of each member of a coalition that forms?

Most studies on TU-games, however, assume that the grand coalition eventually

*Corresponding author. Fax: +81-298-53-6611.
E-mail addresses: hokari@social.tsukuba.ac.jp (T. Hokari), ozgur@sabanciuniv.edu (Ö. Kıbrıs).

0165-4896/02/$ – see front matter
doi:10.1016/S0165-4896(02)00085-9

forms. Then, the analysis reduces to answering question (ii), that is, determining the payoff distribution for the grand coalition.

In this paper, we do not presuppose the formation of the grand coalition and study those solutions that, in a sense, simultaneously answer both questions (i) and (ii). An 'aspiration' for a given game is a payoff vector that summarizes predictions about which coalitions are likely to form and what the resulting payoffs of their members will be. For each agent i, let x_i be a payoff that she demands in return for her cooperation. Then, it is natural to assume that a coalition S forms only if the demands of its members are jointly compatible, namely, $(x_i)_{i\in S}$ is feasible for S. A payoff vector x is an aspiration if no coalition can improve upon its component of x, and for each agent i, there exists at least one coalition for which agent i's demand x_i is jointly compatible with those of other members.

Studies of coalitional games have revealed that the set of aspirations is closely related to the outcomes obtained from two alternative approaches: the 'multi-coalitional bargaining approach' and the 'noncooperative approach'.[1]

In the multi-coalitional bargaining approach, each coalition is assigned a set of attainable payoffs and a 'bargaining solution', which is interpreted as summarizing the bargaining process among the members of that coalition. The disagreement point in each coalition's bargaining problem is determined endogenously as the expectation of each member on what she can obtain from alternative coalitions. A payoff vector x is a 'multi-coalitional bargaining outcome' if, for each coalition S, $(x_i)_{i\in S}$ is chosen by the assigned bargaining solution for the bargaining problem with the disagreement point associated with x. It so happens that every multi-coalitional bargaining outcome is an aspiration. Conversely, each aspiration can be obtained as a multi-coalitional bargaining outcome for some initial specification of bargaining solutions.

In the non-cooperative approach, versions of the following coalition formation game are analyzed: a randomly chosen agent proposes a coalition to be formed and a feasible payoff distribution for its members. The proposal is accepted if every member of the coalition agrees upon it. Otherwise, in the next period the first agent who rejected the proposal makes a new proposal. The game ends when a proposal is accepted. It turns out that the set of aspirations of the original coalitional game is equal to the set of stationary subgame perfect equilibrium proposals of this non-cooperative game.[2]

Its relation to the bargaining and non-cooperative approaches strongly suggests that the set of aspirations is an appropriate object to focus on if one wants to analyze coalitional games without imposing the assumption that the grand coalition eventually forms. Let us refer to the solution that assigns to each game its set of aspirations as the 'aspiration correspondence'. In the literature, several subsolutions of the aspiration correspondence have been studied. We are interested in their properties. In particular, we focus on two properties: 'consistency' and 'converse consistency'.[3]

[1] See Bennett (1983) for more detail.

[2] The particular non-cooperative game described here is due to Selten (1981). The literature following the original paper includes Chatterjee et al. (1993), Perry and Reny (1994), and Moldovanu and Winter (1994b, 1995).

[3] See Thomson (1996) for an extensive survey of studies on these properties applied to various models of game theory and economics.

Consistency deduces from the desirability of a payoff vector in a game the desirability of its restrictions to all subgroups of agents in the associated 'reduced games'. Suppose that a set of agents N is facing a game and a payoff vector x is agreed upon. Suppose then that some agents leave. Then let us reevaluate the situation from the viewpoint of the remaining agents N'. Namely, for each coalition $S \subseteq N'$, let us identify what S can obtain without any help from other members of N'. In this context, since any agent i in $N \backslash N'$ has agreed upon x, it seems natural to assume that she will be willing to cooperate with S if offered x_i. Additionally, suppose that S can choose such 'partners' from $N \backslash N'$. Then the revised worth of S would be the maximal (total) payoff that S can obtain in this manner. This operation defines a game in which the set of agents is N'. We refer to this game as an 'MW-reduced game' since it was introduced by Moldovanu and Winter (1994a) and Winter (1989).[4] 'MW-consistency' states that, in this reduced game, the original agreement should be confirmed, namely, $(x_i)_{i \in N'}$ should be agreed upon.

Converse consistency deduces the desirability of a payoff vector in a game from the desirability of its restrictions to all pairs of agents in the associated two-agent reduced games. Consider a game for N and a payoff vector x under evaluation. Suppose that for each pair of agents $\{i,j\}$ in N, (x_i,x_j) is chosen for the MW-reduced game associated with x and $\{i,j\}$. 'Converse MW-consistency' states that, in such a situation, x should be chosen for the original game.

On the domain of all TU-games, the aspiration correspondence and the partnered aspiration solution satisfy both *MW-consistency* and *converse MW-consistency* (Moldovanu and Winter, 1994a; Winter, 1989). In this paper, we analyze which other subsolutions of the aspiration correspondence satisfy *MW-consistency* and/or *converse MW-consistency*. Moreover, we obtain axiomatic characterizations of two solutions: the aspiration kernel and the aspiration nucleolus.

The paper is organized as follows. In Section 2, we introduce basic concepts and notations used later on. In Section 3, we study which subsolutions of the aspiration correspondence satisfy *MW-consistency* and/or *converse MW-consistency*. In Section 4, we provide axiomatic characterizations of the aspiration kernel and the aspiration nucleolus. In Section 5, we give some remarks on the non-transferable utility case.

2. Preliminary

There is an infinite set of 'potential' agents, indexed by the members of the set $\mathbb{N}$ of natural numbers. Let $\mathcal{N}$ denote the set of non-empty and finite subsets of $\mathbb{N}$. Given a countable set A, let $\mathbb{R}^A$ denote the Cartesian product of $|A|$ copies of the set $\mathbb{R}$ of real numbers, indexed by the members of A. We use $\subset$ for strict set inclusion and $\subseteq$ for weak set inclusion. To simplify the notation, given $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $S \subseteq N$, we often write $x_S \equiv (x_i)_{i \in S}$ and $x(S) \equiv \Sigma_{i \in S} x_i$.

Given $N \in \mathcal{N}$, a *transferable utility coalitional game for N* (TU-game for N, for short)

[4] In the standard approach, the way of reevaluating the situation of the remaining agents we described above was first introduced by Davis and Maschler (1965). For the coalition of all remaining agents, its revised worth in the standard approach is defined to be the worth of the original grand coalition minus the sum of payoffs assigned for those agents who left.

is a function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. For each $S \subseteq N$, the number $v(S)$ represents what coalition S can obtain on its own, its 'worth'. Let $\mathscr{V}^N_{\text{all}}$ denote the class of all TU-games for N, and $\mathscr{V}_{\text{all}} \equiv \cup_{M \in \mathscr{N}} \mathscr{V}^M_{\text{all}}$.

Given $N \in \mathscr{N}$ and $v \in \mathscr{V}^N_{\text{all}}$, a payoff vector $x \in \mathbb{R}^N$ is *individually feasible in* v if for each $i \in N$, there exists $S \subseteq N$ such that $i \in S$ and $x(S) \leq v(S)$. It is *coalitionally rational in* v if for each $S \subseteq N$, $x(S) \geq v(S)$. An *aspiration for* v is a payoff vector in $\mathbb{R}^N$ satisfying individual feasibility and coalitional rationality.

Given $N \in \mathscr{N}$, $v \in \mathscr{V}^N_{\text{all}}$, and $x \in \mathbb{R}^N$, the *set of generating coalitions for* v *and* x, denoted $\mathscr{GC}(v,x)$, contains those coalitions whose members' promised payoffs in x are jointly compatible:

$$\mathscr{GC}(v,x) \equiv \{S \subseteq N \mid x(S) \leq v(S)\}.$$

Thus, if each agent demands her component of the payoff vector, the generating coalitions are the only coalitions that are likely to form. In an aspiration for the game, each agent is a member of at least one generating coalition and each generating coalition distributes payoffs efficiently among its members.

A *solution on* $\mathscr{V}_{\text{all}}$ is a correspondence from $\mathscr{V}_{\text{all}}$ to $\cup_{N \in \mathscr{N}} \mathbb{R}^N$ that associates with each $N \in \mathscr{N}$ and each $v \in \mathscr{V}^N_{\text{all}}$ a non-empty set of payoff vectors satisfying individual feasibility. We use φ to denote a generic solution.

3. Consistency and converse consistency

Given $N \in \mathscr{N}$, $v \in \mathscr{V}^N_{\text{all}}$, $x \in \mathbb{R}^N$, and $N' \subset N$, the *MW-reduced game of* v *relative to* x *and* N', denoted $r^x_{N'}(v)$, is defined by setting for each $S \subseteq N'$

$$r^x_{N'}(v)(S) \equiv \begin{cases} \max_{T \subseteq N \setminus N'} [v(S \cup T) - x(T)], & \text{if } S \neq \emptyset, \\ 0, & \text{if } S = \emptyset. \end{cases}$$

'MW-consistency' (Moldovanu and Winter, 1994a; Winter, 1989) states that if a payoff vector is chosen for a game, then the restriction of it to any subgroup should be chosen for the associated MW-reduced game (see Fig. 1).

MW-consistency. For each $N \in \mathscr{N}$, each $v \in \mathscr{V}^N_{\text{all}}$, each $x \in \varphi(v)$, and each $N' \subset N$, we have $x_{N'} \in \varphi(r^x_{N'}(v))$.

'Converse MW-consistency' (Moldovanu and Winter, 1994a; Winter, 1989) states that if a payoff vector for a game is such that its restriction to any pair of agents is chosen for the associated two-agent MW-reduced game, then it should be chosen for the original game.

Converse MW-consistency. For each $N \in \mathscr{N}$, each $v \in \mathscr{V}^N_{\text{all}}$, and each $x \in \mathbb{R}^N$, if for each $N' \subset N$ with $|N'| = 2$, we have $x_{N'} \in \varphi(r^x_{N'}(v))$, then $x \in \varphi(v)$.

On $\mathscr{V}_{\text{all}}$, the 'aspiration correspondence' (Albers, 1979; Bennett, 1983; Cross, 1967)

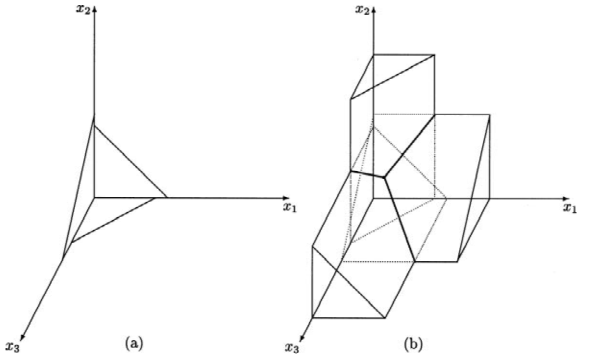

Fig. 1. Aspirations in three-agent TU-games. Let $N=\{1,2,3\}$, $v(\{1\})=v(\{2\})=v(\{3\})=0$, $v(\{1,2\})=6$, $v(\{1,3\})=5$, $v(\{2,3\})=7$, $v(N)=0$. The set of aspirations for v is the union of the three thick line segments in (b).

and the 'partnered aspiration solution' (Albers, 1979; Bennett, 1983) are *MW-consistent* and *conversely MW-consistent* (Moldovanu and Winter, 1994a; Winter, 1989). It follows from this result that each of these two solutions is the unique *MW-consistent* and *conversely MW-consistent* extension of its two-agent version to the *n*-agent case.

In the following subsections, we introduce other subsolutions of the aspiration correspondence that have been studied in the literature, and check which of them satisfies *MW-consistency* and/or *converse MW-consistency*.

3.1. Balanced aspiration solution

Given $N\in\mathcal{N}$, a collection $\mathcal{B}\subseteq 2^N$ of coalitions is *weakly balanced on N* if there exists a list of non-negative weights $(\delta_S)_{S\in\mathcal{B}}$ such that for each $i\in N$

$$\sum_{\substack{S\in\mathcal{B}\\ S\ni i}}\delta_S=1.$$

It is *strictly balanced on N* if, in addition, all weights are positive.

Cross (1967) and Bennett (1983) argue that the competition among the coalitions for 'scarce' agents leads to a balanced structure of the generating coalitions, by driving up the payoff demands of these agents and driving down the payoff demands of others. The following solution is based on this idea:

Balanced aspiration solution, *BalAsp***.** For each $N\in\mathcal{N}$ and each $v\in\mathcal{V}^N_{\text{all}}$, *BalAsp*($v$) is the collection of aspirations x for v such that $\mathcal{GC}(v,x)$ is weakly balanced on N.

Proposition 3.1. *On $\mathcal{V}_{all}$, the balanced aspiration solution is MW-consistent.*

Proof. Let $N\in\mathcal{N}$, $v\in\mathcal{V}^N_{\text{all}}$, $x\in BalAsp(v)$, and $\mathcal{B}\equiv\mathcal{GC}(v,x)$. Then there exists $(\delta_S)_{S\in\mathcal{B}}\in\mathbb{R}^{\mathcal{B}}_+$ such that for each $i\in N$

$$\sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \delta_S = 1.$$

Let $N' \subset N$ and $\mathcal{B}' \equiv \mathcal{GC}(r^x_{N'}(v), x_{N'})$. Since the aspiration correspondence is *MW-consistent*, $x_{N'}$ is an aspiration for $r^x_{N'}(v)$. For each $S \in \mathcal{B}'$, let

$$\lambda_S \equiv \sum_{\substack{T \subseteq N \setminus N' \\ \text{s.t. } S \cup T \in \mathcal{B}}} \delta_{S \cup T}.$$

Then $(\lambda_S)_{S \in \mathcal{B}'} \in \mathbb{R}^{\mathcal{B}'}_+$ and for each $i \in N'$

$$\sum_{\substack{S \in \mathcal{B}' \\ S \ni i}} \lambda_S = \sum_{\substack{S \in \mathcal{B}' \\ S \ni i}} \sum_{\substack{T \subseteq N \setminus N' \\ \text{s.t. } S \cup T \in \mathcal{B}}} \delta_{S \cup T} = \sum_{\substack{R \in \mathcal{B} \\ R \ni i}} \delta_R = 1.$$

Thus, $\mathcal{B}'$ is weakly balanced on N' and, hence, $x_{N'} \in BalAsp(r^x_{N'}(v))$. □

The following example shows that the balanced aspiration solution violates *converse MW-consistency*.

Example 3.1. Let $N \equiv \{1,2,3\}$. Consider the following TU-game: for each $S \subseteq N$

$$v(S) \equiv \begin{cases} 1, & \text{if } |S| = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x \equiv (1,1,0)$. Then x is an aspiration for v and $\mathcal{GC}(v,x) = \{\{1,3\},\{2,3\},\{3\}\}$. Note that the aspiration correspondence and the balanced aspiration solution coincide in the two-agent case (Fig. 2). Therefore, as the aspiration correspondence is *MW-consistent*, for each $N' \subset N$ with $|N'| = 2$, we have $x_{N'} \in BalAsp(r^x_{N'}(v))$. However, since $\mathcal{GC}(v,x)$ is not weakly balanced, we have $x \not\in BalAsp(v)$.

3.2. Aspiration kernel

Next, we turn to a solution that is similar to the prekernel (Maschler et al., 1972). Given $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{\text{all}}$, $x \in \mathbb{R}^N$, and $i,j \in N$ with $i \neq j$, let

$$s_{ij}(v,x) \equiv \max_{\substack{S \ni i \\ S \not\ni j}} [v(S) - x(S)].$$

The number $s_{ij}(v,x)$ represents the maximum 'surplus' that agent i can obtain without the cooperation of agent j, supposing that other agents agree upon x. For each game v, the aspiration kernel (Bennett, 1981) chooses those aspirations that equalize these surpluses for each pair of agents.

Aspiration kernel, *AspKer***.** For each $N \in \mathcal{N}$ and each $v \in \mathcal{V}^N_{\text{all}}$, $AspKer(v)$ is the collection of aspirations x for v such that for each $S \in \mathcal{GC}(v,x)$ and each pair $i,j \in S$, we have $s_{ij}(v,x) = s_{ji}(v,x)$.

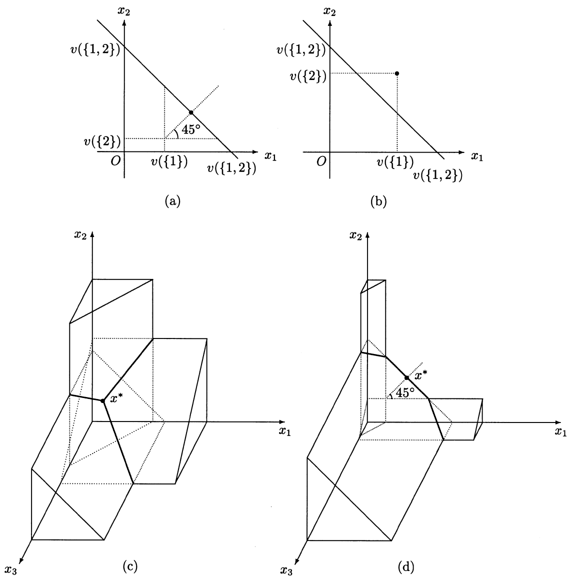

Fig. 2. The aspiration kernel and the aspiration nucleolus in two-agent and three-agent TU-games. In these cases, the aspiration kernel coincides with the aspiration nucleolus. (a) If $v(\{1,2\}) \geq v(\{1\}) + v(\{2\})$, they select the 'standard solution' payoff vector: each agent is given first her individual worth, and then what remains is divided equally. (b) Otherwise, they select $(v(\{1\}),v(\{2\}))$. In panels (c) and (d), $v(\{1,2,3\}) = 0$ and the payoff vector chosen by them is indicated as x^*.

The next lemma is due to Peleg (1986), and it essentially implies that the aspiration kernel is *MW-consistent* and *conversely MW-consistent*.

Lemma 3.1. *For each $N \in \mathcal{N}$, each $v \in \mathcal{V}^N_{all}$, each $x \in \mathbb{R}^N_+$, each $N' \subset N$, and each pair $i,j \in N'$, we have $s_{ij}(r^x_{N'}(v),x_{N'}) = s_{ij}(v,x)$.*

Proposition 3.2. *On $\mathcal{V}_{all}$, the aspiration kernel is MW-consistent and conversely MW-consistent.*

Proof. (*MW-consistency*) Let $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{\text{all}}$, $x \in AspKer(v)$, and $N' \subset N$. Since the aspiration correspondence is *MW-consistent*, $x_{N'}$ is an aspiration for $r^x_{N'}(v)$. Let $S \in \mathcal{GC}(r^x_{N'}(v),x_{N'})$ and $i,j \in S$ with $i \neq j$. Then, by the definition of $r^x_{N'}(v)(S)$, there exists $T \subseteq N \backslash N'$, where T may be the empty set, such that $x(S) \leq r^x_{N'}(v)(S) = v(S \cup T) - x(T)$. Since $x \in AspKer(v)$, $S \cup T \in \mathcal{GC}(v,x)$, and $i,j \in S \cup T$, we have $s_{ij}(v,x) = s_{ji}(v,x)$. By Lemma 3.1, $s_{ij}(r^x_{N'}(v),x_{N'}) = s_{ji}(r^x_{N'}(v),x_{N'})$.

(*Converse MW-consistency*) Let $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{\text{all}}$, and $x \in \mathbb{R}^N$ be such that for each $N' \subset N$ with $|N'| = 2$, $x_{N'} \in AspKer(r^x_{N'}(v))$. Since the aspiration correspondence is *conversely MW-consistent* and the aspiration kernel is its subsolution, x is an aspiration for v. Let $S \in \mathcal{GC}(v,x)$ and $i,j \in S$ with $i \neq j$. Then $x_i + x_j \leq v(S) - x(S \backslash \{i,j\}) \leq$

$r^x_{\{i,j\}}(v)(\{i,j\})$ and, hence, $\{i,j\} \in \mathscr{GC}(r^x_{\{i,j\}}(v),x_i,x_j)$. Since $(x_i,x_j) \in AspKer(r^x_{\{i,j\}}(v))$, we have $s_{ij}(r^x_{\{i,j\}}(v),x_i,x_j) = s_{ji}(r^x_{\{i,j\}}(v),x_i,x_j)$. By Lemma 3.1, $s_{ij}(v,x) = s_{ji}(v,x)$. □

3.3. Aspiration nucleolus

Next, we analyze a solution which is closely related to the prenucleolus (Schmeidler, 1969).

For each $x \in \mathbb{R}^N$, let $e(v,x)$ be the vector in $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ defined by setting for each $S \in 2^N \setminus \{\emptyset\}$, $e_S(v,x) \equiv v(S) - x(S)$. The number $e_S(v,x)$ represents the dissatisfaction of S in v at x. Also, let $\theta(e(v,x)) \in \mathbb{R}^{2^{|N|}-1}$ be obtained by rearranging the coordinates of $e(v,x)$ in non-increasing order.

Given $m \in \mathbb{N}$ and $z,z' \in \mathbb{R}^m$, *z is lexicographically smaller than z'* if either (i) $z_1 < z'_1$ or (ii) there exists $k > 1$ such that $z_k < z'_k$ and for each $\ell < k$, $z_\ell = z'_\ell$.

For each game, the aspiration nucleolus (Bennett, 1981) selects a payoff vector that lexicographically minimizes the dissatisfactions of the coalitions over the set of aspirations.

Aspiration nucleolus, ***AspNuc*****.** For each $N \in \mathscr{N}$ and each $v \in \mathscr{V}^N_{\text{all}}$, $AspNuc(v)$ is the unique aspiration x for v such that for each other aspiration y for v, $\theta(e(v,x))$ is lexicographically smaller than $\theta(e(v,y))$.

The aspiration nucleolus is a subsolution of both the aspiration kernel and the balanced aspiration solution (Sharkey, 1993).

Given $N \in \mathscr{N}$, $v \in \mathscr{V}^N_{\text{all}}$, $x \in \mathbb{R}^N$, and $\alpha \in \mathbb{R}$, let

$$\mathscr{S}_\alpha(v,x) \equiv \{S \subset N \mid e_S(v,x) \geq \alpha\}.$$

The following lemma is a characterization of the aspiration nucleolus.[5]

Lemma 3.2. (Sharkey, 1993) *For each $N \in \mathscr{N}$, each $v \in \mathscr{V}^N_{all}$, and each aspiration x for v, $x = AspNuc(v)$ if and only if for each $\alpha \in \mathbb{R}$ with $\mathscr{S}_\alpha(v,x) \neq \emptyset$, $\mathscr{S}_\alpha(v,x)$ is strictly balanced on N.*

We use Lemma 3.2 to prove the following result:

Proposition 3.3. *On $\mathscr{V}_{all}$, the aspiration nucleolus is MW-consistent.*

Proof. Let $N \in \mathscr{N}$, $v \in \mathscr{V}_{\text{all}}$, $x \equiv AspNuc(v)$, and $N' \subset N$. Since the aspiration correspondence is *MW-consistent*, $x_{N'}$ is an aspiration for $r^x_{N'}(v)$. Let $\alpha \in \mathbb{R}$ be such that $\mathscr{S}_\alpha(r^x_{N'}(v),x_{N'}) \neq \emptyset$. By the definition of $r^x_{N'}(v)$, for each $S \in \mathscr{S}_\alpha(r^x_{N'}(v),x_{N'})$, there exists $T \subseteq N \setminus N'$, where T may be the empty set, such that $r^x_{N'}(v)(S) = v(S \cup T) - x(T)$. Since $v(S \cup T) - x(T) - x(S) \geq \alpha$, $\mathscr{S}_\alpha(v,x) \neq \emptyset$. To simplify the notation, let $\mathscr{B} \equiv \mathscr{S}_\alpha(v,x)$ and

[5] A similar characterization of the (pre)nucleolus is due to Kohlberg (1971).

$\mathcal{B}' \equiv \mathcal{S}_\alpha(r^x_{N'}(v),x_{N'})$. By Lemma 3.2, $\mathcal{B}$ is strictly balanced on N. Thus, there exists a list $(\delta_S)_{S\in\mathcal{B}}$ of positive weights such that for each $i\in N$

$$\sum_{\substack{S\in\mathcal{B}\\ S\ni i}} \delta_S = 1.$$

For each $S\in\mathcal{B}'$, let

$$\lambda_S \equiv \sum_{\substack{T\subseteq N\backslash N'\\ \text{s.t. } S\cup T\in\mathcal{B}}} \delta_{S\cup T}.$$

As shown above, for each $S\in\mathcal{B}'$, there exists $T\subseteq N\backslash N'$ such that $S\cup T\in\mathcal{B}$. Thus, for each $S\in\mathcal{B}'$, $\lambda_S>0$. Note also that, for each $i\in N'$

$$\sum_{\substack{S\in\mathcal{B}'\\ S\ni i}} \lambda_S = \sum_{\substack{S\in\mathcal{B}'\\ S\ni i}} \sum_{\substack{T\subseteq N\backslash N'\\ \text{s.t. } S\cup T\in\mathcal{B}}} \delta_{S\cup T} = \sum_{\substack{R\in\mathcal{B}\\ R\ni i}} \delta_R = 1.$$

Thus, $\mathcal{B}'$ is strictly balanced on N'. Since this is true for each $\alpha\in\mathbb{R}$ with $\mathcal{S}_\alpha(r^x_{N'}(v),x_{N'})\neq\emptyset$, by Lemma 3.2 we have $x_{N'} = AspNuc(r^x_{N'}(v))$. □

As the following example shows, the aspiration nucleolus violates *converse MW-consistency*.

Example 3.2. Let $N\equiv\{1,2,3,4\}$. Consider the following TU-game for N: for each $S\subseteq N$

$$v(S) \equiv \begin{cases} 6, & \text{if } S\in\{\{1,2,3\},\{1,2,4\}\},\\ 0, & \text{otherwise.}\end{cases}$$

Let $x\equiv(3,3,0,0)$ and $y\equiv(2,2,2,2)$. It can be shown that $x\in AspKer(v)$, y is an aspiration for v, and $e(v,y)$ is lexicographically smaller than $e(v,x)$. Thus, $x\neq AspNuc(v)$. Since the aspiration kernel is *MW-consistent* and it coincides with the aspiration nucleolus in the two-agent case, for each pair $i,j\in N$

$$(x_i,x_j)\in AspKer(r^x_{\{i,j\}}(v)) = \{AspNuc(r^x_{\{i,j\}}(v))\}.$$

Thus, x satisfies the hypothesis of *converse MW-consistency* for the aspiration nucleolus. However, we have $x\neq AspNuc(v)$. □

3.4. Equal gains aspiration solution

The next subsolution of the aspiration correspondence is based on the premise that agents when bargaining tend to share the gains equally. In our context, by forming a coalition, the agents forego the payoffs that they could have attained by forming alternative coalitions. Therefore, each agent's largest payoff from alternative coalitions serves as an 'outside option'. Formally, given $N\in\mathcal{N}$, $v\in\mathcal{V}^N_{\text{all}}$, $x\in\mathbb{R}^N$, $S\in\mathcal{GC}(v,x)$, and $i\in S$, the *outside option for i relative to v, x, and S* is defined by

$$d_i^S(v,x) \equiv \max_{\substack{T \ni i \\ T \neq S}} [v(T) - x(T \setminus \{i\})].$$

Equal gains aspiration solution, *EqAsp***.** For each $N \in \mathcal{N}$ and each $v \in \mathcal{V}_{\text{all}}^N$, $EqAsp(v)$ is the collection of aspirations for v such that for each $S \in \mathcal{GC}(v,x)$ and each pair $i,j \in S$, we have $x_i - d_i^S(v,x) = x_j - d_j^S(v,x)$.

Note that the aspiration kernel is a subsolution of the equal gains aspiration solution. As the following example shows the equal gains aspiration solution violates *MW-consistency*.

Example 3.3. Let $N \equiv \{1,2,3\}$. Consider the following TU-game: for each $S \subseteq N$

$$v(S) \equiv \begin{cases} 6, & \text{if } S \in \{\{1,2,3\},\{1,2\}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x \equiv (4,2,0)$. Then x is an aspiration for v and

$$\mathcal{GC}(v,x) = \{\{3\},\{1,2\},\{1,2,3\}\}.$$

Note that

$$x_1 - d_1^{\{1,2\}}(v,x) = -\max_{\substack{S \ni 1 \\ S \neq \{1,2\}}} [v(S) - x(S)] = -\max\{0-4, 0-4, 6-6\} = 0,$$

$$x_2 - d_2^{\{1,2\}}(v,x) = -\max_{\substack{S \ni 2 \\ S \neq \{1,2\}}} [v(S) - x(S)] = -\max\{0-2, 0-2, 6-6\} = 0,$$

$$x_1 - d_1^{\{1,2,3\}}(v,x) = -\max_{\substack{S \ni 1 \\ S \neq \{1,2,3\}}} [v(S) - x(S)] = -\max\{0-4, 6-6, 0-4\} = 0,$$

$$x_2 - d_2^{\{1,2,3\}}(v,x) = -\max_{\substack{S \ni 2 \\ S \neq \{1,2,3\}}} [v(S) - x(S)] = -\max\{0-2, 6-6, 0-2\} = 0,$$

$$x_3 - d_3^{\{1,2,3\}}(v,x) = -\max_{\substack{S \ni 3 \\ S \neq \{1,2,3\}}} [v(S) - x(S)] = -\max\{0-0, 0-4, 0-3\} = 0.$$

Thus, $x \in EqAsp(v)$. Note that

$$r_{\{1,2\}}^x(v)(\{1\}) = \max\{v(\{1\}), v(\{1,3\}) - x_3\} = \max\{0, 0-0\} = 0,$$

$$r_{\{1,2\}}^x(v)(\{2\}) = \max\{v(\{2\}), v(\{2,3\}) - x_3\} = \max\{0, 0-0\} = 0,$$

$$r_{\{1,2\}}^x(v)(\{1,2\}) = \max\{v(\{1,2\}), v(\{1,2,3\}) - x_3\} = \max\{6, 6-0\} = 6.$$

Since $EqAsp(r_{1,2}^x(v)) = \{(3,3)\}$, we have $(x_1,x_2) \notin EqAsp(r_{1,2}^x(v))$.

Proposition 3.4. *On $\mathcal{V}_{all}$, the equal gains aspiration solution is conversely MW-consistent.*

Proof. Let $N \in \mathcal{N}$, $v \in \mathscr{V}^N_{\text{all}}$, and $x \in \mathbb{R}^N$ be such that for each $N' \subset N$ with $|N'| = 2$, $x_{N'} \in EqAsp(r^x_{N'}(v))$. Note that the aspiration kernel and the equal gains aspiration solution coincide in the two-agent case. Since the aspiration kernel is *conversely MW-consistent*, $x \in AspKer(v)$. Since $AspKer(v) \subseteq EqAsp(v)$, we have $x \in EqAsp(v)$. □

4. Two axiomatic characterizations

In this section, we study the implications of *MW-consistency*, *converse MW-consistency*, and the following three basic axioms:

Equal treatment of equals. For each $N \in \mathcal{N}$, each $v \in \mathscr{V}_{\text{all}}$, and each pair $i,j \in N$, if for each $S \subseteq N \backslash \{i,j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$, then for each $x \in \varphi(v)$, $x_i = x_j$.

Anonymity. For each $N,M \in \mathcal{N}$ with $|N| = |M|$, each $v \in \mathscr{V}^N_{\text{all}}$, each $w \in \mathscr{V}^M_{\text{all}}$, and each bijection $b : N \to M$, if for each $S \subseteq N$, $w(\{b(i) \mid i \in S\}) = v(S)$, then

$$\varphi(w) = \{x \in \mathbb{R}^M \mid \text{there exists } y \in \varphi(v) \text{ such that for each } i \in N, x_i = y_{b(i)}\}.$$

Zero-independence. For each $N \in \mathcal{N}$, each pair $v,w \in \mathscr{V}^N_{\text{all}}$, and each $y \in \mathbb{R}^N$, if for each $S \subseteq N$, $w(S) = v(S) + \Sigma_{i \in S}\, y_i$, then for each $x \in \varphi(v)$, $x + y \in \varphi(w)$.

Table 1 summarizes which solutions satisfy which properties.

As mentioned in the Introduction, most studies on coalitional games assume that the grand coalition eventually forms. In this 'standard approach', a notion of reduced games was first introduced by Davis and Maschler (1965). Given $N \in \mathcal{N}$, $v \in \mathscr{V}^N_{\text{all}}$, $x \in \mathbb{R}^N$, and $N' \subset N$, the *DM-reduced game of v relative to x and N'* is defined by setting for each $S \subseteq N'$

Table 1
Properties of subsolutions of the aspiration correspondence on the domain of all TU-games. *Asp* and *ParAsp* denote the aspiration correspondence and the partnered aspiration solution, respectively.

Domain: $\mathscr{V}_{all}$	*Asp*	*ParAsp*	*BalAsp*	*AspNuc*	*AspKer*	*EqAsp*
Equal treatment of equals	No	No	No	Yes	Yes	Yes
Anonymity	Yes	Yes	Yes	Yes	Yes	Yes
Zero-independence	Yes	Yes	Yes	Yes	Yes	Yes
Single-valuedness	No	No	No	Yes	No	No
MW-consistency	Yes	Yes	Yes (Proposition 3.1)	Yes (Proposition 3.3)	Yes (Proposition 3.2)	No (Example 3.3)
Converse MW-consistency	Yes	Yes	No (Example 3.1)	No (Example 3.2)	Yes (Proposition 3.2)	Yes (Proposition 3.4)

$$\hat{r}^x_{N'}(v)(S) \equiv \begin{cases} \max\limits_{T \subseteq N \setminus N'} [v(S \cup T) - x(T)], & \text{if } S \not\in \{N', \emptyset\}, \\ v(N) - x(N \setminus N'), & \text{if } S = N', \\ 0, & \text{if } S = \emptyset. \end{cases}$$

Associated axioms of consistency and its converse are defined as follows.

DM-consistency. For each $N \in \mathcal{N}$, each $v \in \mathcal{V}^N_{\text{all}}$, each $x \in \varphi(v)$, and each $N' \subset N$, we have $x_{N'} \in \varphi(\hat{r}^x_{N'}(v))$.

Converse DM-consistency. For each $N \in \mathcal{N}$, each $v \in \mathcal{V}^N_{\text{all}}$, and each $x \in \mathbb{R}^N$ with $x(N) = v(N)$, if for each $N' \subset N$ with $|N'| = 2$ we have $x_{N'} \in \varphi(\hat{r}^x_{N'}(v))$, then $x \in \varphi(v)$.

Given a game v for N, a *preimputation for* v is a payoff vector $x \in \mathbb{R}^N$ with $x(N) = v(N)$. For each game v, the prekernel (Maschler et al., 1972) chooses those preimputations that equalize the surpluses for each pair of agents. On $\mathcal{V}_{\text{all}}$, the prekernel is the only subsolution of the preimputation correspondence satisfying *equal treatment of equals*, *zero-independence*, *DM-consistency*, and *converse DM-consistency* (Peleg, 1986). It so happens that a similar result holds for the aspiration kernel.

Theorem 4.1. *On $\mathcal{V}_{all}$, the aspiration kernel is the only subsolution of the aspiration correspondence satisfying equal treatment of equals, zero-independence, MW-consistency, and converse MW-consistency.*

Proof. Clearly, the aspiration kernel is a subsolution of the aspiration correspondence satisfying *equal treatment of equals* and *zero-independence*. By Proposition 3.2, it also satisfies *MW-consistency* and *converse MW-consistency*.

Conversely, let φ be a subsolution of the aspiration correspondence satisfying the four axioms. Clearly, φ coincides with the aspiration kernel in the two-agent case. Let $N \in \mathcal{N}$ with $|N| \geq 3$, and $v \in \mathcal{V}^N_{\text{all}}$. First, we show that $\varphi(v) \subseteq AspKer(v)$. Let $x \in \varphi(v)$. By *MW-consistency* of φ, for each $N' \subset N$ with $|N'| = 2$, we have $x_{N'} \in \varphi(r^x_{N'}(v)) = AspKer(r^x_{N'}(v))$. Since the aspiration kernel is *conversely MW-consistent*, $x \in AspKer(v)$.

Next, we show that $AspKer(v) \subseteq \varphi(v)$. Let $y \in AspKer(v)$. Since the aspiration kernel is *MW-consistent*, for each $N' \subset N$ with $|N'| = 2$

$$y_{N'} \in AspKer(r^y_{N'}(v)) = \varphi(r^y_{N'}(v)).$$

Since φ is *conversely MW-consistent*, $y \in \varphi(v)$.

Altogether, $\varphi(v) = AspKer(v)$. □

For each game, the prenucleolus (Schmeidler, 1969) selects a payoff vector that lexicographically minimizes the dissatisfactions of the coalitions over the set of preimputations. On $\mathcal{V}_{\text{all}}$, the prenucleolus is the only subsolution of the preimputation correspondence satisfying *single-valuedness*, *anonymity*, *zero-independence*, and *DM-consistency* (Sobolev, 1975).

It turns out that, by using Lemma 3.2 and by following the argument in Sobolev

(1975), one can obtain a similar axiomatic characterization of the aspiration nucleolus. (Since the proof is very long, we provide it in Appendix A.)

Theorem 4.2. *On $\mathscr{V}_{all}$, the aspiration nucleolus is the only subsolution of the aspiration correspondence satisfying single-valuedness, anonymity, zero-independence, and MW-consistency.*

5. Remarks on the NTU-case

The definitions of the aspiration correspondence, the partnered aspiration solution, the balanced aspiration solution, and the equal gains aspiration solution have been generalized to define corresponding solutions for non-transferable utility coalitional games (NTU-games, for short).

Moldovanu and Winter (1994a) study *MW-consistency* and *converse MW-consistency* on the domain of all NTU-games. They show that, on this domain, both the aspiration correspondence and the partnered aspiration solution satisfy these two properties. Here, we report, without proofs, two additional results:[6]

- On the domain of all NTU-games, the balanced aspiration solution satisfies *MW-consistency*.
- On the domain of all NTU-games, the equal gains aspiration solution violates *converse MW-consistency*.

Acknowledgements

We thank William Thomson, François Maniquet, and an anonymous referee for helpful comments. We are responsible for any remaining errors. Financial support from the Murata Science Foundation is gratefully acknowledged.

Appendix A

In this appendix, we provide the proof of Theorem 4.2. As mentioned before, the proof is similar to that of a theorem in Sobolev (1975), which is written in Russian. The proof of Sobolev's theorem (in English) can be found in Peleg (1988). Essential parts of Peleg's proof are reproduced in Snijders (1995).

Proof of Theorem 4.2. Clearly, the aspiration nucleolus is a subsolution of the

[6] The proofs are available from the authors on request.

aspiration correspondence satisfying *single-valuedness*, *anonymity*, and *zero-independence*. By Proposition 3.3, it is also *MW-consistent*.

Conversely, let φ be a subsolution of the aspiration correspondence satisfying the four axioms. Let $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{\text{all}}$, and $x \equiv AspNuc(v)$. We show, in seven steps, that $x = \varphi(v)$.

Let

$$A \equiv \{\alpha \in \mathbb{R} \mid \text{there exists } S \subset N \text{ such that } \alpha = e_S(v,x)\},$$

and $(\alpha_1, \alpha_2, \ldots, \alpha_{|A|}) \in \mathbb{R}^{|A|}$ be the enumeration of A with $\alpha_1 > \alpha_2 > \cdots > \alpha_{|A|}$. To simplify the notation, for each $k \in \{1,2,\ldots,|A|\}$, we write $\mathcal{S}_k \equiv \mathcal{S}_{\alpha_k}(v,x)$. Also, for each $k \in \{1,2,\ldots,|A|\}$ and each $i \in N$, let $\mathcal{S}^i_k \equiv \{S \in \mathcal{S}_k \mid i \in S\}$. Given $k \in \{1,2,\ldots,|A|\}$, by Lemma 3.2, $\mathcal{S}_k$ is strictly balanced on N. Moreover, the associated weights can be chosen to be rational. Thus, there exist a natural number μ_k and a list of natural numbers $(\mu_S)_{S \in \mathcal{S}_k}$ such that for each $i \in N$, $\Sigma_{S \in \mathcal{S}^i_k} \mu_S = \mu_k$. Let $\mathcal{B}_k$ be the partition of N such that for each pair $i,j \in N$, there exists $B \in \mathcal{B}_k$ with $i,j \in B$ if and only if $\mathcal{S}^i_k = \mathcal{S}^j_k$. Let $\beta_k \equiv \max_{B \in \mathcal{B}_k} |B|$, $\gamma_k \equiv \Sigma_{S \in \mathcal{S}_k} \mu_S$, and

$$\lambda_k \equiv \binom{\gamma_k}{\mu_k}.$$

Step 1. Given $k \in \{1,2,\ldots,|A|\}$, we construct $M_k \in \mathcal{N}$ and $\mathcal{T}_k \subseteq 2^{M_k} \backslash \{M_k, \emptyset\}$ that satisfy the following conditions:

(i) $N \subseteq M_k$;
(ii) $|M_k| = \beta_k \cdot \lambda_k$;
(iii) for each $S \in \mathcal{S}_k$, there exists $T \in \mathcal{T}_k$ such that $S \subseteq T$;
(iv) for each $S \subseteq N$ and each $T \in \mathcal{T}_k$, if $S \subseteq T$, then $T \cap N = S$ and $S \in S_k$;
(v) for each $i \in M_k$, we have $|\mathcal{T}^i_k| = \mu_k$ and $|\{j \in M_k \mid \mathcal{T}^j_k = \mathcal{T}^i_k\}| = \beta_k$, where $\mathcal{T}^i_k \equiv \{T \in \mathcal{T}_k \mid i \in T\}$ and $\mathcal{T}^j_k \equiv \{T \in \mathcal{T}_k \mid j \in T\}$.

Let $(B_1, B_2, \ldots, B_{|\mathcal{B}_k|})$ be an enumeration of $\mathcal{B}_k$. For each $h \in \{1,2,\ldots,\lambda_k\}$, we construct a set D_h of agents as follows:

- if $h \leq |\mathcal{B}_k|$ and $|B_h| = \beta_k$, then let $D_h \equiv B_h$;
- if $h \leq |\mathcal{B}_k|$ and $|B_h| < \beta_k$, then let D_h be the union of B_h and $(\beta_k - |B_h|)$ agents chosen from $\mathbb{N} \backslash N$;
- if $h > |\mathcal{B}_k|$, then let D_h be a set of β_k agents chosen from $\mathbb{N} \backslash N$.

Since $\mathcal{B}_k$ is a partition of N and the set of potential agents is countably infinite, it is clear that, in the above construction of D_h's, we can make them mutually exclusive. Then, let

$$M_k \equiv D_1 \cup D_2 \cup \cdots \cup D_{\lambda_k}.$$

Note that $\{D_1, D_2, \ldots, D_{\lambda_k}\}$ is a partition of M_k. By construction, M_k satisfies conditions (i) and (ii).

Next, imagine that there are γ_k empty 'rooms.' We will fill these rooms with (appropriately replicated) groups in $\{D_1, D_2, \ldots, D_{\lambda_k}\}$, and each room will correspond to an element of $\mathcal{T}_k$. For each $S \in \mathcal{S}_k$, create μ_S copies of the set

$$\bigcup_{\substack{h \in \{1,\ldots,|\mathcal{B}_k|\} \\ \text{s.t. } D_h \cap S \neq \emptyset}} D_h.$$

Since the total number of these copies is $\gamma_k = \Sigma_{S \in \mathcal{S}_k} \mu_S$, we can put them into different rooms. Recall that for each $i \in N$, $\Sigma_{S \in \mathcal{S}_k^i} \mu_S = \mu_k$. This implies that for each $h \in \{1, \ldots, |\mathcal{B}_k|\}$, group D_h belongs to exactly μ_k rooms. Next, for each $h \in \{|\mathcal{B}_k| + 1, \ldots, \lambda_k\}$, create μ_k copies of D_h. Since

$$\lambda_k = \binom{\gamma_k}{\mu_k},$$

we can place these copies of $D_{|\mathcal{B}_k|+1}, \ldots, D_{\lambda_k}$ into the rooms so that all γ_k rooms contain the same number of groups and for each $h \in \{|\mathcal{B}_k| + 1, \ldots, \lambda_k\}$, group D_h belongs to exactly μ_k rooms. It is easy to see that $\mathcal{T}_k$ thus constructed satisfies conditions (iii), (iv), and (v). (The above construction of M_k and $\mathcal{T}_k$ is illustrated in Fig. 3 for a simple case.)

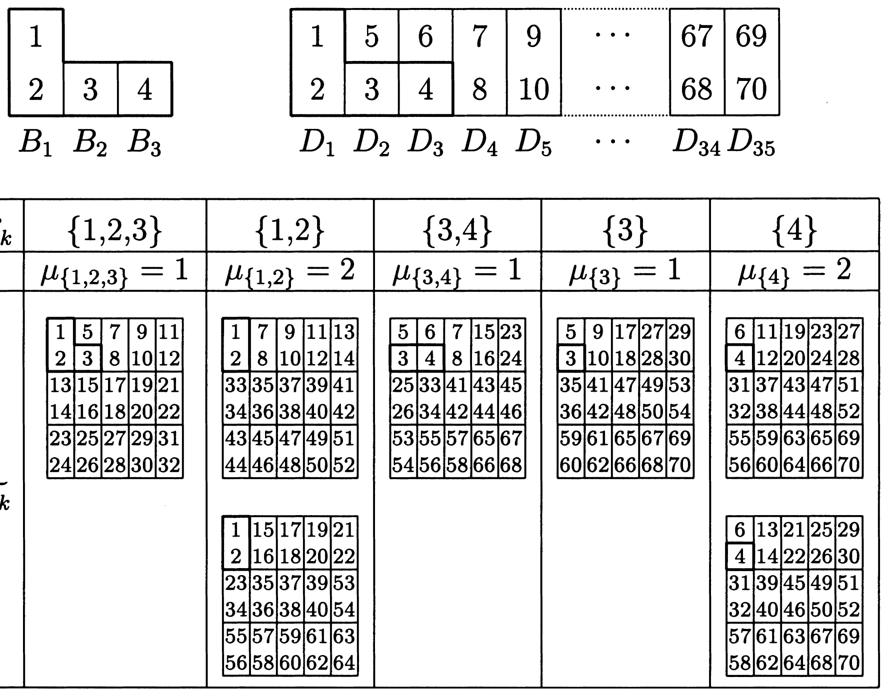

Fig. 3. Step 1 of the proof of Theorem 4.2. In the above example, $N = \{1,2,3,4\}$, $\mathcal{S}_k = \{\{1,2,3\},\{1,2\},\{3,4\},\{3\},\{4\}\}$, $\mu_k = 3$, $\mu_{\{1,2,3\}} = \mu_{\{3,4\}} = \mu_{\{3\}} = 1$, $\mu_{\{12\}} = \mu_{\{4\}} = 2$. Thus, $\beta_k = \max\{2,1\} = 2$, $\gamma_k = 1 + 1 + 1 + 2 + 2 = 7$, and

$$\lambda_k = \binom{\gamma_k}{\mu_k} = \binom{7}{3} = 35.$$

Step 2. Given $k \in \{1,2,\ldots,|A|\}$, let $\mathscr{D}_k \equiv \{D_1,D_2,\ldots,D_{\lambda_k}\}$, M_k, and $\mathscr{T}_k$ be constructed as in Step 1. We show that, for each pair $i,j \in M_k$, there exists a permutation π_{M_k} on M_k such that $\pi_{M_k}(i)=j$ and for each $T \in \mathscr{T}_k$, $\pi_{M_k}(T) \in \mathscr{T}_k$.

Let $i,j \in M_k$. By condition (v) of T_k in Step 1, $|\mathscr{T}^i_k| = |\mathscr{T}^j_k| = \mu_k$. Thus, there exists a permutation $\pi_{\mathscr{T}_k}$ on $\mathscr{T}_k$ such that $\pi_{\mathscr{T}_k}(\mathscr{T}^i_k) = \mathscr{T}^j_k$. Note that $\mathscr{T}_k$ has a property that if μ_k distinct coalitions in $\mathscr{T}_k$ are chosen, then there exists exactly one group in $\mathscr{D}_k$ that is included in all of these μ_k coalitions. Thus, $\pi_{\mathscr{T}_k}$ induces a permutation on $\mathscr{D}_k$. Formally, this permutation, denoted $\pi_{\mathscr{D}_k}$, is defined by setting for each $D \in \mathscr{D}_k$

$$\pi_{\mathscr{D}_k}(D) \equiv \bigcap_{\substack{T \in \mathscr{T}_k \\ T \supseteq D}} \pi_{\mathscr{T}_k}(T).$$

Note that each coalition in $\mathscr{T}_k$ can be viewed as a coalition of groups. For each $T \in \mathscr{T}_k$, its image under $\pi_{\mathscr{D}_k}$ is defined by

$$\pi_{\mathscr{D}_k}(T) \equiv \bigcup_{\substack{D \in \mathscr{D}_k \\ D \subseteq T}} \pi_{\mathscr{D}_k}(D).$$

Now, we show that for each $T \in \mathscr{T}_k$, $\pi_{\mathscr{D}_k}(T) \in \mathscr{T}_k$. Let $T \in \mathscr{T}_k$. Then, by the definition of $\pi_{\mathscr{D}_k}$, for each $D \in \mathscr{D}_k$ with $D \subseteq T$, we have $\pi_{\mathscr{D}_k}(D) \subseteq \pi_{\mathscr{T}_k}(T)$. Thus,

$$\pi_{\mathscr{D}_k}(T) = \bigcup_{\substack{D \in \mathscr{D}_k \\ D \subseteq T}} \pi_{\mathscr{D}_k}(D) \subseteq \pi_{\mathscr{T}_k}(T).$$

Since

$$|T| = |\pi_{\mathscr{D}_k}(T)| = |\pi_{\mathscr{T}_k}(T)| = \binom{\gamma_k - 1}{\mu_k - 1},$$

we have $\pi_{\mathscr{D}_k}(T) = \pi_{\mathscr{T}_k}(T)$. Thus, $\pi_{\mathscr{D}_k}(T) \in \mathscr{T}_k$.

By construction, each group in $\mathscr{D}_k$ contains exactly β_k agents. For each $D \in \mathscr{D}_k$ with $i \in D$, choose a bijection $\pi_D : D \to \pi_{\mathscr{D}_k}(D)$ such that $\pi_D(i)=j$. For each $D \in \mathscr{D}_k$ with $i \notin D$, choose an arbitrary bijection $\pi_D : D \to \pi_{\mathscr{D}_k}(D)$. Given the list $(\pi_D)_{D \in \mathscr{D}_k}$ of such bijections, define the permutation π_{M_k} on M_k by setting for each $D \in \mathscr{D}_k$ and each $h \in D$, $\pi_{M_k}(h) \equiv \pi_D(h)$. Clearly, $\pi_{M_k}(i)=j$. Let $T \in \mathscr{T}_k$. Then

$$\pi_{M_k}(T) = \bigcup_{h \in T} \pi_{M_k}(h) = \bigcup_{D \subseteq T} \bigcup_{h \in D} \pi_D(h) = \bigcup_{D \subseteq T} \pi_{\mathscr{D}_k}(D) = \pi_{\mathscr{D}_k}(T) \in \mathscr{T}_k.$$

Thus, π_{M_k} is a desired permutation on M_k.

Step 3. For each $k \in \{1,2,\ldots,|A|\}$, let M_k and $\mathscr{T}_k$ be constructed as in Step 1. Here, we construct a set $\boldsymbol{M}$ and a partition of $2^{\boldsymbol{M}} \backslash \{\boldsymbol{M},\emptyset\}$.

Let

$$\boldsymbol{M} \equiv M_1 \times M_2 \times \cdots \times M_{|A|}.$$

In order to stress the fact that $\boldsymbol{M}$ is a Cartesian product of the sets of agents, we write its subsets and its elements in bold face. Note that, since the set of potential agents is countably infinite, in the presence of *anonymity*, $\boldsymbol{M}$ can be viewed as an element of $\mathcal{N}$.

For each $k \in \{1,2,\ldots,|A|\}$, let

$$\mathbb{S}_{|A|+1} \equiv \{\boldsymbol{S} \subset \boldsymbol{M} \mid \text{there exists } T \in \mathcal{T}_k \text{ such that}$$
$$\boldsymbol{S} = M_1 \times \cdots \times M_{k-1} \times T \times M_{k+1} \times \cdots \times M_{|A|}\}.$$

Let

$$\mathbb{S}_{|A|+1} \equiv 2^{\boldsymbol{M}} \setminus \left(\{\boldsymbol{M},\emptyset\} \cup \left(\bigcup_{k=1}^{|A|} \mathbb{S}_k \right) \right).$$

Step 4. We show that for each pair $\boldsymbol{i},\boldsymbol{j} \in \boldsymbol{M}$, there exists a permutation $\pi_{\boldsymbol{M}}$ on $\boldsymbol{M}$ such that (i) $\pi_{\boldsymbol{M}}(\boldsymbol{i}) = \boldsymbol{j}$ and (ii) for each $k \in \{1,2,\ldots,|A|+1\}$ and each $\boldsymbol{S} \in \mathbb{S}_k$, we have $\pi_{\boldsymbol{M}}(\boldsymbol{S}) \in \mathbb{S}_k$.

Let $\boldsymbol{i} = (i_1,i_2,\ldots,i_{|A|}) \in \boldsymbol{M}$ and $\boldsymbol{j} = (j_1,j_2,\ldots,j_{|A|}) \in \boldsymbol{M}$. By Step 2, for each $k \in \{1, 2,\ldots,|A|\}$, there exists a permutation π_{M_k} on M_k such that $\pi_{M_k}(i_k) = j_k$ and for each $T \in \mathcal{T}_k$, $\pi_{M_k}(T) \in \mathcal{T}_k$. Define the permutation $\pi_{\boldsymbol{M}}$ on $\boldsymbol{M}$ by setting for each $\boldsymbol{h} = (h_1,h_2,\ldots,h_{|A|}) \in \boldsymbol{M}$

$$\pi_{\boldsymbol{M}}(\boldsymbol{h}) \equiv (\pi_{M_1}(h_1),\pi_{M_2}(h_2),\ldots,\pi_{M_{|A|}}(h_{|A|})).$$

Clearly, $\pi_{\boldsymbol{M}}(\boldsymbol{i}) = \boldsymbol{j}$ and for each $k \in \{1,2,\ldots,|A|\}$ and each $\boldsymbol{S} \in \mathbb{S}_k$, we have $\pi_{\boldsymbol{M}}(\boldsymbol{S}) \in \mathbb{S}_k$. Note that $\pi_{\boldsymbol{M}}$ induces a permutation on $\cup_{k=1}^{|A|} \mathbb{S}_k$. Thus, also for each $\boldsymbol{S} \in \mathbb{S}_{|A|+1}$, $\pi_{\boldsymbol{M}}(\boldsymbol{S}) \in \mathbb{S}_{|A|+1}$.

Step 5. Let $w \in \mathcal{V}_{\text{all}}^{\boldsymbol{M}}$ be defined as follows: (i) $w(\boldsymbol{M}) \equiv v(N) - x(N)$; (ii) for each $k \in \{1,2,\ldots,|A|\}$ and each $\boldsymbol{S} \in \mathbb{S}_k$, $w(\boldsymbol{S}) \equiv \alpha_k$; and (iii) for each $\boldsymbol{S} \in \mathbb{S}_{|A|+1}$, $w(\boldsymbol{S}) \equiv \min\{\alpha_{|A|},v(N) - x(N)\}$. We show that for each $\boldsymbol{i} \in \boldsymbol{M}$, $\varphi_{\boldsymbol{i}}(w) = 0$.

Let $\boldsymbol{i},\boldsymbol{j} \in \boldsymbol{M}$. By Step 4, there exists a permutation $\pi_{\boldsymbol{M}}$ on $\boldsymbol{M}$ such that $\pi_{\boldsymbol{M}}(\boldsymbol{i}) = \boldsymbol{j}$, and for each $k \in \{1,2,\ldots,|A|+1\}$ and each $\boldsymbol{S} \in \mathbb{S}_k$, we have $\pi_{\boldsymbol{M}}(\boldsymbol{S}) \in \mathbb{S}_k$. Let $w' \in \mathcal{V}_{\text{all}}^{\boldsymbol{M}}$ be defined by setting for each $\boldsymbol{S} \subseteq \boldsymbol{M}$, $w'(\boldsymbol{S}) \equiv w((\pi_{\boldsymbol{M}})^{-1}(\boldsymbol{S}))$. By *anonymity*, $\varphi_{\boldsymbol{i}}(w) = \varphi_{\boldsymbol{j}}(w')$. Let $\boldsymbol{S} \subset \boldsymbol{M}$. Since $\cup_{k=1}^{|A|+1} \mathbb{S}_k = 2^{\boldsymbol{M}} \setminus \{\boldsymbol{M},\emptyset\}$, there exists $k \in \{1,2,\ldots,|A|+1\}$ such that $\boldsymbol{S} \in \mathbb{S}_k$. Since $(\pi_{\boldsymbol{M}})^{-1}(\boldsymbol{S}) \in \mathbb{S}_k$, by the definitions of w and w', $w'(\boldsymbol{S}) = w(\boldsymbol{S})$. Clearly, $w'(\boldsymbol{M}) = w(\boldsymbol{M})$. Thus, $w' = w$. Therefore, $\varphi_{\boldsymbol{i}}(w) = \varphi_{\boldsymbol{j}}(w') = \varphi_{\boldsymbol{j}}(w)$.

The above argument can be applied to all $\boldsymbol{i},\boldsymbol{j} \in \boldsymbol{M}$. Since x is an aspiration for v, x is individually feasible and coalitionally rational. By coalitional rationality, for each $k \in \{1,2,\ldots,|A|\}$, $\alpha_k \le 0$. Together with individual feasibility, we have either $\alpha_1 = 0$ or $v(N) = x(N)$. Thus, we have (i) for each $\boldsymbol{S} \subseteq \boldsymbol{M}$, $w(\boldsymbol{S}) \le 0$, and (ii) there exists $\boldsymbol{T} \subseteq \boldsymbol{M}$ such that $\boldsymbol{T} \ne \emptyset$ and $w(\boldsymbol{T}) = 0$. Let $\boldsymbol{i} \in \boldsymbol{T}$. Then, since $\varphi(w)$ is coalitionally rational in w,

$$0 = w(\boldsymbol{T}) \le \sum_{\boldsymbol{j} \in \boldsymbol{T}} \varphi_{\boldsymbol{j}}(w) = |\boldsymbol{T}| \cdot \varphi_{\boldsymbol{i}}(w),$$

so that $\varphi_{\boldsymbol{i}}(w) \ge 0$. By individual feasibility of $\varphi(w)$, $\varphi_{\boldsymbol{i}}(w) = 0$.

Thus, for each $\boldsymbol{i} \in \boldsymbol{M}$, $\varphi_{\boldsymbol{i}}(w) = 0$.

Step 6. Let

$$\boldsymbol{M}' \equiv \{(i,i,\ldots,i) \in M \mid i \in N\}.$$

Clearly, $|\boldsymbol{M}'| = |N|$. Let $\boldsymbol{b} : N \to \boldsymbol{M}'$ be the bijection defined by setting for each $i \in N$, $\boldsymbol{b}(i) \equiv (i,i,\ldots,i)$. We show that for each $S \subseteq N$, $r_{\boldsymbol{M}'}^{\varphi(w)}(w)(\boldsymbol{b}(S)) = v(S) - x(S)$.

Let $S \subset N$. Then there exists $k \in \{1,2,\ldots,|A|\}$ such that $v(S) - x(S) = \alpha_k$. By properties (iii) and (iv) of $\mathcal{T}_k$ in Step 1, there exists $T \in \mathcal{T}_k$ such that $T \cap N = S$. Let

$$\boldsymbol{R} \equiv M_1 \times \cdots \times M_{k-1} \times T \times M_{k+1} \times \cdots M_{|A|}.$$

Then $\boldsymbol{b}(S) \subseteq \boldsymbol{R}$. Moreover, since $T \cap N = S$, we have $\boldsymbol{R} \backslash \boldsymbol{b}(S) \subseteq \boldsymbol{M} \backslash \boldsymbol{M}'$. Thus,

$$r_{\boldsymbol{M}'}^{\varphi(w)}(w)(\boldsymbol{b}(S)) = \max_{\boldsymbol{Q} \subseteq \boldsymbol{M} \backslash \boldsymbol{M}'} w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) \geq w(\boldsymbol{b}(S) \cup (\boldsymbol{R} \backslash \boldsymbol{b}(S))) = w(\boldsymbol{R}).$$

Since $\boldsymbol{R} \in \mathbb{S}_k$, $w(\boldsymbol{R}) = \alpha_k$. Thus,

$$r_{\boldsymbol{M}'}^{\varphi(w)}(w)(\boldsymbol{b}(S)) \geq w(\boldsymbol{R}) = \alpha_k = v(S) - x(S).$$

Now, we claim that the opposite (weak) inequality also holds. Let $\boldsymbol{Q} \subseteq \boldsymbol{M} \backslash \boldsymbol{M}'$. If $\boldsymbol{b}(S) \cup \boldsymbol{Q} \in \mathbb{S}_{|A|+1}$, then $w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) = \alpha_{|A|} \leq \alpha_k = v(S) - x(S)$. If there exists $\ell \leq |A|$ such that $\boldsymbol{b}(S) \cup \boldsymbol{Q} \in \mathbb{S}_\ell$, then there exists $T' \in \mathcal{T}_\ell$ such that

$$\boldsymbol{b}(S) \cup \boldsymbol{Q} = M_1 \times \cdots \times M_{\ell-1} \times T' \times M_{\ell+1} \times \cdots \times M_{|A|}.$$

Since $\boldsymbol{Q} \subseteq \boldsymbol{M} \backslash \boldsymbol{M}'$ and $S \subseteq T'$, we have $T' \cap N = S$. Thus, by property (iv) of $\mathcal{T}_\ell$ in Step 1, we have $S \in \mathcal{S}_\ell$, so that $w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) = \alpha_\ell = v(S) - x(S)$.

Thus, for each $\boldsymbol{Q} \subseteq \boldsymbol{M} \backslash \boldsymbol{M}'$, we have $w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) \leq v(S) - x(S)$. This implies that

$$r_{\boldsymbol{M}'}^{\varphi(w)}(w)(\boldsymbol{b}(S) = \max_{\boldsymbol{Q} \subseteq \boldsymbol{M} \backslash \boldsymbol{M}'} w(\boldsymbol{b}(S) \cup \boldsymbol{Q}) \leq v(S) - x(S).$$

By the definition of w, $w(\boldsymbol{M}) = v(N) - x(N)$. For each $k \in \{1,2,\ldots,|A|\}$, by the definition of $\mathcal{S}_k$, we have $N \notin \mathcal{S}_k$. Thus, for each $k \in \{1,2,\ldots,|A|\}$, by property (iv) of $\mathcal{T}_k$ in Step 1, there exists no $T \in \mathcal{T}_k$ such that $T \supseteq N$. This implies that, for each $\boldsymbol{S} \supseteq \boldsymbol{M}'$, we have $\boldsymbol{S} \in \mathbb{S}_{|A|+1}$, so that

$$w(\boldsymbol{S}) = \min\{\alpha_{|A|}, v(N) - x(N)\} \leq v(N) - x(N).$$

Thus,

$$r_{\boldsymbol{M}'}^{\varphi(w)}(w)(\boldsymbol{b}(N)) = r_{\boldsymbol{M}'}^{\varphi(w)}(w)(\boldsymbol{M}') = \max_{\boldsymbol{Q} \subseteq \boldsymbol{M} \backslash \boldsymbol{M}'} w(\boldsymbol{M}' \cup \boldsymbol{Q}) = v(N) - x(N).$$

Therefore, for each $S \subseteq N$

$$r_{\boldsymbol{M}'}^{\varphi(w)}(w)(\boldsymbol{b}(S)) = w(S) - x(S).$$

Step 7. By *max* consistency, for each $\boldsymbol{i} \in \boldsymbol{M}'$, $\varphi_{\boldsymbol{i}}(r_{\boldsymbol{M}'}^{\varphi(w)}(w)) = 0$.

Finally, by *anonymity* and *zero-independence* of φ, we deduce that, for each $i \in N$,

$$\varphi_i(v) = \varphi_{\boldsymbol{b}(i)}(r_{\boldsymbol{M}'}^{\varphi(w)}(w)) + x_i = 0 + x_i = x_i.$$

Thus, $\varphi(v) = AspNuc(v)$. $\square$

References

Albers, W., 1979. Core and kernel-variants based on imputation and demand profiles. In: Moeschlin, O., Pallaschke, D. (Eds.), Game Theory and Related Topics. North-Holland, Amsterdam, pp. 3–16.

Bennett, E., 1981. The aspiration core, bargaining set, kernel and nucleolus. Working Paper 488, School of Management, State University of New York at Buffalo.

Bennett, E., 1983. The aspiration approach to predicting coalition formation and payoff distribution in sidepayment games. International Journal of Game Theory 12, 1–28.

Chatterjee, K., Dutta, B., Ray, D., Sengupta, K., 1993. A noncooperative theory of coalitional bargaining. Review of Economic Studies 60, 463–477.

Cross, J.G., 1967. Some theoretic characteristics of economic and political coalitions. Journal of Conflict Resolution 11, 184–195.

Davis, M., Maschler, M., 1965. The kernel of a cooperative game. Naval Research Logistics Quarterly 12, 223–259.

Kohlberg, E., 1971. On the nucleolus of a characteristic function game. SIAM Journal on Applied Mathematics 20, 62–66.

Maschler, M., Peleg, B., Shapley, L.S., 1972. The kernel and bargaining set for convex games. International Journal of Game Theory 1, 73–93.

Moldovanu, B., Winter, E., 1994a. Consistent demands for coalition formation. In: Megiddo, N. (Ed.), Essays in Game Theory in Honor of Michael Maschler. Springer, New York, pp. 129–140.

Moldovanu, B., Winter, E., 1994b. Core implementation and increasing returns to scale for cooperation. Journal of Mathematical Economics 23, 533–548.

Moldovanu, B., winter, E., 1995. Order independent equilibria. Games and Economic Behavior 9, 21–34.

Peleg, B., 1986. On the reduced game property and its converse. International Journal of Game Theory 15, 187–200.

Peleg, B., 1988. Introduction to the theory of cooperative games, Chapter 6: The prenucleolus. Research Memorandum 85, Center for Research in Mathematical Economics and Game Theory, The Hebrew University, Jerusalem, Israel.

Perry, M., Reny, P.J., 1994. A noncooperative view of coalition formation and the core. Econometrica 62, 795–817.

Schmeidler, D., 1969. The nucleolus of a characteristic function game. SIAM Journal on Applied Mathematics 17, 1163–1170.

Selten, R., 1981. A non-cooperative model of characteristic function bargaining. In: Böhm, V., Nachtkamp, H. (Eds.), Essays in Game Theory and Mathematical Economics in Honor of O. Morgenstern. Bibliographishes Institut, Mannheim, pp. 245–282.

Sharkey, W.W., 1993. A characterization of some aspirations with an application to spatial games. Bellcore Economics Working Paper 95, Bellcore.

Snijders, C., 1995. Axiomatization of the nucleolus. Mathematics of Operations Research 20, 189–196.

Sobolev, A.I., 1975. Characterization of the principle of optimality for cooperative games through functional equations. Mathematical Methods in Social Sciences 6, 94–151, in Russian.

Thomson, W., 1996. Consistent allocation rules. RCER Working Paper 418, University of Rochester.

Winter, E., 1989. An axiomatization of the stable and semistable demand vectors by the reduced game property. Discussion Paper A-254, University of Bonn.