Games and Economic Behavior **39,** 91–110 (2002)
doi:10.1006/game.2001.0884, available online at http://www.idealibrary.com on IDEAL®

Misrepresentation of Utilities in Bargaining: Pure Exchange and Public Good Economies

Özgür Kıbrıs

Faculty of Arts and Social Sciences, Sabancı University, Orhanlı, Tuzla 81474, İstanbul, Turkey, and CORE, Université Catholique de Louvain, 34 Voie du Roman Pays, 1348, Louvain-la-Neuve, Belgium
E-mail: ozgur@sabanciuniv.edu

Received July 3, 2000; published online February 7, 2002

In order to analyze bargaining in pure exchange and public good economies when the agents are not informed about their opponents' payoffs, we embed each bargaining problem into a noncooperative game of misrepresentation. In pure exchange (public good) economies with an arbitrary number of agents whose true utilities satisfy a mild assumption, the set of allocations obtained at the linear-strategies Nash equilibria of this game is equal to the set of constrained Walrasian (Lindahl) allocations with respect to the agents' true utilities. Without this assumption, the result holds for two-agent pure-exchange economies and, under alternative assumptions, for public good economies. *Journal of Economic Literature* Classification Number: C72, C78. © 2002 Elsevier Science (USA)

Key Words: bargaining; Walrasian rule; Lindahl rule; distortion game; interiority.

1. INTRODUCTION

Many solution rules for economic problems are manipulable by misrepresentation of private information. Understanding the "real" outcomes of such rules, therefore, requires taking strategic behavior into account. A standard technique for this is to embed the original problem into a noncooperative game (in which agents strategically "distort" their private information) and to analyze its equilibrium outcomes. In this paper, we use this technique to analyze bargaining in pure exchange and public good economies when the agents are not informed about their opponents' utility information.

This paper is the second chapter of my Ph.D. thesis submitted to the University of Rochester. I thank the seminar participants at the International Conference on Game Theory in SUNY Stony Brook, Larry Epstein, Joel Sobel, and especially my advisor, William Thomson, for helpful comments and suggestions. The usual disclaimer applies.

0899-8256/02 $35.00
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First assume that the agents' (ordinal) preferences are publicly known. Then, manipulation can only take place through misrepresentation of cardinal utility information. In two-agent bargaining with the Nash (1950) or the Kalai–Smorodinsky (1975) rules, an agent's utility increases if his opponent is replaced with another that has the same preferences but a more concave utility function[1] (Kihlstrom *et al.*, 1981). This finding extends to n agents (Nielsen, 1984) as well as to noncooperative models (Roth, 1985; Binmore *et al.*, 1986; Harrington, 1986). On allocation problems, this implies that an agent can increase his payoff by declaring a less concave utility function. For the Nash bargaining rule, it is a dominant strategy for each agent to declare the least concave representation of his preferences (Kannai, 1977; Crawford and Varian, 1979). For a single good, the equilibrium outcome is an equal division.

If preferences are not publicly known, however, their misrepresentation can also be used for manipulation. The resulting game does not have dominant strategy equilibria. Nevertheless, for a large class of two-agent bargaining rules, the set of allocations obtained at Nash equilibria in which agents declare linear utilities is equal to the set of "constrained" Walrasian allocations from equal division (with respect to the agents' true utilities) (Sobel, 1981 and 1998). This equivalence also holds for more general resource allocation rules in n-agent quasilinear problems (Thomson, 1984 and 1988). Following this branch of the literature, we also assume that the agents are not informed about their opponents' preferences.

The existing literature focuses on allocating a social endowment of private goods. However, there are many exchange and public good economies where bargaining takes place and agents strategically distort private information (such as firm–union negotiations or bargaining between interest groups on government projects). We, therefore, extend the analysis to such economies.

Second, the literature is mostly restricted to two-agent problems. If there are more agents the analysis gets very complicated since then each agent's attainable set is jointly determined by all of his opponents' declarations. Thomson (1984, 1988) overcomes this difficulty by restricting the true preferences to be quasilinear. We discover an alternative restriction (*interiority*): *interior bundles are strictly preferred to boundary bundles*. Any economy that satisfies Inada conditions also satisfies this property. The class of economies that satisfy *interiority* has an empty intersection with quasilinear economies. Therefore, our results apply to a class not analyzed by Thomson. Third, our results hold for all Pareto optimal and individually rational bargaining

[1]Assuming that the agents' risk preferences satisfy Savage's axioms, the concavity of their (Bernoulli) utility functions determine their risk attitudes.

rules. This class contains the Nash and Kalai–Smorodinsky rules on which most of the literature is based as well as the class analyzed by Sobel (1981).[2]

For pure exchange economies (Section 3), our conclusions are similar to those of Sobel (1981, 1998) and Thomson (1984, 1988). *Interiority* only plays a role for the n-agent case. In public good economies (Section 4), however, this conclusion fails even for two agents, unless *interiority* is assumed. Since the two-agent case is still tractable without *interiority*, we also explore the possibility of replacing it with other assumptions. First, we analyze the implications of the Nash equilibrium outcomes being Pareto optimal (Subsection 4.1). For two agents, this is equivalent to strengthening the equilibrium concept to a strong Nash equilibrium. Next, we analyze the implications of the bargaining rule being continuous (Subsection 4.2). Supplementary results and proofs are contained in the Appendix (Section 6).

2. MODEL

The vector inequalities are $\leq$, $<$, and $\ll$. There are m commodities. Let $N = \{1, \ldots, i, \ldots, n\}$ be the set of *agents*. Each $i \in N$ has an *endowment*, $\omega_i \in \mathbb{R}^m_+$, and a **true utility function**, $u_i : \mathbb{R}^m_+ \to \mathbb{R}$, which is *concave, nondecreasing*, and *increasing on* $\mathbb{R}^m_{++}$.[3] A utility function u_i satisfies **interiority** if, for each $x \in \mathbb{R}^m_{++}$ and for each $y \in \mathbb{R}^m_+ \backslash \mathbb{R}^m_{++}$, $u_i(x) > u_i(y)$. Let $u = (u_1, \ldots, u_n)$ and $\omega = (\omega_1, \ldots, \omega_n)$. Let $P(u)$ and $I(u, \omega)$ represent the sets of all *Pareto optimal* and *individually rational* allocations, respectively.

An n-person **bargaining problem** is a pair (S, d) where $d \in \mathbb{R}^n$ is the *disagreement point* and $S \subset \mathbb{R}^n$, the *bargaining set*, is nonempty, compact, convex, and contains d. Let $\mathcal{B}$ be the class of all bargaining problems. A **bargaining rule** F assigns each bargaining problem $(S, d) \in \mathcal{B}$ to a payoff profile $F(S, d) \in S$. It is **Pareto optimal** if, for each $(S, d) \in \mathcal{B}$, $s > F(S, d)$ implies $s \notin S$. It is **individually rational** if, for each $(S, d) \in \mathcal{B}$, $F(S, d) \geq d$. It is **continuous** if, for each sequence of problems $\{(S^k, d^k)\}$ in $\mathcal{B}$ converging to some $(S, d) \in \mathcal{B}$, the sequence $\{F(S^k, d^k)\}$ converges to $F(S, d)$.[4] A bargaining rule is **admissible** if it is *Pareto optimal* and *individually rational*.

Let X be a set of feasible allocations. Let F be an *admissible* bargaining rule. In the *game*, each agent i declares a utility function $v_i : \mathbb{R}^m_+ \to \mathbb{R}$

[2]Sobel assumes Pareto optimality, symmetry, scale invariance, and symmetric monotonicity. The last two properties imply individual rationality.

[3]A function $f : \mathbb{R}^m_+ \to \mathbb{R}$ is nondecreasing [increasing] if, for each $x, y \in \mathbb{R}^m_+$, $x < y$ implies $f(x) \leq f(y)[f(x) < f(y)]$.

[4]The sequence $\{F(S^k, d^k)\}$ converges to (S, d) if $\{S^k\}$ converges to S with respect to the Hausdorff metric and $\{d^k\}$ converges to d with respect to the Euclidean metric.

from the set of *credible declarations* **V** and a tie-breaking action f_i in $\mathbb{R}^{m\times n}$. Each $v_i \in V$ is *continuous, concave, nondecreasing,* and *increasing on* $\mathbb{R}^m_{++}$. The concavity assumption can be interpreted as the agents being known to be risk-averse. Let $v = (v_1, \ldots, v_n)$ and $f = (f_1, \ldots, f_n)$. The resulting *bargaining set* is $S(v) = \{s \in \mathbb{R}^n \mid s = v(x) \text{ for some } x \in X\}$, and the *disagreement point* is $d(v) = v(\omega)$. The solution $F(S(v), d(v))$ to this problem corresponds to the set of allocations $B(v) = \{x \in X \mid v(x) = F(S(v), d(v))\}$. A tie-breaking rule uses f to make a single-valued selection from $B(v)$ as follows:

$$\overline{B}(v, f) = \begin{cases} \frac{1}{n}\sum f_i & \text{if } \frac{1}{n}\sum f_i \in B(v), \\ \text{a fixed element of } B(v) & \text{otherwise.} \end{cases}$$

For each F, u, and ω, this procedure defines a **distortion game**

$$\mathcal{D}_F(u) = (u, (V \times \mathbb{R}^{m\times n})^n, \overline{B}).$$

Sobel (1981) shows that any such game is well-defined. He also shows that a strategy tuple, $((v_1^*, f_1^*), \ldots, (v_n^*, f_n^*))$, is a Nash equilibrium of the game if and only if there is an $x^* \in B(v^*)$ such that, for each $i \in N$, x_i^* maximizes $u_i(x_i)$ subject to $x \in B(V, v_{-i}^*)$.[5] It is straightforward to generalize these results to pure exchange and public good economies with an arbitrary number of agents. From now on we will refer to the pair (v^*, x^*) as a **Nash equilibrium pair** of the distortion game. Let $\mathcal{NE}(\mathcal{D}_F(u))$ denote the set of *Nash equilibrium pairs* of $\mathcal{D}_F(u)$. Let

$$\mathcal{NE}_v(\mathcal{D}_F(u)) = \{v \in V^n \mid \text{ for some } x \in X, (v, x) \in \mathcal{NE}(\mathcal{D}_F(u))\}$$

denote the set of **Nash equilibrium declarations** and let

$$\mathcal{NE}_x(\mathcal{D}_F(u)) = \{x \in X \mid \text{ for some } v \in V^n, (v, x) \in \mathcal{NE}(\mathcal{D}_F(u))\}$$

denote the set of **Nash equilibrium allocations**.

3. PURE EXCHANGE ECONOMIES

Each agent i has an *endowment* $\omega_i \gg 0$ of private goods. For simplicity assume $\sum \omega_i = 1$.[6] A **consumption bundle of agent** i is a vector $x_i \in \mathbb{R}^m_+$. A **feasible allocation** is a list of *consumption bundles* $x = (x_1, \ldots, x_n) \in \mathbb{R}^{m\times n}_+$ satisfying $\sum x_i \leq 1$. The **feasible set** X_e is the set of all *feasible allocations*. An allocation $x^* \in X_e$ is a **constrained Walrasian allocation**,

[5] From here on, $B(V, v_{-i}) = \bigcup_{v_i \in V} B(v_i, v_{-i})$.
[6] From here on, $\mathbf{b} \in \mathbb{R}^m$ denotes the vector whose every coordinate is equal to b.

$x^* \in W^c(u, \omega)$, if there is a price vector $p \in \mathrm{int}(\Delta^{m-1})$ such that, for each $i \in N$, x_i^* maximizes $u_i(x_i)$ subject to $px_i \leq p\omega_i$ and $\mathbf{0} \leq x_i \leq \mathbf{1}$.[7] For each $i \in N$, let $l_i[p] \in V$ be a **linear utility function** associated with $p \in \mathbb{R}^m_{++}$: given $\alpha_i \in \mathbb{R}_{++}$ and $\beta_i \in \mathbb{R}$, $l_i[p](x_i) = \alpha_i(px_i) + \beta_i$ for each $x_i \in \mathbb{R}^m_+$.

The agents bargain to reallocate their endowments. However, each agent manipulates the process by strategically distorting his utility information. Even though the bargaining outcomes under truthful declaration satisfy many desirable properties, after manipulation they may violate even very basic properties such as Pareto optimality. Nevertheless, certain Nash equilibrium outcomes still satisfy many desirable properties: they are constrained Walrasian allocations with respect to the agents' true utilities. Our first result states that every constrained Walrasian allocation is also a Nash equilibrium outcome of the distortion game. Moreover, the price vector associated with each such allocation determines the agents' Nash equilibrium strategies.

THEOREM 1. *If* $x^* \in W^c(u, \omega)$, *with associated price vector* $p^* \in \Delta^{m-1}$, *then* $(l_1[p^*], \ldots, l_n[p^*], x^*)$ *is a Nash equilibrium pair of* $\mathcal{D}_F(u)$.

Theorem 1 is a straightforward generalization of a similar result by Sobel (1981).[8] Its proof is based on the observation that at Nash equilibria in linear declarations each agent's set of attainable bundles is a subset of his (constrained) budget set. Also note that, since any Walrasian allocation is also a constrained Walrasian allocation, Theorem 1 holds for the Walrasian rule as well.

Now that we know that certain manipulation outcomes are desirable, we ask if the agents can be guaranteed to receive such allocations. If declaring linear utilities at a Nash equilibrium always leads to a constrained Walrasian allocation, a social planner can guarantee such an outcome by publishing this information (or by restricting the agents' strategy spaces to linear functions). This turns out to be the case in Sobel (1981). His proof can be divided into two steps, the first of which states that at any Nash equilibrium in linear utilities agents' declared preferences have the same slope.

LEMMA 2. *Let* $N = \{1, 2\}$. *If* $(l_1[p_1], l_2[p_2], x^*)$ *is a Nash equilibrium pair of* $\mathcal{D}_F(u)$, *then* $p_1 = p_2$.

Due to the aforementioned reasons, this lemma does not straightforwardly generalize to n agents. However, note that if two agents declare different slopes, (by *Pareto optimality*) one of them has to receive a boundary

[7]We use $\Delta^{m-1} = \{x \in \mathbb{R}^m_+ \mid \sum x_i = 1\}$ to denote the $(m-1)$-dimensional simplex. We use $\mathrm{int}(S)$ to denote the interior of the set S.

[8]Sobel also notes that his result straightforwardly extends to n agents.

bundle. Under *interiority*, this agent is strictly better off by truthful declaration which (by *individual rationality*) at least gives him his (interior) endowment. This observation leads to the following generalization of Lemma 2.

LEMMA 3. *Assume that* $u_1, \ldots, u_n$ *satisfy* interiority. *If* $(l_1[p_1], \ldots, l_n[p_n], x^*)$ *is a Nash equilibrium pair of* $\mathcal{D}_F(u)$, *then* $p_1 = \cdots = p_n$.

The following lemma completes the argument. It states that any allocation obtained at a Nash equilibrium in which agents declare linear preferences with identical slopes is a constrained Walrasian allocation (with respect to the true utilities). This result generalizes the second step in Sobel's argument.

LEMMA 4. *If* $(l_1[p^*], \ldots, l_n[p^*], x^*)$ *is a Nash equilibrium pair of* $\mathcal{D}_F(u)$, *then* $x^* \in W^c(u, \omega)$ *with the associated price vector* p^*.

Lemmas 3 and 4 together lead to the conclusion that under *interiority* any "linear-strategies" Nash equilibrium outcome of the distortion game is also a constrained Walrasian allocation with respect to the agents' true utilities. Due to Lemmas 2 and 4, this conclusion is also true for two-agent economies that violate *interiority*.

THEOREM 5. *Assume that* $u_1, \ldots, u_n$ *satisfy* interiority. *If* $(l_1[p_1], \ldots, l_n[p_n], x^*)$ *is a Nash equilibrium pair of* $\mathcal{D}_F(u)$, *then* $x^* \in W^c(u, \omega)$.

Under *interiority*, the constrained Walrasian rule coincides with the Walrasian rule. Therefore, Theorem 5 holds for the Walrasian rule as well.

4. PUBLIC GOOD ECONOMIES

There is a single private good and a single public good. The initial level of the public good is 0. Each agent i has a positive endowment of the private good, $\omega_{x,i} > 0$. Therefore, agent i's endowment is $\omega_i = (\omega_{x,i}, 0)$. For simplicity assume $\sum \omega_{x,i} = 1$. The public good is produced from the private good via a constant returns-to-scale technology. To produce y units of the public good, at least y units of the private good must be used. A **consumption bundle of agent** i is $z_i = (x_i, y) \in \mathbb{R}^2_+$, where x_i denotes his consumption of the private good and y that of the public good. A **feasible allocation** is a list of *consumption bundles* $z = (z_1, \ldots, z_n) \in \mathbb{R}^{2\times n}_+$ satisfying $y + \sum x_i \leq \sum \omega_{x,i} = 1$. The **feasible set** X_p is the set of all *feasible allocations*.

An allocation $z^* \in X_p$ is a **constrained Lindahl allocation**, $z^* \in L^c(u, \omega)$, if for each $i \in N$ there is a *Lindahl individualized price* $\pi_i \geq 0$ such that (i) z^* maximizes $y \sum \pi_k - \sum(\omega_{x,k} - x_k)$ subject to $z \in X_p$ and (ii), for each $i \in N$, z_i^* maximizes $u_i(z_i)$ subject to $x_i + \pi_i y \leq \omega_{x,i}$ and $x_i + y \leq 1$.

For each $i \in N$, let $l_i[\pi_i] \in V$ be a **linear utility function** associated with $\pi_i \in \mathbb{R}_{++}$: given $\alpha_i \in \mathbb{R}_{++}$ and $\beta_i \in \mathbb{R}$, $l_i[\pi_i](z_i) = \alpha_i(x_i + \pi_i y) + \beta_i$ for each $z_i \in \mathbb{R}^2_+$.

The results obtained for public good economies are different due to some basic differences between the two models. Bargaining problems associated with pure exchange economies are comprehensive. This is not true for public good economies unless the utility functions satisfy *interiority*. In pure exchange economies, agents have endowments of all goods. In public good economies agents only have endowments of the private good. Therefore, *interiority* does not imply that the agents prefer their endowments to boundary bundles. Due to the monotonicity of preferences, in pure-exchange economies, any utility maximizing bundle satisfies the budget constraint with equality. This is not necessarily the case in public-good economies. However, every constrained Lindahl allocation satisfies this property (see Lemma 17). Moreover, at constrained Lindahl allocations, the sum of the agents' Lindahl prices never exceed one; if the public good is produced ($y > 0$), these prices are uniquely defined and add up to one (see Proposition 18).

The agents bargain over the amount of the public good to produce and the allocation of the production cost. Even though each agent manipulates the process by strategically distorting his utility information, certain Nash equilibrium outcomes still satisfy desirable properties: they are constrained Lindahl allocations with respect to the agents' true utilities. Our next result states that every constrained Lindahl allocation is also a Nash equilibrium outcome of the distortion game. Moreover, for such allocations, each agent's individualized Lindahl price determines his equilibrium strategy.

THEOREM 6. *If $z^* \in L^c(u, \omega)$ with associated prices $\pi = (\pi_1, \ldots, \pi_n)$, then $(l_1[\pi_1], \ldots, l_n[\pi_n], z^*)$ is a Nash equilibrium pair of $\mathcal{D}_F(u)$.*

Since any Lindahl allocation is also a constrained Lindahl allocation, Theorem 6 holds for the Lindahl rule as well.

Next we ask if Nash equilibria in linear strategies always lead to constrained Lindahl allocations. As in pure exchange economies, such a result guarantees that by focusing on Nash equilibria in linear utilities the agents will end up at a desirable allocation. We present our argument in two steps. The first step, making explicit use of the *interiority* assumption, establishes that at every Nash equilibrium in linear utilities the slopes of the agents' declared preferences add up to one and the resulting allocation is Pareto optimal with respect to the agents' true utilities.

LEMMA 7. *Assume that $u_1, \ldots, u_n$ satisfy interiority. If $(l_1[\pi_1], \ldots, l_n[\pi_n], z^*)$ is a Nash equilibrium pair of $\mathcal{D}_F(u)$, then $\sum \pi_k = 1$ and $z^* \in P(u)$.*

The second step states that if these properties are satisfied at a Nash equilibrium, z^* is a constrained Lindahl allocation. This step does not use the *interiority* assumption and, therefore, is true for any profile of true utilities.

LEMMA 8. *If* $(l_1[\pi_1], \ldots, l_n[\pi_n], z^*)$ *is a Nash equilibrium pair of* $\mathcal{D}_F(u)$ *where* $\sum \pi_k = 1$ *and* $z^* \in P(u)$, *then* $z^* \in L^c(u, \omega)$ *with associated prices* π.

Lemmas 7 and 8 together lead to the conclusion that under *interiority* any "linear-strategies" Nash equilibrium outcome of the distortion game is also a constrained Lindahl allocation with respect to the agents' true utilities.

THEOREM 9. *Assume that* $u_1, \ldots, u_n$ *satisfy* interiority. *If* $(l_1[\pi_1], \ldots, l_n[\pi_n], z^*)$ *is a Nash equilibrium pair of* $\mathcal{D}_F(u)$, *then* $z^* \in L^c(u, \omega)$.

Under *interiority*, the constrained Lindahl rule coincides with the Lindahl rule. Therefore, Theorem 9 holds for the Lindahl rule as well.

In pure exchange economies, the basic motivation for *interiority* was that it enabled us to generalize the two-agent conclusion of Lemma 2 to an arbitrary number of agents (see Lemma 3). The rest of the argument did not utilize this assumption (see Lemma 4). Similarly, in the proof of Theorem 9, only Lemma 7 utilizes *interiority*. If this lemma continues to hold for two-agent public-good economies that violate *interiority*, we can conclude that the assumption plays the same role in both models. Surprisingly, as the following example demonstrates, even when there is a single agent whose true utility violates *interiority*, the two-agent version of Lemma 7 fails. That is, for such economies, there are distortion games with linear Nash equilibrium strategies $(l[\pi_1], l[\pi_2])$ violating $\pi_1 + \pi_2 = 1$. Moreover, even for Nash equilibria satisfying $\pi_1 + \pi_2 = 1$, the resulting allocation does not have to be Pareto optimal (and, therefore, not a constrained Lindahl allocation) with respect to the agents' true utilities. This example suggests that *interiority* plays a much more central role in public good economies.

EXAMPLE 10. Let $N = \{1, 2\}$. Let D_i^* denote the benevolent dictatorial rule where agent i is the dictator.[9] Let $\pi_1^* = \frac{2}{7}$ and

$$F = \begin{cases} D_1^* & \text{if } \pi_1 > \pi_1^*, \\ D_2^* & \text{if } \pi_1 \leq \pi_1^*. \end{cases}$$

Let $\omega_1 = \omega_2 = 0.5$. For each $z \in X_p$, let $u_1(x_1, y) = x_1^{1/5} y^{4/5}$ and $u_2(x_2, y) = x_2 + 2y$. Let $\pi_2 \in [\frac{5}{7}, 2)$. Let $z^* \in X_p$ be such that $x_2^* = 0$ and $x_1^* + \pi_1^* y^* = \omega_1$. Then $(l_1[\pi_1^*], l_2[\pi_2], z^*) \in \mathcal{NE}(\mathcal{D}_F(u))$. However, $z^* \notin L^c(u, \omega)$ (Fig. 1).

[9] Given $(S, d) \in \mathcal{B}$, $D_i^*(S, d)$ chooses the payoff profile that maximizes agent i's payoff subject to individual rationality and Pareto optimality constraints.

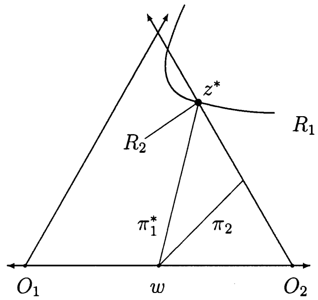

FIG. 1. In Example 10, z^* is not a constrained Lindahl allocation.

Note that the bargaining rule used in the above example is discontinuous. Moreover, the allocation z^* is not Pareto optimal with respect to the true utilities. Thus, we ask how important these properties are to our conclusion; we analyze the implications of using a continuous and admissible bargaining rule and strengthening the Nash equilibrium concept. Since *interiority* is not assumed, the results are restricted to the two-agent case.

4.1. *Implications of Pareto Optimality*

First we analyze the implications of the Nash equilibrium outcome being Pareto optimal with respect to the agents' true utilities. For two agents, this requirement coincides with strengthening the Nash equilibrium concept to strong Nash (Aumann, 1959) or coalition-proof Nash (Bernheim *et al.*, 1987) equilibria. The conclusion highly depends on the sum of the slopes of the agents' equilibrium declarations, $\pi_1 + \pi_2$. If it is equal to one, it follows from Lemma 8 that the outcomes of any such Nash equilibria are also constrained Lindahl allocations.

COROLLARY 11. *Let* $N = \{1, 2\}$. *If* $(l_1[\pi_1], l_2[\pi_2], z^*)$ *is a Nash equilibrium pair of* $\mathcal{D}_F(u)$ *such that* $\pi_1 + \pi_2 = 1$ *and* $z^* \in P(u)$, *then* $z^* \in L^c(u, \omega)$ *with associated prices* π_1, π_2.

If $\pi_1 + \pi_2 < 1$, Pareto optimality implies that $z^* = \omega$. That is, the public good is not produced. However, this allocation being obtained at a Nash equilibrium sufficiently informs us about the agents' true utilities to conclude that it is also a constrained Lindahl allocation.

PROPOSITION 12. *Let* $N = \{1, 2\}$. *If* $(l_1[\pi_1], l_2[\pi_2], z^*)$ *is a Nash equilibrium pair of* $\mathcal{D}_F(u)$ *such that* $\pi_1 + \pi_2 < 1$ *and* $z^* \in P(u)$, *then* $z^* = \omega \in L^c(u, \omega)$ *for some prices* π_1', π_2' *such that* $\pi_1' + \pi_2' = 1$.

Unfortunately, *Pareto optimal Nash equilibrium outcomes at which* $\pi_1 + \pi_2 > 1$ *are not necessarily constrained Lindahl allocations.* The following example demonstrates this point.

EXAMPLE 13. Let $N = \{1, 2\}$. Let D_i^* denote the benevolent dictatorial rule where agent i is the dictator. Let $\pi_1^* = \frac{6}{8}$ and

$$F = \begin{cases} D_1^* & \text{if } \pi_1 < \pi_1^*, \\ D_2^* & \text{if } \pi_1 \geq \pi_1^*. \end{cases}$$

Let $\omega_1 = \frac{8}{10}, \omega_2 = \frac{2}{10}$. For each $z \in X_p$, let $u_1(x_1, y) = x_1^{1/3} y^{2/3}$ and $u_2(x_2, y) = x_2 + 2y$. Let $\pi_2 \in (\frac{2}{5}, 2)$. Let $z^* \in X_p$ be such that $x_2^* = 0$ and $x_1^* + \pi_1^* y^* = \omega_1$. Then $(l_1[\pi_1^*], l_2[\pi_2], z^*) \in \mathcal{NE}(\mathcal{D}_F(u))$. Moreover, $z^* \in P(u)$. However, $z^* \notin L^c(u, \omega)$ (Fig. 2).

4.2. *Implications of Continuity*

Next, we analyze the implications of restricting the class of distortion games to those obtained from a *continuous* and *admissible* bargaining rule. If the bargaining rule F is *continuous*, the outcome correspondence B is *upper hemicontinuous* (see Lemma 19) even though it is *not lower hemicontinuous* (see Example 20). This observation plays an important rule in this section.

Once again, the conclusion depends on the sum of the slopes of the agents' equilibrium declarations, $\pi_1 + \pi_2$. If it is equal to one, the corresponding Nash equilibrium outcomes are also constrained Lindahl allocations.

PROPOSITION 14. *Let $N = \{1, 2\}$. Let F be a continuous and admissible bargaining rule. If $(l_1[\pi_1], l_2[\pi_2], z^*)$ is a Nash equilibrium pair of $\mathcal{D}_F(u)$ such that $\pi_1 + \pi_2 = 1$, then $z^* \in L^c(u, \omega)$ with associated prices π_1, π_2.*

If $\pi_1 + \pi_2 > 1$, the public good is produced maximally subject to the feasibility, Pareto optimality, and individual rationality constraints. Such equilibrium outcomes also turn out to be constrained Lindahl allocations.

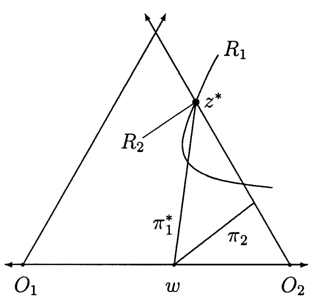

FIG. 2. In Example 13, z^* is not a constrained Lindahl allocation.

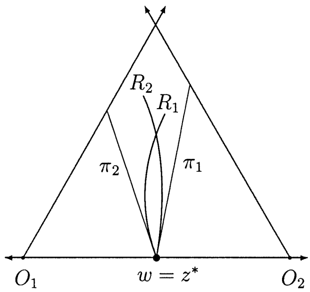

FIG. 3. Construction of Example 16.

PROPOSITION 15. *Let $N = \{1, 2\}$. Let F be a continuous and admissible bargaining rule. Let $(l_1[\pi_1], l_2[\pi_2], z^*)$ be a Nash equilibrium pair of $\mathscr{D}_F(u)$ such that $\pi_1 + \pi_2 > 1$. Let $\bar{z} = ((0, 1), (0, 1))$. If $\bar{z} \in P(l_1[\pi_1], l_2[\pi_2]) \cap I(l_1[\pi_1], l_2[\pi_2], \omega)$, then $z^* = \bar{z}$. Otherwise, z^* is the closest point to $\bar{z}$ in $P(l_1[\pi_1], l_2[\pi_2]) \cap I(l_1[\pi_1], l_2[\pi_2], \omega)$. Moreover, in each case $z^* \in L^c(u, \omega)$ with associated prices $\pi_1' \leq \pi_1$ and $\pi_2' \leq \pi_2$.*

Unfortunately, *continuity of F is not sufficient to ensure that Nash equilibrium outcomes at which $\pi_1 + \pi_2 < 1$ are constrained Lindahl allocations*. The following example demonstrates this point.

EXAMPLE 16. Let $N = \{1, 2\}$. Let F be a continuous and admissible bargaining rule. Since $\pi_1 + \pi_2 < 1$, $B(l_1[\pi_1], l_2[\pi_2]) = \{\omega\}$. Let ω and u be such that $\omega \notin P(u)$. Let $P(u) \cap I(u, \omega) \subset \text{int}(X_p)$. Then $(l_1[\pi_1], l_2[\pi_2], \omega) \in \mathscr{N}\mathscr{E}(\mathscr{D}_F(u))$, but $\omega \notin L^c(u, \omega)$ (Fig. 3).

5. CONCLUSION

Our results basically state that the set of allocations obtained at a Nash equilibrium in which agents declare linear utilities is equal to the set of constrained Walrasian/Lindahl allocations (with respect to the agents' true utilities). Assuming *interiority*, we obtain results for an arbitrary number of agents. Moreover, under this assumption, the Walrasian/Lindahl rules coincide with their constrained versions. Therefore, unlike in the previous literature, the above equivalence also holds for the (unconstrained) Walrasian/Lindahl rules. Also, note that the *interiority* assumption plays a more central role in public good economies.

Some of the results have trivial extensions. It is straightforward to extend the proofs of Theorems 1 and 6 to show that any constrained Walrasian/Lindahl allocation is obtained at a strong Nash equilibrium in which agents declare linear utilities. Also, the proofs of Lemmas 3, 4, 7, and 8 make no explicit use of the bargaining framework. Therefore, Theorems 5 and 9 hold for any distortion game derived from an allocation rule that is Pareto optimal and individually rational.

Possible extensions of this work are as follows. In our model agents are involved in a single bargaining process. In the case of disagreement, trade does not occur and each agent gets his endowment. If agents are involved in several bargaining processes simultaneously, the conclusions may dramatically change. Our preliminary results on this issue are that each agent is indifferent between declaring alternative representations of his preferences. Next, note that in the case that the endowments are not observable, the agents could manipulate the outcome via misrepresentation of this information. Many solution rules, such as the Walrasian rule in exchange economies, are known to be manipulable through misrepresentation of endowment information. An analysis of the allocations that result from such manipulation may prove to be very useful.

APPENDIX

A.1. *Supplementary Results*

The following lemma establishes that at every constrained Lindahl allocation the budget constraints are satisfied with equality.

LEMMA 17. *Let $z^* \in L^c(u, \omega)$. Let $\pi = (\pi_1, \ldots, \pi_n)$ be associated with z^*. Then, for each $i \in N$, $x_i^* + \pi_i y^* = \omega_{x,i}$.*

Proof. If $z^* = \omega$ the result trivially holds. So, assume $z^* \neq \omega$. Then $y^* > 0$. Suppose there is $i \in N$ such that $x_i^* + \pi_i y^* < \omega_{x,i}$. Then, by the utility maximization of agent i, $x_i^* + y^* = 1$. Therefore, for each $j \in N\backslash\{i\}$, $x_j^* = 0$. By the utility maximization of agent j, $\pi_j y^* = \omega_{x,j}$. Adding up these equalities over all $j \in N\backslash\{i\}$, $y^* \sum_{N\backslash i} \pi_j = \sum_{N\backslash i} \omega_{x,j}$. Adding up the inequality for agent i, $x_i^* + y^* \sum \pi_k < 1$. Since $x_i^* + y^* = 1$ we have $\sum \pi_k < 1$. But then, the profit-maximizing output of the firm is $y^* = 0$, contradicting $y^* > 0$. ■

We use Lemma 17 to prove the following result.

PROPOSITION 18. *Let $z^* \in L^c(u, \omega)$. If $z^* \neq \omega$, there is a unique associated price vector π and it satisfies $\sum \pi_i = 1$. If $z* = \omega$, any associated price vector π satisfies $\sum \pi_i \leq 1$ and there is an associated price vector that satisfies $\sum \pi_i = 1$.*

Proof. Let $z^* \in L^c(u, \omega)$. First assume $z^* \neq \omega$. Then $y^* > 0$. By Lemma 6, for each $i \in N$, $x_i^* + \pi_i y^* = \omega_{x,i}$. Adding up over the agents,

$\sum x_i^* + y^* \sum \pi_i = \sum \omega_{x,i} = 1$. Therefore,

$$y^* \sum \pi_i = 1 - \sum x_i^* = \sum(\omega_{x,i} - x_i^*) = y^*.$$

Since $y^* > 0$, $\sum \pi_i = 1$. However, there is a unique π associated with z^*, and that satisfies $\sum \pi_i = 1$.

Now assume $z^* = \omega$. Then $y^* = 0$. Let π be a price vector associated with z^*. Then $\sum \pi_i \leq 1$. To see this suppose $\sum \pi_i > 1$. Then, by the profit maximization of the firm, $y^* = 1$ and therefore, for each $i \in N$, $x_i^* = 0$. Since for each $i \in N$, z_i^* satisfies agent i's budget constraint, $\pi_i y^* = \pi_i \leq \omega_{x,i}$. Adding up over the agents we have $\sum \pi_i \leq \sum \omega_{x,i} = 1$, a contradiction.

To show the existence of a π' such that $\sum \pi'_k = 1$, let $\sum \pi_k < 1$. For some $i \in N$, let $\pi'_i = 1 - \sum_{N \setminus i} \pi_j$. For each $j \in N \setminus \{i\}$, let $\pi'_j = \pi_j$. Then, $\pi'_i > \pi_i$ implies that z_i^* maximizes u_i subject to $x_i + \pi'_i y \leq \omega_{x,i}$. Moreover, since $\sum \pi'_k = 1$, z^* is a profit-maximizing bundle with respect to π'. Therefore, π' is also associated with z^*. ■

The following lemma establishes the relation between the continuity of the bargaining rule F and the outcome correspondence B. Note that the function space V is equipped with the sup-norm topology.

LEMMA 19. *Let $F: \mathscr{B} \to \mathbb{R}^n$ be a continuous and admissible bargaining rule. Let $X \subseteq \mathbb{R}^{m \times n}$ be compact. Let $B: V^n \to X$ be defined as*

$$B(v) = \{x \in X \mid v(x) = F(v(X), v(\omega))\}.$$

Then, B is upper hemicontinuous.

Proof. First we will show that B is compact-valued. Let $v \in V$. Since $B(v) \subseteq X$, it is bounded. Let $\{x^k\}_{k \in \mathbb{N}}$ be a sequence in $B(v)$ converging to $x \in X$. By the continuity of v, $\{v(x^k)\}_{k \in \mathbb{N}}$ converges to $v(x)$. Since, for each $k \in \mathbb{N}$, $v(x^k) = F(v(X), v(\omega))$ we have $v(x) = F(v(X), v(\omega))$. Therefore, $x \in B(v)$. This establishes the closedness of $B(v)$.

Let $\{v^k\}_{k \in \mathbb{N}}$ be a sequence in V that converges to a $v \in V$. Let $\{x^k\}_{k \in \mathbb{N}}$ be a sequence in X such that, for each $k \in \mathbb{N}$, $x^k \in B(v^k)$. Since X is compact, $\{x^k\}_{k \in \mathbb{N}}$ has a convergent subsequence $\{x^{r(k)}\}_{k \in \mathbb{N}}$ that converges to an $x \in X$. We need to show that $x \in B(v)$. That is, $v(x) = F(v(X), v(\omega))$.

First, we show that $\{v^{r(k)}(x^{r(k)})\}_{k \in \mathbb{N}}$ converges to $v(x)$. To see this, note that it is a subsequence of $\{v^l(x^m)\}_{l, m \in \mathbb{N}}$. If this sequence converges to $v(x)$ so does its subsequence. For each $l \in \mathbb{N}$, by the continuity of v^l, $\{v^l(x^m)\}_{m \in \mathbb{N}}$ converges to $v^l(x)$. Now $\{v^l\}$ converges to v with respect to the sup-norm topology. Moreover, convergence in the sup-norm metric implies pointwise convergence. Therefore, $\{v^l(x)\}_{l \in \mathbb{N}}$ converges to $v(x)$. This establishes the result.

Next, we show that $\{F(v^{\gamma(k)}(X), v^{\gamma(k)}(\omega))\}$ converges to $F(v(X), v(\omega))$. Let d, δ, and D denote the Euclidean, sup-norm, and Hausdorff metrics, respectively. Let $v, v' \in V$. Then, for each $x \in X$,

$$d(v(x), v'(x)) \geq \inf\{d(v(x), v'(y)) \mid y \in X\}.$$

Therefore,

$$\begin{aligned}\delta(v, v') &= \sup\{d(v(x), v'(x)) \mid x \in X\} \\ &\geq \sup\{\inf\{d(v(x), v'(y)) \mid y \in X\} \mid x \in X\}.\end{aligned}$$

Similarly,

$$\delta(v, v') = \delta(v', v) \geq \sup\{\inf\{d(v'(x), v(y)) \mid y \in X\} \mid x \in X\}.$$

Thus, $\delta(v, v') \geq D(v(X), v'(X))$. However, this implies that if the sequence $\{v^{\gamma(k)}\}$ converges to v with respect to sup-norm topology, the sequence $\{v^{\gamma(k)}(X)\}$ converges to $v(X)$ with respect to Hausdorff topology. Moreover, since sup-norm convergence implies pointwise convergence, the sequence $\{v^{\gamma(k)}(\omega)\}$ converges to $v(\omega)$. Therefore, by continuity of F, the sequence $\{F(v^{\gamma(k)}(X), v^{\gamma(k)}(\omega))\}$ converges to $F(v(X), v(\omega))$.

We showed that $\{F(v^{r(k)}(X), v^{\gamma(k)}(\omega))\}_{k\in\mathbb{N}}$ converges to $F(v(X), v(\omega))$ and $\{v^{r(k)}(x^{r(k)})\}_{k\in\mathbb{N}}$ converges to $v(x)$. Moreover, for each $k \in \mathbb{N}$, $v^{r(k)}(x^{r(k)}) = F(v^{r(k)}(X), v^{\gamma(k)}(\omega))$. Therefore, we have $v(x) = F(v(X), v(\omega))$. This establishes upper hemicontinuity of B. ■

Note that continuity of F does not imply lower hemicontinuity of B. The following example establishes this point.

EXAMPLE 20. Let $N = \{1, 2\}$. Note that, for $\pi_i + \pi_j = 1$, $B(l[\pi_i], l[\pi_j])$ is the whole budget line. Let $\{\pi_i^k\}$ be a sequence that converges to π_i and is such that, for each $k \in \mathbb{N}$, $\pi_i^k + \pi_j > 1$. Then, for each $k \in \mathbb{N}$, $B(l[\pi_i^k], l[\pi_j])$ is a singleton. Let $z \in X_p$ be such that $z \gg 0$ and $x_i + \pi_i y = \omega_{x,i}$. For each $k \in \mathbb{N}$, let $\{z^k\} = B(l[\pi_i^k], l[\pi_j])$. Then, $\{z^k\}$ does not converge to z. Therefore B is not lower hemicontinuous at $(l[\pi_i], l[\pi_j])$.

A.2. *Proofs*

In the proofs of Theorem 1 and Lemma 4, we use $l[p^*]$ to denote $(l_1[p^*], \ldots, l_n[p^*])$ and $l[p^*_{-i}]$ to denote $(l_j[p^*])_{j\neq i}$.

Proof of Theorem 1. Let $d = d(l[p^*])$ and $S = S(l[p^*])$. It is straightforward to check that d is Pareto optimal in S. This, by the individual rationality of F, implies $d = F(S, d)$. Therefore, $\omega \in B(l[p^*])$. Since $x^* \in W^c(u, \omega)$ and, for each $i \in N$, u_i is monotonic, $p^* x_i^* = p^* \omega_i$. Therefore, for each $i \in N$, $l_i[p^*](x_i^*) = l_i[p^*](\omega_i)$. This implies $x^* \in B(l[p^*])$.

Let $x \in B(V, l[p^*_{-i}])$. By the individual rationality of F, for each $j \in N\backslash\{i\}$ $l_j[p^*](x_j) \geq l_j[p^*](\omega_j)$ and therefore $p^* x_j \geq p^* \omega_j$. Since $\sum p^* x_k \leq \sum p^* \omega_k$, this implies $p^* x_i \leq p^* \omega_i$.

Since $x^* \in W^c(u, \omega)$, x^*_i maximizes $u_i(x_i)$ subject to $\mathbf{0} \leq x_i \leq \mathbf{1}$ and $p^* x_i \leq p^* \omega_i$. By previous paragraphs, $x^* \in B(l[p^*]) \subseteq B(V, l[p^*_{-i}])$ and $B(V, l[p^*_{-i}])$ is a subset of agent i's constrained budget set. Therefore, x^*_i maximizes $u_i(x_i)$ subject to $x \in B(V, l[p^*_{-i}])$. ■

The proof of Lemma 2 is identical to that of Sobel (1981) and therefore is omitted.

Proof of Lemma 3. Suppose there are $i, j \in N$ such that $p_i \neq p_j$. Since $x^* \in B(l_1[p_1], \ldots, l_n[p_n])$, $x^* \in P(l_1[p_1], \ldots, l_n[p_n])$. Therefore, i or j receives a boundary bundle. Without loss of generality assume that $x^*_{ik} = 0$ for some $k \in \{1, \ldots, m\}$. Then, by *interiority* $u_i(\omega_i) > u_i(x^*_i)$. But agent i can declare u_i and, since F is individually rational, obtain a share x'_i such that $u_i(x'_i) \geq u_i(\omega_i) > u_i(x_i)$. Therefore, $(l_1[p_1], \ldots, l_n[p_n], x^*) \notin \mathcal{NE}(\mathcal{D}_F(u))$. ■

Proof of Lemma 4. Let $(l[p^*], x^*) \in \mathcal{NE}(\mathcal{D}_F(u))$. Let $i \in N$. Then, x^*_i maximizes $u_i(x_i)$ subject to $x \in B(V, l[p^*_{-i}])$. Note that $x^* \in B(l_i[p^*], l[p^*_{-i}]) \subseteq B(V, l[p^*_{-i}])$. Moreover, $x \in B(l_i[p^*], l[p^*_{-i}])$ if and only if $x \in X_e$ and, for each $k \in N$, $p^* x_k = p^* \omega_k$. Therefore, x^*_i maximizes $u_i(x_i)$ subject to $p^* x_i = p^* \omega_i$ and $x_i \in [0, 1]^m$. Since u_i is monotonic, this implies that x^*_i maximizes $u_i(x_i)$ subject to $p^* x_i \leq p^* \omega_i$ and $x_i \in [0, 1]^m$. ■

In the proofs of the remaining results, we use $l[\pi]$ to denote $(l_1[\pi_1], \ldots, l_n[\pi_n])$ and $l[\pi_{-i}]$ to denote $(l_j[\pi_j])_{j \neq i}$.

Proof of Theorem 6. Let $i \in N$. By Lemma 6, $x^*_i + \pi_i y^* = \omega_{x,i}$. Moreover, since $l[\pi](\omega) = d(l[\pi])$ is Pareto optimal in $S(l[\pi])$, $l[\pi](\omega) = F(S(l[\pi]), d(l[\pi]))$. Therefore $\omega \in B(l[\pi])$. For each $i \in N$, $x^*_i + \pi_i y^* = \omega_{x,i}$ and, thus, $l_i[\pi_i](z^*_i) = l_i[\pi_i](\omega_i)$. This implies that $z^* \in B(l[\pi])$.

Let $z \in B(V, l[\pi_{-i}])$. Then, by the individual rationality of F, for each $j \in N\backslash\{i\}$, $l_j[\pi_j](z_j) \geq l_j[\pi_j](\omega_j)$ and, thus, $x_j + \pi_j y \geq \omega_{x,j}$. Adding up these inequalities over all $j \in N\backslash\{i\}$ yields

$$\sum_{N\backslash\{i\}} x_j + \sum_{N\backslash\{i\}} \pi_j y \geq \sum_{N\backslash\{i\}} \omega_{x,j} = 1 - \omega_{x,i}.$$

By Proposition 18, $\sum \pi_k \leq 1$. Therefore, $\pi_i \leq 1 - \sum_{N\backslash\{i\}} \pi_j$ and, since $y + \sum x_k \leq 1$,

$$x_i + \pi_i y \leq x_i + \left(1 - \sum_{N\backslash\{i\}} \pi_j\right) y \leq \omega_{x,i}.$$

By assumption z_i^* maximizes $u_i(z_i)$ subject to $x_i + \pi_i y \leq \omega_{x,i}$ and $x_i + y \leq 1$. Note that $z^* \in B(l[\pi]) \subseteq B(V, l[\pi_{-i}])$. Also, $z \in B(V, l[\pi_{-i}])$ implies $x_i + \pi_i y \leq \omega_{x,i}$ and $x_i + y \leq 1$. Therefore, z^* maximizes $u_i(z_i)$ subject to $z \in B(V, l[\pi_{-i}])$. ■

Proof of Lemma 7. First suppose that $\sum \pi_k < 1$. Then $z^* = \omega$. Let $i \in N$ declare $l_i[\pi_i']$ where $\pi_i' = 1 - \sum_{j \neq 1} \pi_j$. Then, since

$$B(l_i[\pi_i'], l[\pi_{-i}]) = \{z \in X_p \mid \text{ for each } k \in N, l_k[\pi_k](z_k) = l_k[\pi_k](\omega_k)\},$$

there is $z' \in B(l_i[\pi_i'], k[\pi_{-i}])$ such that $z_i' \gg 0$. Since u_i satisfies *interiority*, $u_i(z_i') > u_i(z_i^*) = u_i(\omega_i)$, contradicting $Z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$.

Next, suppose that $\sum \pi_k > 1$. Then $y^* > 0$ and there is $i \in N$ such that $x_i^* = 0$. There are two possible cases.

Case 1: $\sum_{j \neq i} \pi_j < 1$. Let $\pi_i' = 1 - \sum_{j \neq i} \pi_j$. There is $z' \in B(l_i[\pi_i'], l[\pi_{-i}])$ such that $z_i' \gg 0$. Since u_i satisfies *interiority*, $u_i(z_i') > u_i(z_i^*)$, contradicting $z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$.

Case 2: $\sum_{j \neq i} \pi_j \geq 1$. Let $\pi_i' < \omega_{x,i}$. Then, $l_i[\pi_i'](\omega_i) > l_i[\pi_i'](0, 1)$. Therefore, for each $z \in X_p$ such that $x_i = 0$, $l_i[\pi_i'](\omega_i) > l_i[\pi_i'](z_i)$. Let $z' \in B(l_i[\pi_i'], l[\pi_{-i}])$. By Pareto optimality, $y' > 0$ and, by individual rationality, $x_i' > 0$. Therefore, $u_i(z_i') > u_i(z_i^*)$, contradicting $z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$.

Since $\sum \pi_k = 1$, $B(l[\pi]) = \{z \in X_p \mid \text{ for each } i \in N, x_i + \pi_i y = \omega_{x,i}\}$. By *interiority*, $z^* \in B(l[\pi])$ implies that, for each $i \in N$, $z_i^* \gg 0$. Then, for each $i \in N$, $MRS_i(z_i^*) = \pi_i$ and, since $\sum MRS_i(z_i^*) = \sum \pi_i = 1$, $z^* \in P(u)$. ■

Proof of Lemma 8. Since $z^* \in B(l[\pi])$ and $\sum \pi_k = 1$, for each $k \in N$, $x_k^* + \pi_k y^* = \omega_{x,k}$. Let $i \in N$. Suppose z_i^* does not maximize u_i subject to $x_i + \pi_i y \leq \omega_{x,i}$ and $x_i + y \leq 1$. Then, there is $z' \in X_p$ such that $x_i' + \pi_i y' \leq \omega_{x,i}$, $x_i' + y' \leq 1$, and $u_i(z_i') > u_i(z_i^*)$. Since $z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$, $x_i' + \pi_i y' < \omega_{x,i}$. Since u_i is nondecreasing, without loss of generality $x_i' + y' = 1$. Therefore, $y' > y^*$ and, for each $j \in N \backslash \{i\}$, $x_j' = 0$.

Case 1: $x_i^* + y^* < 1$ (Fig. 4). Let z_i'' be such that $x_i'' + \pi_i y'' = \omega_{x,i}$ and $z_i'' > \alpha z_i' + (1 - \alpha) z_i^*$ for some $\alpha \in (0, 1)$. Since $z^* \in \operatorname{int}(X_p)$ or $z^* = \omega$, such a z_i'' exists. By the concavity of u_i, $u_i(z_i'') > u_i(z_i^*)$. But since $x_i'' + \pi_i y'' = \omega_{x,i}$, $l[\pi_i](z_i'') = l[\pi_i](\omega_i)$. Therefore, $z'' \in B(l[\pi])$. This contradicts $z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$.

Case 2: $x_i^* + y^* = 1$. Then, for each $j \in N \backslash \{i\}$, $z_j' \geq z_j^*$, and thus $u_j(z_j') \geq u_j(z_j^*)$. This contradicts $z^* \in P(u)$. ■

Proof of Proposition 12. Since $\pi_1 + \pi_2 < 1$, $B(l_1[\pi_1], l_2[\pi_2]) = \{z^*\} = \{\omega\}$. Since $\omega \in P(u)$, by the Second Welfare Theorem $z^* = \omega \in L^c(u, \omega)$. By Proposition 18, if $\omega \in L^c(u, \omega)$ then there are associated prices π' satisfying $\pi_1' + \pi_2' = 1$. ■

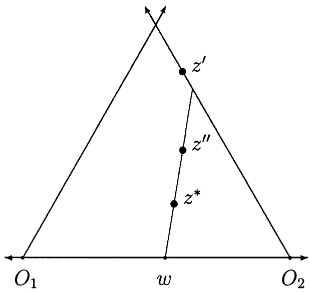

FIG. 4. Construction of z^*, z', and z'' in Case 1 of Lemma 8.

Proof of Proposition 14 (Fig. 5). If $z^* \in \text{int}(X_p)$, $z^* = ((0, 1), (0, 1))$, or $z^* = \omega$, then $z^* \in P(u)$. Therefore, the result follows from Lemma 8. Let z^* be such that $x_i^* > 0$ and $x_j^* = 0$. Suppose $z^* \notin L^c(u, \omega)$. Then, there is $z^{**} \in X_p$ satisfying $x_i^{**} + \pi_i y^{**} < \omega_{x,i}$ and $u_i(z_i^{**}) > u_i(z_i^*)$. Since u_i is nondecreasing, it is no loss of generality to assume that $x_i^{**} + y^{**} = 1$ and therefore $x_j^{**} = 0$.

Note that, for each $\pi_i' > 1 - \pi_j$, $B(l_i[\pi_i'], l_j[\pi_j])$ is a singleton. Moreover, by Lemma 19, B is an upper hemicontinuous correspondence. Therefore, $B(l_i[.], l_j[\pi_j])$ is a continuous function on $(1 - \pi_j, \infty)$. Let b: $[1 - \pi_j, \infty] \to X_p$ be defined as

$$b(\pi_i') = \begin{cases} B(l_1[\pi_i'], l_j[\pi_j]) & \text{if } \pi_i' > 1 - \pi_j, \\ z^* & \text{if } \pi_i' = 1 - \pi_j. \end{cases}$$

Since B is upper hemicontinuous, b is continuous. For $\pi_i' = 1$, $b(\pi_i') = z'$ is such that $x_i' = 0$. Therefore, there is $\pi_i^{**} \in (1 - \pi_j, 1)$ such that $b(\pi_i^{**}) = B(l_i[\pi_i^{**}], l_j[\pi_j]) = \{z^{**}\}$. This contradicts $z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$. ■

Proof of Proposition 15. Since $\pi_1 + \pi_2 > 1$, $\bar{z} \in P(l_1[\pi_1], l_2[\pi_2])$.

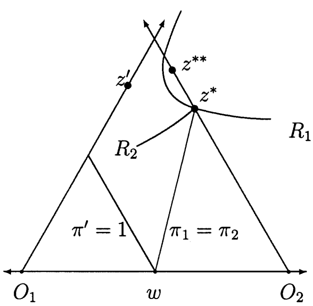

FIG. 5. Construction of z^*, z^{**}, and z' in the proof of Proposition 14.

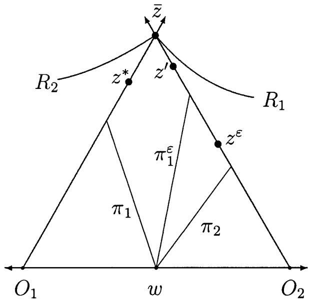

FIG. 6. Construction of Case 1 in the proof of Proposition 15.

Case 1 (Fig. 6). Assume that $\bar{z} \in I(l_1[\pi_1], l_2[\pi_2], \omega)$. Suppose $z^* \neq \bar{z}$. Then, there are $i \in N$ and $z' \in X_p$ such that $z_i' \gg z_i^*$ and $x_i' + \max\{0, (1-\pi_j)\}y' < \omega_{x,i}$.

Since $z_i' \gg z_i^*$, $u_i(z_i') > u_i(z_i^*)$. Note that, for each $\pi_i' > \max\{0, 1, -\pi_j\}$, $B(l_i[\pi_j'], l_j[\pi_j])$ is a singleton. Moreover, B is an upper hemicontinuous correspondence. Therefore, $B(l_i[.], l_j[\pi_j])$ is a continuous function on $(\max\{0, 1-\pi_j\}, \pi_i]$. For each $\varepsilon > 0$, let $\pi_i^\varepsilon = \varepsilon + \max\{0, 1-\pi_j\}$. Let $\{z^\varepsilon\} = B(l_i[\pi_i^\varepsilon], l_j[\pi_j])$. Then, for sufficiently small ε, $x_i' + \pi_i^\varepsilon y' < \omega_{x,i}$. That is, $z' \notin I(l_i[\pi_i^\varepsilon], l_j[\pi_j], \omega)$. Therefore, by the continuity of $B(l_i[.], l_j[\pi_j])$ there is $\pi_i' \in (\pi_i^\varepsilon, \pi_i)$ such that $B(l_i[\pi_i'], l_j[\pi_j]) = \{z'\}$. Since $u_i(z_i') > u_i(z_i^*)$, this contradicts $z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$. Therefore $z^* = \bar{z}$. Let (π_1', π_2') be such that $\pi_1' + \pi_2' = 1$ and, for each $i \in N$, $x_i^* + \pi_i' y^* = \omega_{x,i}$. Now suppose there is $i \in N$ such that z_i^* does not maximize u_i subject to $x_i + \pi_i' y \leq \omega_{x,i}$ and $x_i + y \leq 1$. Then, there is $z' \in X_p$ such that $u_i(z_i') > u_i(z_i^*)$, $x_j' = 0$, and

$$x_i' + \max\{0, (1-\pi_j)\}y' < \omega_{x,i}.$$

The same argument as that used above shows that there is π_i' such that $B(l_i[\pi_i'], l_j[\pi_j]) = \{z'\}$. This contradicts $z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$.

Case 2 (Fig. 7). Assume that $\bar{z} \notin I(l_1[\pi_1], l_2[\pi_2], \omega)$. Let z' be the closest point to $\bar{z}$ in $P(l_1[\pi_1], l_2[\pi_2]) \cap I(l_1[\pi_1], l_2[\pi_2], \omega)$. Then, there is $i \in N$ such that $x_i' = 0$. Note that then $z' \in B(l_i[1-\pi_j], l_j[\pi_j])$.

Suppose $z' \neq z^*$. Note that $x_i' = x_i^* = 0$ and $y' > y^*$. Then, there is $z^{**} \in B(l_i[1-\pi_j], l_j[\pi_j])$ such that $z_i^{**} \gg z_i^*$. Therefore $u_i(z_i^{**}) > u_i(z_i^*)$. Moreover, agent i can declare $l_i[\pi_i^{**}]$ such that $\pi_i^{**} = 1 - \pi_j$ and get z_i^{**}. This contradicts $z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$. Therefore $z^* = z'$. Note that $x_i^* = 0$. Let (π_i', π_j') be such that $\pi_i' = 1 - \pi_j$ and $\pi_j' = \pi_j$. Next we will show that $z^* \in L^c(u, \omega)$ with associated prices π_i', π_j'.

Since the feasibility constraint is not binding for agent i's budget set, for each $z \in X_p$ such that $x_i + \pi_i' y < \omega_i$ there is $z'' \in X_p$ such

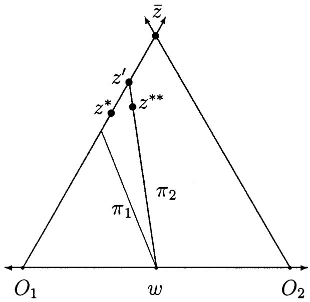

FIG. 7. Construction of Case 2 in the proof of Proposition 15.

that $x_i'' + \pi_i' y'' = \omega_i$ and $z_i'' \gg z_i$. Moreover, $z'' \in B(l_i[\pi_i'], l_j[\pi_j']) \subseteq B(V, l_j[\pi_j'])$. Therefore, if z_i^* maximizes u_i subject to $z \in B(V, l_j[\pi_j'])$, then z_i^* maximizes u_i subject to $x_i + \pi_i' y \le \omega_i$ and $z \in X_p$.

Treatment for agent j is more complicated since his budget set does not satisfy this property. Suppose that z_j^* does not maximize u_j subject to $x_j + \pi_j' y \le \omega_j$ and $z \in X_p$. Let $z'' \in X_p$ be such that z_j'' is a maximizer of u_j subject to $x_j + \pi_j' y \le \omega_j$ and $z \in X_p$. Then $x_j'' + \pi_j' y'' < \omega_j$ and $x_j'' + y'' = 1$. Let $\tilde{\pi}_j = 1$ and $\{\tilde{z}\} = B(l_i[\pi_i], l_j[\tilde{\pi}_j])$. Then $\tilde{x}_j = 0$. Finally, note that $B(l_i[\pi_i], l_j[.])$ is a continuous function on $[\pi_j, \tilde{\pi}_j]$. Therefore, there is $\pi_j'' \in [\pi_j, \tilde{\pi}_j]$ such that $B(l_i[\pi_i], l_j[\pi_j'']) = \{z''\}$. But this contradicts $z^* \in \mathcal{NE}_x(\mathcal{D}_F(u))$. ■

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