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Constrained allocation problems with single-peaked preferences: An axiomatic analysis

Özgür Kıbrıs

Sabancı University, Faculty of Arts and Social Sciences, 81474, Orhanlı, Tuzla, İstanbul, Turkey (e-mail: ozgur@sabanciuniv.edu)

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Abstract. We introduce a new class of problems that contains two existing classes: allocation problems with single-peaked preferences and bankruptcy problems. On this class, we analyze the implications of well-known properties such as Pareto optimality, strategy-proofness, resource-monotonicity, no-envy, equal treatment of equals, and two new properties we introduce, hierarchical no-envy and independence of nonbinding constraints. Unlike earlier literature, we consider rules that allow free-disposability. We present characterizations of a rule we introduce on this domain. We relate this rule to well-known rules on the aforementioned subdomains. Based on this relation, we present a characterization of a well-known bankruptcy rule called the constrained equal awards rule.

1 Introduction

We look for a “good” solution to the following problem: a perfectly divisible commodity is to be allocated among a set of agents each having an exogenous constraint and a single-peaked preference relation on his consumption. The (federal) Environmental Protection Agency (EPA) of the US government regularly faces this problem when allocating pollution permits. Each permit gives the bearer firm the right to a certain amount of pollution. A firm’s right to pollute is also bounded by the constraints imposed by local governments. E.g. very strict local limits may be imposed on polluters in densely populated or

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overpolluted areas.[1] Assume each firm's profit to be a strictly concave function of its output and therefore, a single-peaked function of its emissions.[2]

Permit allocation in US started in 1899 and is currently used in the regulation of various pollutants.[3] The rules used in practice are highly variable and depend on the industry being regulated. Joskow et al. [6] give a detailed summary of how in the US Acid Rain Program, Title *IV* of the 1990 Clean Air Act regulates the allocation of SO_2 permits among coal burning electric-generating units. To summarize, a firm's "basic" share is assigned proportionally to an estimate of its profit maximizing emission level. Then, the shares are adjusted through several additional provisions. A practical implication of our analysis is the provision of a basis to evaluate such rules.

We look for *Pareto optimal* rules that satisfy two main properties. *Strategy-proofness* requires that revealing true preferences is always a dominant strategy for the agents. Strategy-proof rules are not informationally demanding and thus cheaper and easier to implement. The significance of such features can be highlighted with the fact that in US in 1980 there were up to 55000 major sources of industrial water pollution alone [7]. With a strategy-proof permit allocation rule, the EPA can trust the polluter firms to truthfully declare their profit maximizing emission levels; the firms have no incentive to misreport. *Resource-monotonicity* is a solidarity property which requires a change in the social endowment to affect all agents in the same direction (i.e., all gain or lose together). In permit allocation, the amount of pollution rights to be distributed is periodically updated according to the current environmental conditions. With a resource-monotonic rule, such updates affect all firms' profits in the same way; in effect, a firm can not have a larger permit as a result of worsening environmental conditions.

Amongst the rules that satisfy these properties, some (such as the dictatorial rules) are not normatively appealing. To formalize this concern, we introduce standard fairness properties such as *no-envy* [5] and *equal treatment of equals*. Note, however, that the asymmetries in the consumption constraints may sometimes represent a hierarchical relation among the agents. In permit allocation, firms' constraints may be related to their importance for the local economy. Thus, we also analyze the implications of a *hierarchical no-envy* property that gives priority to agents with less restrictive constraints.

Two special cases of our problem have already been center of considerable attention: in **unconstrained allocation problems** [9] the agents do not have consumption constraints, in **bankruptcy problems** [8] all agents have monotonic preferences. In both cases, rules based on equal division (the *uniform rule* [9]

[1] On the other hand, industries that are important for the local economies or that have strong lobbies may not face so strict limits.

[2] The relation follows since each firm's emissions increase in its output.

[3] The Refuse Act of 1899 required that all industrial waste dischargers have permits from the US Army Corps of Engineers [7]. This law was rediscovered in 1960's and the Congress passed a new water quality law in 1972 which required that permits be issued for all point-source waste discharges to watercourses.

and the *constrained equal awards rule* [1], respectively) satisfy many desirable properties. The analysis of this paper also reveals an "equal-division type" rule (hereafter called the *constrained uniform rule*).

On bankruptcy problems, this rule coincides with the constrained equal awards rule which has been adopted as law by most major codifiers in history[4] and which is the only "symmetric" bankruptcy rule with a strategy-proof extension to the whole domain (see Theorem 6). On unconstrained allocation problems, however, our rule coincides with the uniform rule only when there is excess demand. The difference is due to the following reason.

In the literature on unconstrained allocation problems, feasibility requires *all* of the social endowment to be allocated, even when there is excess supply of the commodity. In our framework, however, rules satisfying this feasibility condition violate even very basic fairness properties such as no-envy. Moreover, (unlike in unconstrained problems) excess supply is not solely due to the agents' preferences; even when the sum of the agents' most preferred amounts exceed the social endowment, there might be excess supply due to the exogenous constraints. Both points are demonstrated in Example 1.

We enlarge the set of feasible allocations by assuming free-disposability of the commodity. This is natural in permit allocation: if the amount of permits to be distributed is larger than the sum of the firms' profit maximizing emission levels, rules that do not satisfy free-disposability force the firms to pollute more than their profit maximizing levels. The free-disposability assumption enlarges the class of admissible rules: rules satisfying the "strict" feasibility constraint are still admissible, but there are many more. It also makes Pareto optimality more demanding since an allocation is now compared to a larger set: when there is too much of the commodity, to allocate it all is now Pareto dominated.

Finally note that due to domain differences and free-disposability, our results are not logically related to previous results obtained for the two subclasses. Keeping this in mind, however, we discuss each result in relation to previous findings and compare the implications of the properties in question.

2 The model

Let $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$. There is a **social endowment** of $E \in \mathbb{R}_+$ units of a perfectly divisible commodity to be allocated among a set N of agents. Each $i \in N$ is characterized by an exogenous **constraint** $c_i \in \overline{\mathbb{R}}_+$ and a **single-peaked preference relation** R_i on his consumption.[5] Let P_i $[I_i]$ be the strict preference [indifference] relation associated with R_i. Let $p(R_i) \in \overline{\mathbb{R}}_+$ be the

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[4] As an example, Aumann and Maschler [1] mention Maimonides (1135–1204): *The Laws of Lending and Borrowing*, Ch. 20, Sect. 4.

[5] The preference relation R_i is single-peaked if there is a unique consumption level $p(R_i) \in \overline{\mathbb{R}}_+$ such that for any $x, y \in \mathbb{R}_+$, $[x < y \leq p(R_i)$ or $p(R_i) \leq y < x]$ implies $y \, P_i \, x$. Note that monotonic preferences are considered single-peaked.

most preferred consumption level for R_i. Let $\mathscr{R}_{sp}$ be the set of all single-peaked preferences on $\overline{\mathbb{R}}_+$. Let $c = (c_1, \ldots, c_n)$, $R = (R_1, \ldots, R_n)$, and $p(R) = (p(R_1), \ldots, p(R_n))$. A **constrained allocation problem** (with single-peaked preferences) is a triple $(R, c, E) \in \mathscr{R}_{sp}^N \times \overline{\mathbb{R}}_+^N \times \mathbb{R}_+$ satisfying $\sum c_i \geq E$.[6] Let $\mathbb{P}$ denote the set of all such problems.

An **unconstrained allocation problem** is an $(R, c, E) \in \mathbb{P}$ such that $c = \infty^N$. A **bankruptcy problem** is an $(R, c, E) \in \mathbb{P}$ such that $p(R) = \infty^N$. Let $\mathbb{P}_b \subset \mathbb{P}$ denote the set of all bankruptcy problems.

An allocation $x \in \mathbb{R}_+^N$ is **feasible** for $(R, c, E) \in \mathbb{P}$ if $\sum x_k \leq E$ and for each $i \in N$, $x_i \leq c_i$. Let $X(R, c, E)$ denote the set of feasible allocations for (R, c, E). Note that our definition of feasibility assumes free-disposability and is weaker than the one used in the earlier literature (which requires $\sum x_k = E$). We start Section 3 with a discussion of this issue.

A **constrained allocation rule** is a function $F : \mathbb{P} \to \mathbb{R}_+^N$ that assigns to each $(R, c, E) \in \mathbb{P}$, a feasible allocation $x \in X(R, c, E)$. The **constrained uniform rule**, CU, is defined as follows: for all $(R, c, E) \in \mathbb{P}$ and all $i \in N$, (i) if $E \leq \sum \min\{c_k, p(R_k)\}$, $CU_i(R, c, E) = \min\{c_i, p(R_i), \lambda\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum \min\{c_k, p(R_k), \lambda\} = E$ and (ii) if $E > \sum \min\{c_k, p(R_k)\}$, $CU_i(R, c, E) = \min\{c_i, p(R_i)\}$.

A constrained allocation rule F is **Pareto optimal** if for all $(R, c, E) \in \mathbb{P}$ and all $x \in X(R, c, E)$, if there is $i \in N$ such that $x_i \, P_i \, F_i(R, c, E)$, then there is $j \in N$ such that $F_j(R, c, E) \, P_j \, x_j$. It is **strategy-proof** if for all $(R, c, E) \in \mathbb{P}$, all $i \in N$, and all $R_i' \in \mathscr{R}_{sp}$, $F_i(R_i, R_{-i}, c, E) \, R_i \, F_i(R_i', R_{-i}, c, E)$. It satisfies **resource-monotonicity** if for all $(R, c, E) \in \mathbb{P}$ and all $E' \in \mathbb{R}_+$ either for all $i \in N$, $F_i(R, c, E) \, R_i \, F_i(R, c, E')$ or for all $i \in N$, $F_i(R, c, E') \, R_i \, F_i(R, c, E)$. These properties are already explained in the introduction.

A constrained allocation rule F satisfies **no-envy** (is *envy-free*) [5] if each agent prefers his share to the share of another agent (subject to his consumption constraint): for all $(R, c, E) \in \mathbb{P}$ and $i, j \in N$, $F_i(R, c, E) \, R_i \min\{c_i, F_j(R, c, E)\}$. It satisfies **hierarchical no-envy** if each agent prefers his share to the share of another agent with a smaller constraint: for all $(R, c, E) \in \mathbb{P}$ and $i, j \in N$ such that $c_j \leq c_i$, $F_i(R, c, E) \, R_i \, F_j(R, c, E)$.[7] It satisfies **equal treatment of equals** if identical agents obtain similar shares: for all $(R, c, E) \in \mathbb{P}$ and $i, j \in N$ such that $R_i = R_j$ and $c_i = c_j$, $F_i(R, c, E) \, I_i \, F_j(R, c, E)$. A constrained allocation rule F satisfies **independence of nonbinding constraints** if an agent's share is not affected from an increase in his consumption constraint if this constraint was not binding in the first place: for all $(R, c, E) \in \mathbb{P}$, all $i \in N$ such that $p(R_i) \leq c_i$, and all $c_i' \geq c_i$, $F_i(R, c, E) = F_i(R, c_i', c_{-i}, E)$. It is **continuous with respect to E** if small changes in the social endowment do not have a big effect on the agents' shares: for each $(R, c, E) \in \mathbb{P}$ and for each sequence of constrained allocation problems $\{(R, c, E^v)\}$ in $\mathbb{P}$ such that $\{E^v\} \to E$, $\{F(R, c, E^v)\} \to F(R, c, E)$.

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[6] Since no agent i can obtain a share $x_i > c_i$, assuming $\sum c_i \geq E$ does not cause any loss in generality.

[7] Note that since $F_j(R, c, E) \leq c_j \leq c_i$, $F_j(R, c, E) = \min\{c_i, F_j(R, c, E)\}$.

3 Results

We first demonstrate that there are constrained allocation problems in which no *envy-free* allocation satisfies the strict feasibility constraint, $\sum x_k = E$.

Example 1. *Let* $N = \{1, 2\}$. *Let* $c = (4, 10)$, $E = 10$, *and* $R \in \mathscr{R}_{sp}^N$ *such that* $p(R) = (9, 2)$. *The feasible shares for Agent 1 are* $x_1 \in [0, 4]$. *Since* $x_1 + x_2 = 10$, $x_2 \in [6, 10]$. *For any* $x_1 < 4$, $x_2 > 6$, *and* $\min\{x_2, 4\} = 4$. *Then,* $4\, P_1\, x_1$ *rules out all* $x_1 < 4$ *by no-envy applied to Agent 1. This implies* $x_1 = 4$. *But, since* $4\, P_2\, 6$ *this case is ruled out by no-envy applied to Agent 2.*

This example implies that constrained allocation rules which satisfy this strict feasibility constraint are bound to violate *no-envy*. In this paper we use the weaker feasibility constraint, $\sum x_k \leq E$ (i.e., we assume free-disposability). This strenghtens *Pareto optimality*: if $\sum \min\{p(R_k), c_k\} \leq E$, the unique *Pareto optimal* allocation is $x_i = \min\{p(R_i), c_i\}$ for all $i \in N$.[8] If $\sum \min\{p(R_k), c_k\} > E$, *Pareto optimality* implies $\sum x_k = E$ and that each agent i get an $x_i \leq \min\{p(R_i), c_i\}$.

Next we describe the class of allocation rules that are *Pareto optimal* and *strategy-proof*:[9] A constrained allocation rule $F : \mathbb{P} \to \mathbb{R}_+^N$ satisfies *Pareto optimality* and *strategy-proofness* if and only if for all $i \in N$, there is $b_i : \mathscr{R}_{sp}^{N \setminus \{i\}} \times \overline{\mathbb{R}}_+^N \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $(R, c, E) \in \mathbb{P}$,

(i) $b_i(R_{-i}, c, E) \in [0, c_i]$,
(ii) $\sum \min\{b_k(R_{-k}, c, E), p(R_k)\} = E$, and

$$F_i(R, c, E) = \begin{cases} \min\{b_i(R_{-i}, c, E), p(R_i)\} & \text{if } \sum \min\{c_k, p(R_k)\} \geq E; \\ \min\{c_i, p(R_i)\} & \text{if } \sum \min\{c_k, p(R_k)\} < E. \end{cases}$$

This large class, however, has a single *envy-free* member.

Theorem 1. *The* constrained uniform rule *is the only constrained allocation rule that satisfies* Pareto optimality, strategy-proofness, *and* no-envy.

The proof is similar to that of Ching [3] who shows the uniform rule to be the only *Pareto optimal* (with respect to strict feasibility), *strategy-proof*, and *envy-free* rule on unconstrained allocation problems.[10] On this subdomain, replacing *no-envy* with *anonymity* [9] or *equal treatment of equals* [4] does not change the conclusion. This, however, is not the case in our domain.

The constrained uniform rule satisfies *anonymity*, *equal treatment of equals*, and *hierarchical no-envy*. Unfortunately, some very unappealing rules also

[8] Without free-disposal, Pareto optimality implies the weaker $x_i \geq \min\{p(R_i), c_i\}$.
[9] This result is a direct modification of the characterization in Barberà et al. [2]. Its proof follows from a modification of Lemmas 1 and 2 in Sprumont [9]. Here, due to the free-disposal assumption, it is sufficient to specify one function per agent (instead of two).
[10] However, the two results are not logically related due to the domain difference and the free-disposal assumption.

satisfy these properties as well as *Pareto optimality* and *strategy-proofness*. Next, we present such a rule which is "almost dictatorial".

Example 2. *Let $N = \{1, 2\}$. Define the constrained allocation rule as follows: if $c_1 = c_2$, apply the constrained uniform rule, CU; if $c_i > c_j$, let $x_i = \min\{c_i, p(R_i), E\}$ and $x_j = \min\{E - x_i, c_j, p(R_j)\}$. This rule satisfies* Pareto optimality, strategy-proofness, equal treatment of equals, hierarchical no-envy, *and even* anonymity.

Unlike in other models, these properties do not rule out undesirable rules since none require consistency in the way a constrained allocation rule reacts to changes in the exogenous constraints. Next, we analyze the implications of such a property, *independence of nonbinding constraints*.

Theorem 2. *The* constrained uniform rule *is the only constrained allocation rule that satisfies* Pareto optimality, strategy-proofness, equal treatment of equals, *and* independence of nonbinding constraints.

If a constrained allocation rule satisfies *hierarchical no-envy*, it also satisfies *equal treatment of equals.*[11] Therefore, the following result is a direct corollary of Theorem 2.

Corollary 3. *The* constrained uniform rule *is the only constrained allocation rule that satisfies* Pareto optimality, strategy-proofness, hierarchical no-envy, *and* independence of nonbinding constraints.

On bankruptcy problems, most allocation rules are *resource-monotonic* [11]. On unconstrained allocation problems, no *envy-free* rule satisfies this property, but a weaker property is widely satisfied [10]. On our domain, the *constrained uniform rule* is *resource-monotonic*. Moreover, *Pareto optimal* rules satisfying this property are *continuous with respect to E.*

Lemma 4. *If a constrained allocation rule satisfies* Pareto optimality, *and* resource-monotonicity, *it is* continuous with respect to E.

This lemma leads to the following characterization.

Theorem 5. *The* constrained uniform rule *is the only constrained allocation rule that satisfies* Pareto optimality, no-envy, *and* resource-monotonicity.

The proof is similar to that of Thomson [10] who shows the uniform rule to be the only *Pareto optimal* (with respect to strict feasibility) and *envy-free* rule which satisfies a weaker "one-sided resource monotonicity" property on unconstrained allocation problems.

In bankruptcy problems, all agents have identical monotonic preferences. Therefore, bankruptcy rules are not designed to handle preference information. However, any bankruptcy rule $B : \mathbb{P}_b \to \mathbb{R}^N_+$ has a natural exten-

[11] However, the converse is not true. A simple example to demonstrate this can be obtained by modifying the rule in Example 2 so that when $c_i > c_j$, agent j (instead of i) is the dictator.

sion $F_B : \mathbb{P} \to \mathbb{R}_+^N$ to the domain of constrained allocation problems: given $(R, c, E) \in \mathbb{P}$, for each $i \in N$, update i's constraint to $c_i^* = \min\{c_i, p(R_i)\}$. Since *Pareto optimality* implies $x_i \leq p(R_i)$, this operation basically embeds *Pareto optimality* in agent i's consumption constraint. Let,

$$F_B(R, c, E) = \begin{cases} B(R_m^N, c^*, E) & \text{if } \sum c_k^* \geq E, \\ c^* & \text{otherwise.} \end{cases}$$

The **constrained equal awards rule** is defined as follows: for all $(R, c, E) \in \mathbb{P}_b$ and all $i \in N$, $CEA_i(R, c, E) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum \min\{c_k, \lambda\} = E$. It is straightforward to check that $F_{CEA} = CU$.

The restriction of *equal treatment of equals* to bankruptcy problems requires agents with identical constraints to receive identical shares; we call this property **equal treatment of equal constraints**.[12] It is satisfied by most of the well known bankruptcy rules, including the proportional rule, the Talmudic rule [1], and the constrained equal loss rule. However, the extension (to $\mathbb{P}$) of only a single one of these rules is strategy-proof.

Theorem 6. Constrained equal awards rule *is the only bankruptcy rule on* $\mathbb{P}_b$ *that satisfies* equal treatment of equal constraints *and which has a* strategy-proof extension *to* $\mathbb{P}$.

This result also implies that among *Pareto optimal* and *strategy-proof* rules, constrained uniform rule is the only one that satisfies *equal treatment of equals* and which is consistent in the sense that it is preserved under the (aforementioned) extension of its restriction to bankruptcy problems.

4 Proofs

In the proofs of Theorems 1, 2, and 5, it is straightforward to show that CU satisfies the given properties. Conversely, suppose $F \neq CU$ also satisfies them. Let $(R, c, E) \in \mathbb{P}$ be such that $F(R, c, E) \neq CU(R, c, E)$. By *Pareto optimality*, $\sum \min\{p(R_k), c_k\} > E$. Each proof then continues as follows.

Theorem 1. *Since $F(R, c, E) \neq CU(R, c, E)$, there is $i \in N$ such that $F_i(R, c, E) < CU_i(R, c, E) \leq \min\{p(R_i), c_i\}$. Let $R_i' \in \mathscr{R}_{sp}$ be such that $p(R_i') = p(R_i)$ and $E\, P_i'\, F_i(R, c, E)$. Let $R' = (R_i', R_{-i})$. Then, by* strategy-proofness, $F_i(R, c, E) = F_i(R', c, E)$. *Since CU is also* strategy-proof, $CU_i(R, c, E) = CU_i(R', c, E)$. *Thus, $F_i(R', c, E) < CU_i(R', c, E) \leq \min\{p(R_i'), c_i\}$. Since $\sum \min\{p(R_k'), c_k\} > E$, then $\sum F_k(R', c, E) = E$. Thus, there is $j \in N$ such that $CU_j(R', c, E) < F_j(R', c, E) \leq \min\{p(R_j'), c_j\}$. By definition of CU, $CU_j(R', c, E) < \min\{p(R_j'), c_j\}$ implies that for all $k \in N$, $CU_j(R', c, E) \geq CU_k(R', c, E)$. Then $F_i(R', c, E) < CU_i(R', c, E) \leq CU_j(R', c, E) < F_j(R', c, E)$ implies $F_i(R', c, E) <$*

[12] Formally, a bankruptcy rule $B : \mathbb{P} \to \mathbb{R}_+^N$ satisfies **equal treatment of equal constraints** if for all $(R_m^N, c, E) \in \mathbb{P}_b$ and all $i, j \in N$ s.t. $c_i = c_j$, $F_i(R, c, E) = F_j(R, c, E)$.

$F_j(R', c, E)$. Thus $F_j(R', c, E)\, P_i'\, F_i(R', c, E)$. Since $F_i(R', c, E) < \min\{p(R_i'), c_i\}$, $\min\{c_i, F_j(R', c', E)\}\, P_i'\, F_i(R', c, E)$, contradicting no-envy. ■

Theorem 2. *Let $c^* = \max\{c_i : i \in N\}$ and $R^* \in \arg\max\{p(R_i) : i \in N\}$. If $R = (R^*, \ldots, R^*)$ and $c = (c^*, \ldots, c^*)$, then by* equal treatment of equals *and* Pareto optimality *$E = \sum F_k(R, c, E) \neq \sum CU_k(R, c, E) = E$, a contradiction. If $R \neq (R^*, \ldots, R^*)$ or $c \neq (c^*, \ldots, c^*)$ we proceed as follows.*

Step 1. Since $F(R, c, E) \neq CU(R, c, E)$, by *Pareto optimality*, there is $i \in N$ such that $CU_i(R, c, E) < F_i(R, c, E) \leq \min\{p(R_i), c_i\}$. If $R_i = R^*$ and $c_i = c^*$, then skip to Step 2. Otherwise, there are two possible cases:

Case 1. $[p(R_i) \leq c_i]$ Let $c_i' = c^*$. Then, by *independence of nonbinding constraints*, $CU_i(R, c, E) = CU_i(R, c_i', c_{-i}, E)$ and $F_i(R, c, E) = F_i(R, c_i', c_{-i}, E)$. Next, let $R_i' = R^*$. Then, $CU_i(R, c_i', c_{-i}, E) = CU_i(R_i', R_{-i}, c_i', c_{-i}, E)$. Moreover, as we show next, by *Pareto optimality* and *strategy-proofness* of F, $F_i(R, c_i', c_{-i}, E) = F_i(R_i', R_{-i}, c_i', c_{-i}, E)$. To see this, suppose R_i is such that $E\, P_i\, F_i(R, c_i', c_{-i}, E)$. By *strategy-proofness* applied to R_i, $F_i(R, c_i', c_{-i}, E)\, R_i\, F_i(R_i', R_{-i}, c_i', c_{-i}, E)$. This implies, $F_i(R_i', R_{-i}, c_i', c_{-i}, E) \leq F_i(R, c_i', c_{-i}, E)$. Also by *strategy-proofness* applied to R_i', $F_i(R_i', R_{-i}, c_i', c_{-i}, E)\, R_i'\, F_i(R, c_i', c_{-i}, E)$. Therefore,

$$F_i(R, c_i', c_{-i}, E) \leq F_i(R_i', R_{-i}, c_i', c_{-i}, E).$$

If $F_i(R, c_i', c_{-i}, E)\, R_i\, E$, let R_i^E be such that $p(R_i^E) = p(R_i)$ and $E\, P_i^E\, F_i(R, c_i', c_{-i}, E)$. By *Pareto optimality* and *strategy-proofness*, $F_i(R, c_i', c_{-i}, E) = F_i(R_i^E, R_{-i}, c_i', c_{-i}, E)$. Then, one can apply the above reasoning to show the desired equality. Altogether, we have $CU_i(R_i', R_{-i}, c_i', c_{-i}, E) < F_i(R_i', R_{-i}, c_i', c_{-i}, E)$.

Case 2. $[c_i < p(R_i)]$ Let R_i^c be such that $p(R_i^c) = c_i$. Then, $CU_i(R, c, E) = CU_i(R_i^c, R_{-i}, c, E)$ and, by *Pareto optimality* and *strategy-proofness* $F_i(R, c, E) = F_i(R_i^c, R_{-i}, c, E)$. Thus, $CU_i(R_i^c, R_{-i}, c, E) < F_i(R_i^c, R_{-i}, c, E) \leq \min\{p(R_i^c), c_i\}$. Since $p(R_i^c) \leq c_i$, we are in Case 1 and, repeating the same steps, we have $CU_i(R_i', R_{-i}, c_i', c_{-i}, E) < F_i(R_i', R_{-i}, c_i', c_{-i}, E)$.

In either case, we change R_i to $R_i' = R^*$ and c_i to $c_i' = c^*$. If $(R_i', R_{-i}) = (R^*, \ldots, R^*)$ and $(c_i', c_{-i}) = (c^*, \ldots, c^*)$. Then by *equal treatment of equals* and *Pareto optimality*, $E = \sum F_k(R, c, E) > \sum CU_k(R, c, E) = E$, a contradiction. Otherwise, we proceed to Step 2.

Step 2. Since $\sum F_k(R, c, E) = \sum CU_k(R, c, E) = E$, there is $j \in N$ such that $F_j(R_i', R_{-i}, c_i', c_{-i}, E) < CU_j(R_i', R_{-i}, c_i', c_{-i}, E) \leq \min\{p(R_j), c_j\}$. If $R_j = R^*$ and $c_j = c^*$, then since $R_i' = R^*$, $c_i' = c^*$, $F_j(R_i', R_{-i}, c_i', c_{-i}, E) < CU_j(R_i', R_{-i}, c_i', c_{-i}, E)$ and $CU_i(R_i', R_{-i}, c_i', c_{-i}, E) < F_i(R_i', R_{-i}, c_i', c_{-i}, E)$ contradicts *equal treatment of equals*. Otherwise, let $R_j' = R^*$ and $c_j' = c^*$. Following the same steps in Cases 1 and 2 above, we obtain $F_j(R_{ij}', R_{-ij}, c_{ij}', c_{-ij}, E) < CU_j(R_{ij}', R_{-ij}, c_{ij}', c_{-ij}, E)$.

If $(R'_{ij}, R_{-ij}) = (R^*, \dots, R^*)$ and $(c'_{ij}, c_{-ij}) = (c^*, \dots, c^*)$, then by *equal treatment of equals* and *Pareto optimality*, $E = \sum F_k(R, c, E) < \sum CU_k(R, c, E) = E$, a contradiction. Otherwise, we apply Step 1 to $(R'_{ij}, R_{-ij}, c'_{ij}, c_{-ij}, E)$. At each of Steps 1 and 2, we replace the preferences of a new agent by R^* and his constraint by c^*. Since N is finite, such repeated replacement eventually leads to a contradiction. ■

Theorem 5. *Since $F(R, c, E) \neq CU(R, c, E)$ and F is* Pareto optimal, *there are $i, j \in N$ such that $F_i(R, c, E) < \min\{p(R_i), c_i\}$ and $F_i(R, c, E) < F_j(R, c, E)$. By* no-envy, $F_i(R, c, E)\ R_i \min\{F_j(R, c, E), c_i\}$. *Since F satisfies* Pareto optimality *and* resource-monotonicity, *it is continuous with respect to E, and thus, as E goes to 0, the shares of both agents change continuously. Then there is $E' \leq E$ such that $F_j(R, c, E') = \min\{p(R_i), c_i\}$. By* no-envy, $F_i(R, c, E') = F_j(R, c, E')$. *Then, from E to E', i is better off and j is worse off, contradicting* resource-monotonicity. ■

The remaining proofs do not use the supposition made at the beginning of this section.

Lemma 4. *Let $F : \mathbb{P} \to \mathbb{R}^N_+$ be a constrained allocation rule that satisfies the given properties. Let $\{E^t\}$ be a sequence in $\mathbb{R}_+$ converging E. For each $t \in \mathbb{N}$, let $F(R, c, E^t) = x^t$ and let $F(R, c, E) = y$. Note that $\{x^t\}$ is a subset of the compact set $\prod_{k \in N}[0, \min\{p(R_k), c_k\}]$, and let X be the set of its limit points. Let $x \in X$. Then there is a subsequence $\{x^\upsilon\}$ of $\{x^t\}$ converging x.*

Step 1. Note that $\{\sum x^\nu_k\} \to \sum x_k$ and $\{\min\{E^\nu, \sum \min\{p(R_k), c_k\}\}\} \to \min\{E, \sum \min\{p(R_k), c_k\}\}$. By *Pareto optimality*, for all $\nu \in \mathbb{N} \sum x^\nu_k = \min\{E^\nu, \sum \min\{p(R_k), c_k\}\}$. Thus, $\sum x_k = \min\{E, \sum \min\{p(R_k), c_k\}\}$.

Step 2. For all $i \in N$, $x_i \leq \min\{p(R_i), c_i\}$. Otherwise, there is a $\nu^* \in \mathbb{N}$ such that for all $\nu \geq \nu^*$, $x^\nu_i > \min\{p(R_i), c_i\}$. However, *Pareto optimality* implies $x^\nu_i \leq \min\{p(R_i), c_i\}$, a contradiction.

Next, we will show that $x = y$. Since $x \in X$ is arbitrary, this implies $X = \{y\}$, the desired result. First assume $E \geq \sum \min\{p(R_k), c_k\}$. Then by Step 1, $\sum x_k = \sum \min\{p(R_k), c_k\}$, and by Step 2, for all $i \in N$, $x_i \leq \min\{p(R_i), c_i\}$. Thus, by *Pareto optimality* of F, for all $i \in N$, $x_i = \min\{p(R_i), c_i\} = y_i$. Alternatively assume $E < \sum \min\{p(R_k), c_k\}$. Then by Step 1, $\sum x_k = E$, and by Step 2, for all $k \in N$, $x_k \leq \min\{p(R_k), c_k\}$. Suppose $x \neq y$. Then, there are $i, j \in N$ such that $x_i < y_i \leq \min\{p(R_i), c_i\}$ and $y_j < x_j \leq \min\{p(R_j), c_j\}$. Then, there is $\nu^* \in \mathbb{N}$ such that for all $\nu \geq \nu^*$, $E^\nu \leq \sum \min\{p(R_k), c_k\}$, also $x^\nu_i < y_i \leq \min\{p(R_i), c_i\}$, and $y_j < x^\nu_j \leq \min\{p(R_j), c_j\}$. But then, from E^ν to E, i is strictly better off and j is strictly worse off, contradicting *resource-monotonicity*. ■

Theorem 6. *It is straightforward to show that CEA satisfies the given properties. Conversely, let $B : \mathbb{P}_b \to \mathbb{R}^N_+$ be a bankruptcy rule that satisfies the properties. It is straightforward to check that then F_B also satisfies* Pareto optimality, equal treatment of equals, *and* independence of nonbinding constraints. *But then, by Theorem 2, $F_B = CU$. Therefore, $B = CEA$.* ■

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