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# Egalitarianism in ordinal bargaining: the Shapley–Shubik rule

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## Abstract

A bargaining rule is *ordinally invariant* if its solutions are independent of which utility functions are chosen to represent the agents' preferences. For two agents, only dictatorial bargaining rules satisfy this property (Shapley, L., *La Décision: Agrégation et Dynamique des Ordres de Préférence*, Editions du CNRS (1969) 251). For three agents, we construct a “normalized subclass” of problems through which an infinite variety of such rules can be defined. We then analyze the implications of various properties on these rules. We show that a class of monotone path rules uniquely satisfy *ordinal invariance*, *Pareto optimality*, and “monotonicity” and that the Shapley–Shubik rule is the only *symmetric* member of this class. We also show that the only ordinal rules to satisfy a stronger monotonicity property are the dictatorial ones.

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## 1. Introduction

We analyze the implications of standard bargaining axioms in ordinal environments, that is, in bargaining situations where the agents' preferences are only restricted to be complete, transitive, and continuous (as opposed to being of von Neumann–Morgenstern type as in most of the literature). For ordinal environments, the *scale invariance* axiom of Nash (1950) is not sufficient to ensure the invariance of the physical bargaining outcome with respect to utility-representation changes that leave the underlying preferences intact.

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It needs to be replaced with a stronger axiom called *ordinal invariance*. Unfortunately, all of the well-known bargaining rules violate this axiom.

Shapley (1969) showed that for two-agent problems, only dictatorial bargaining rules satisfy this property. For three agents, however, an ordinally invariant and strongly individually rational bargaining rule appeared in Shubik (1982). We will refer to it as the *Shapley–Shubik rule*.<sup>1</sup> Safra and Samet (2001a, 2001b) recently proposed generalizations of the Shapley–Shubik formula to finitely many agents. The literature following Shapley (1969) mainly analyzed the implications of weakening the ordinal invariance requirement on two-agent bargaining rules. Myerson (1977) and Roth (1979) showed that such weakenings and some basic properties characterize Egalitarian type rules. The joint implications of ordinal invariance and other basic properties, however, have not been studied further.<sup>2</sup> In this paper, we attempt to fill this gap for three-agent bargaining problems.

To isolate the implications of ordinal invariance from other axioms, one can construct a “normalized” class of bargaining problems so that via increasing transformations of utilities,

- (i) any bargaining problem can be transformed into a (normalized) problem in this class and
- (ii) a normalized problem can not be transformed into another normalized problem.

Due to (i), any physical problem can be represented as a normalized bargaining problem. Once solutions to normalized problems are given, ordinal invariance completely determines the solutions to non-normalized problems. Due to (ii), ordinal invariance has no implications on relating solutions to two different normalized problems. Therefore, any property, once restricted to apply only to normalized problems, does not run the risk of comparing alternative utility representations of the same physical problem (which is, in itself, a desirable consequence), and therefore, becomes compatible with ordinal invariance. Using this special class of problems, one can also construct an infinite variety of ordinally invariant bargaining rules by first arbitrarily specifying solutions to normalized problems and then, using ordinal invariance to obtain solutions of arbitrary problems. If this normalized class admits a minimal set of symmetric problems and if the normalization procedure is anonymous, this method easily delivers symmetric (or anonymous) rules that are ordinal.

For Pareto surfaces with more than two agents, Sprumont (2000) proved the existence of a “normalized” subclass of surfaces satisfying requirements (i) and (ii) (see Theorem 7 in Appendix A). For three-agent surfaces he also used the Shapley–Shubik formula to construct a “sufficiently symmetric” class. In Section 3, we generalize his construction

<sup>1</sup> There is no reference on the origin of this rule in Shubik (1982). However, Thomson attributes it to Shapley. Furthermore, Roth (1979, pp. 72–73) mentions a three-agent ordinal bargaining rule proposed by Shapley and Shubik (1974, Rand Corporation, R-904/4) which, considering the scarcity of ordinal rules in the literature, is most probably the same bargaining rule.

<sup>2</sup> There is also a body of literature which demonstrates that in alternative approaches to modeling bargaining problems, ordinality can be recovered (see Rubinstein et al., 1992; O’neill et al., 2001).

to a normalized class of three-agent bargaining problems (hereafter, *ordinally normalized problems*).

In Section 4, we show that by using ordinally normalized problems, one can construct an infinite number ordinal rules. For example, it is possible to apply any one of the various objective functions in the literature (such as Nash, Utilitarian, or Egalitarian) to ordinally normalized problems and construct an ordinal rule. We observe that among these, the ordinal rule constructed by the Egalitarian objective coincides with Shapley–Shubik rule.

In Section 5, we look for ordinal rules that are “monotonic” on ordinally normalized problems. We show that a class of “ordinal monotone path rules” uniquely satisfy *Pareto optimality*, *ordinal invariance*, and *monotonicity*. The Shapley–Shubik rule is the only *symmetric* member of this class. All other members thus violate anonymity. These two results are similar in nature to previous findings on the Egalitarian (Kalai, 1977) and the Kalai–Smorodinsky (1975) rules, as well as monotone path rules (see Kalai, 1977; Thomson, 1980, and Thomson and Myerson, 1980). Finally, we show that the only ordinal rules that satisfy a stronger monotonicity property are the dictatorial ones.

## 2. Model

Let  $N = \{1, 2, 3\}$  be the set of *agents*. Vector inequalities are defined as:  $x \leq y$  iff  $x_i \leq y_i$  for each  $i \in N$ ;  $x \leq y$  iff  $x \leq y$  and  $x \neq y$ ;  $x < y$  iff  $x_i < y_i$  for each  $i \in N$ . For each  $i \in N$ ,  $e(i)$  stands for the vector in  $\mathbb{R}_+^N$  whose  $i$ th coordinate is 1 and all other coordinates are 0. For each  $X \subset \mathbb{R}^N$  and  $x \in \mathbb{R}^N$ ,  $\text{conv}\{X\}$  is the convex hull of  $X$  and  $\text{comp}\{X \mid x\} = \{y \in \mathbb{R}^N : y \geq x \text{ and } y \leq z \text{ for some } z \in X\}$  is the comprehensive hull of  $X$  down to  $x$ .

A pair  $(S, d) \in 2^{\mathbb{R}^N} \times \mathbb{R}^N$  is a *bargaining problem* if  $S$  is compact,  $d \in S$ , and there is  $x \in S$  with  $x > d$ . A bargaining problem  $(S, d)$  is *strictly  $d$ -comprehensive* if for each  $x \in S$  and  $y \in \mathbb{R}^N$  such that  $d \leq y \leq x$ ,  $y \in S$  and there is  $z \in S$  such that  $z > y$ . Let  $\mathcal{B}$  denote the set of all strictly  $d$ -comprehensive bargaining problems.<sup>3</sup> Let the sets  $P(S, d) = \{x \in S \mid y \geq x \Rightarrow y \notin S\}$  and  $I(S, d) = \{x \in S \mid x \geq d\}$  denote the *Pareto optimal* and *individually rational* profiles in  $S$ . Let  $IP(S, d) = I(S, d) \cap P(S, d)$ . Let  $\mathcal{B}_I$  be the class of bargaining problems  $(S, d) \in \mathcal{B}$  where every profile is individually rational, i.e.  $S = I(S, d)$ . The *ideal payoff* of an agent  $i \in N$  in  $(S, d) \in \mathcal{B}$  is  $m_i(S, d) = \max\{x_i \in \mathbb{R} : x \in I(S, d)\}$ . The *ideal point* of  $(S, d)$  is  $m(S, d) = (m_i(S, d))_{i \in N}$  (see Fig. 1). A bargaining problem  $(S, d)$  is *0–1 normalized* if  $d = 0$  and  $m(S, d) = (1, 1, 1)$ . Let  $\mathcal{B}_{0-1}$  denote the set of all such problems in  $\mathcal{B}$ .

For each  $i \in N$ , let  $f_i$  be an increasing continuous function on  $\mathbb{R}$  and define  $f = (f_i)_{i \in N} : \mathbb{R}^N \rightarrow \mathbb{R}^N$  by  $f(x) = (f_i(x_i))_{i \in N}$  for each  $x \in \mathbb{R}^N$ . Let  $\mathcal{F}$  denote the set of such functions. Two problems  $(S, d), (S', d') \in \mathcal{B}$  are *ordinally equivalent* if there is  $f \in \mathcal{F}$  such that  $f(S) = S'$  and  $f(d) = d'$ . Otherwise, they are called *ordinally distinct*. A subclass

<sup>3</sup> This property implies, first, that utility is disposable down to the disagreement point, and second, that any individually rational and weakly Pareto optimal point is also Pareto optimal. Roth (1979) shows that when weakly comprehensive problems are admitted, Shapley’s (1969) impossibility result generalizes to an arbitrary number of agents.

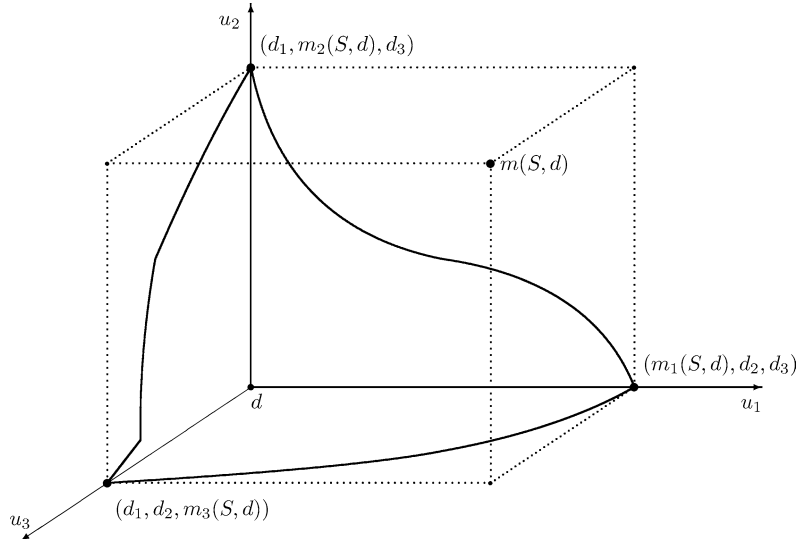


Fig. 1. The ideal point of an arbitrary problem.

$\mathcal{B}' \subseteq \mathcal{B}$  *ordinally spans*  $\mathcal{B}$  if for any problem  $(S, d) \in \mathcal{B}$ , there is a problem  $(S', d') \in \mathcal{B}'$  which is *ordinally equivalent* to it. The subclass  $\mathcal{B}'$  is an *ordinal basis* of  $\mathcal{B}$  if it *ordinally spans*  $\mathcal{B}$  and all problems in  $\mathcal{B}'$  are *ordinally distinct*.

A *bargaining rule*  $F: \mathcal{B} \rightarrow \mathbb{R}_+^N$  assigns each  $(S, d) \in \mathcal{B}$  to a feasible payoff profile  $F(S, d) \in S$ . It is *Pareto optimal* if for each  $(S, d) \in \mathcal{B}$ ,  $F(S, d) \in P(S, d)$ . It is *strongly individually rational* if for each  $(S, d) \in \mathcal{B}$ ,  $F(S, d) > d$ . The next property is based on the principle of anonymity but is considerably weaker. Let  $\Pi$  be the set of all permutations  $\pi$  on  $N$ . For  $S \subset \mathbb{R}^N$ , let  $\pi(S) = \{\pi(s) \mid s \in S\}$ . A bargaining rule  $F$  is *symmetric* if given  $(S, d) \in \mathcal{B}$  that satisfies  $\pi(S) = S$  and  $\pi(d) = d$  for each  $\pi \in \Pi$ , we have  $F_i(S, d) = F_j(S, d)$  for each  $i, j \in N$ . Our final property requires the physical bargaining outcome to be invariant under utility changes as long as the underlying preference information is unchanged:  $F$  is *ordinally invariant* if for each  $(S, d) \in \mathcal{B}$  and  $f \in \mathcal{F}$ ,  $F(f(S), f(d)) = f(F(S, d))$ .

### 3. Brace and ordinally normalized problems

In this section, we construct a class of normalized problems and establish its properties. Let  $(S, d) \in \mathcal{B}$ . Define  $p^{-1}(S, d) = d$  and for each  $n \in \mathbb{N}$  define  $p^n(S, d) \in \mathbb{R}^N$  to be such that

$$\begin{aligned} b^{n,1}(S, d) &= (p_1^{n-1}(S, d), p_2^n(S, d), p_3^n(S, d)) \in P(S, d), \\ b^{n,2}(S, d) &= (p_1^n(S, d), p_2^{n-1}(S, d), p_3^n(S, d)) \in P(S, d), \quad \text{and} \\ b^{n,3}(S, d) &= (p_1^n(S, d), p_2^n(S, d), p_3^{n-1}(S, d)) \in P(S, d). \end{aligned}$$

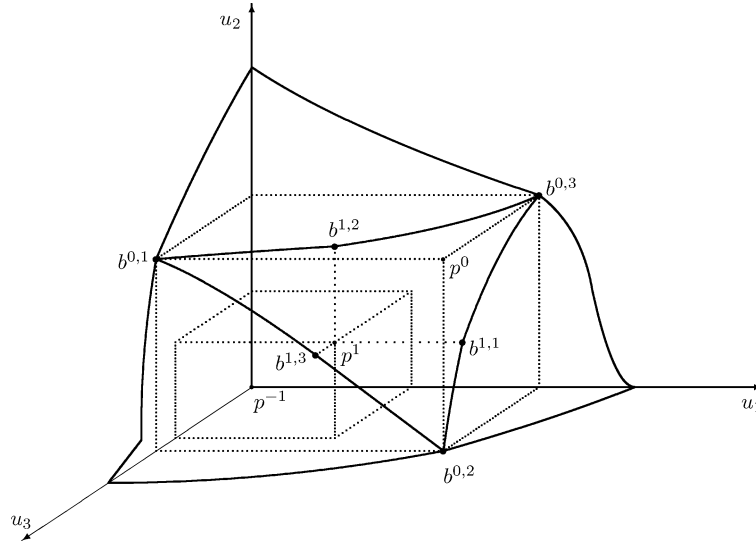


Fig. 2. Constructing the brace of an arbitrary problem.

The sequence  $\{p^n(S, d)\}_{n \in \mathbb{N}}$  is uniquely defined and it is convergent (Shubik, 1982). Also note that for each  $i \in N$ ,  $\lim_{n \rightarrow \infty} p^n(S, d) = \lim_{n \rightarrow \infty} b^{n,i}(S, d)$ . The *brace* of  $(S, d)$  is defined as follows (and it is a subset of  $IP(S, d)$ ) (see Fig. 2):

$$\text{br}(S, d) = \bigcup_{n \in \mathbb{N}} \bigcup_{i \in N} \{b^{n,i}(S, d)\}.$$

Each brace point represents a “coalitional ideal payoff” for a two-agent coalition (Kibris, 2004). We use the *brace* to define a subclass of  $\mathcal{B}$ . Let  $(S, d) \in \mathcal{B}$  and define  $b^{-1,1}(S, d) = (m_1(S, d), d_2, d_3)$ ,  $b^{-1,2}(S, d) = (d_1, m_2(S, d), d_3)$ , and  $b^{-1,3}(S, d) = (d_1, d_2, m_3(S, d))$ . For each  $n \in \mathbb{N}$  and  $i \in N$ , let  $A^{n,i}(S, d)$  be the Pareto optimal curve that connects  $b^{n-1,i}(S, d)$  and  $b^{n,i+1}(S, d)$  (with the convention that for  $i = 3, i + 1 = 1$ ) as follows:

$$A^{n,i}(S, d) = \left\{ \begin{array}{l} x \in P(S, d): \text{ for each } j \in N, \\ \min\{b_j^{n-1,i}(S, d), b_j^{n,i+1}(S, d)\} \leq x_j \\ \leq \max\{b_j^{n-1,i}(S, d), b_j^{n,i+1}(S, d)\} \end{array} \right\}.$$

The *extended brace* of  $(S, d)$  is  $A(S, d) = \bigcup_{n \in \mathbb{N}} \bigcup_{i \in N} A^{n,i}(S, d)$  (Fig. 3).

Let  $d^* = 0$  and  $S^* = \text{comp}\{\text{conv}\{e(1), e(2), e(3)\} \mid d^*\}$ . For this “unit” problem  $(S^*, d^*) \in \mathcal{B}$ , the *brace* and the *extended brace* are trivially defined:  $p^0(S^*, d^*) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ ,  $p^1(S^*, d^*) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ , and for  $n \geq 2$  we have  $p^n(S^*, d^*) = \frac{1}{2}(p^{n-1}(S, d) + p^{n-2}(S, d)) = \frac{1}{3} + \frac{1}{6}(\frac{-1}{2})^n \rightarrow \frac{1}{3}$ . Each brace point of this problem is the midpoint average of two others. The *normalized extended brace* is the *extended brace* of this problem:  $A^* = A(S^*, d^*)$  (Fig. 4).

A bargaining problem  $(S, d) \in \mathcal{B}$  is *ordinally normalized* if  $d = 0$ ,  $m(S, d) = (1, 1, 1)$ , and  $A(S, d) = A^*$  (Fig. 5). Let  $\mathcal{B}_{\text{ord}}$  denote the set of all such problems. Note that  $\mathcal{B}_{\text{ord}} \subset$

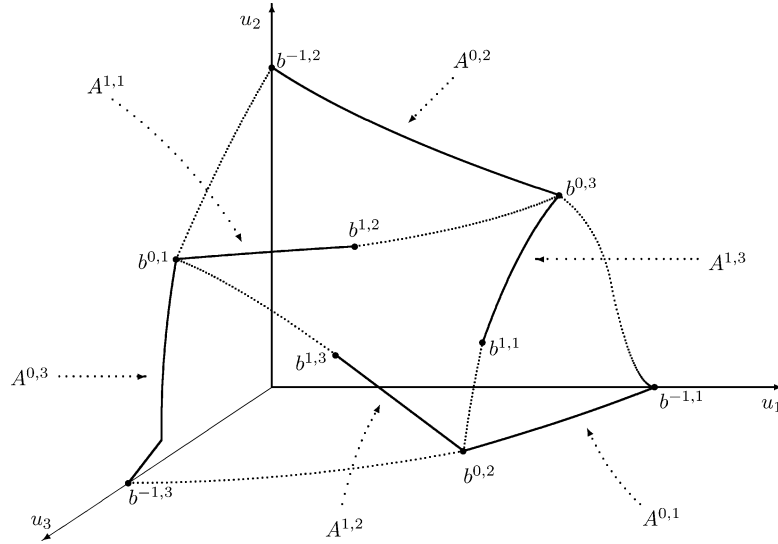


Fig. 3. Constructing the extended brace of an arbitrary problem.

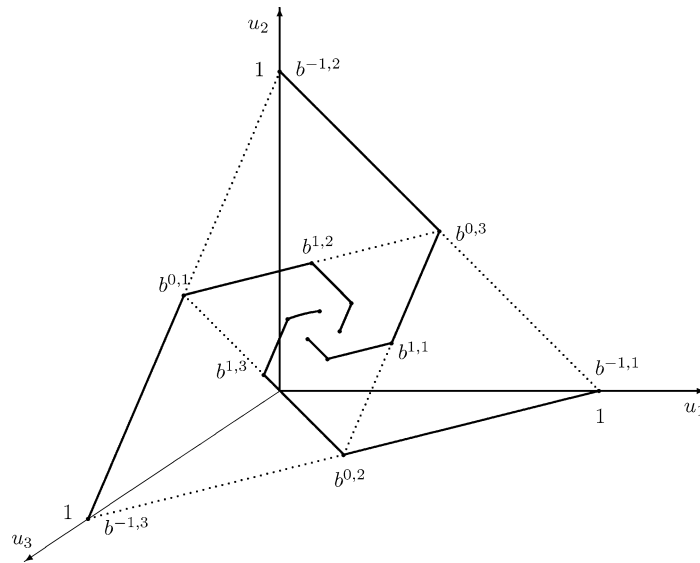


Fig. 4. The normalized extended brace.

$\mathcal{B}_{0-1}$ . Our first result is based on Sprumont’s (2000) theorem (Theorem 7, Appendix A). It states that ordinally normalized problems *ordinally span* the whole class.<sup>4</sup> Also, they form an *ordinal basis* for the class in which every payoff profile is individually rational.

<sup>4</sup> The class  $\mathcal{B}_{\text{ord}}$  contains at least one basis of  $\mathcal{B}$ , even though it is not one.

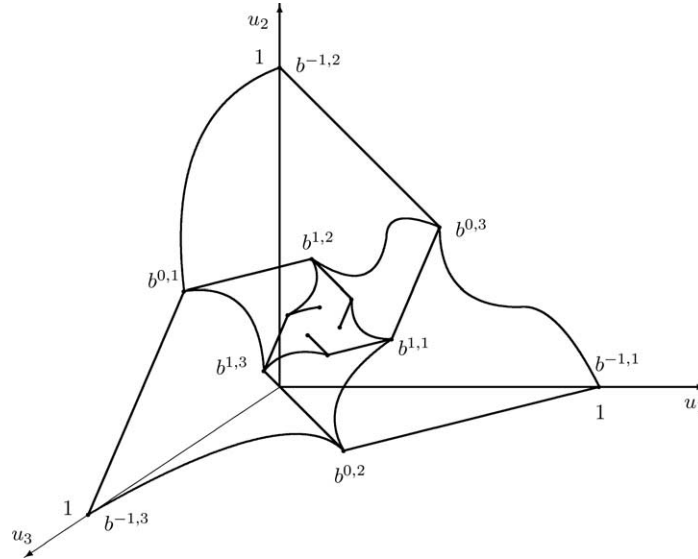


Fig. 5. A typical ordinally normalized problem.

**Proposition 1.** *The subclass  $\mathcal{B}_{\text{ord}}$  ordinally spans  $\mathcal{B}$ . Moreover,  $\mathcal{B}_I \cap \mathcal{B}_{\text{ord}}$  is an ordinal basis of  $\mathcal{B}_I$ .*

#### 4. Constructing ordinal bargaining rules

Given the class  $\mathcal{B}_{\text{ord}}$ , any bargaining rule defined via the following procedure is *ordinally invariant*, *Pareto optimal*, and *individually rational*: for each  $(S, d) \in \mathcal{B}_{\text{ord}}$ , arbitrarily fix an  $x \in IP(S, d)$  and let  $F(S, d) = x$ . For each  $(S, d) \in \mathcal{B}$ , let  $f \in \mathcal{F}$  be such that  $(f(S), f(d)) \in \mathcal{B}_{\text{ord}}$  (by Proposition 1, such an  $f$  exists and is unique on  $I(S, d)$ ) and let  $F(S, d) = f^{-1}(F(f(S), f(d)))$ . This procedure, furthermore, generates bargaining rules that are “independent of non-individually rational alternatives.”<sup>5</sup>

A more desirable selection from this very large class can be obtained as follows. A *monotone path* on  $[0, 1]^N$  is the image  $G$  of a function  $\psi : [0, 1] \rightarrow [0, 1]^N$  such that for each  $i \in N$   $\psi_i$  is continuous, non-decreasing, and satisfies  $\psi_i(0) = 0$ ; moreover, there is  $j \in N$  such that  $\psi_j(1) = 1$ . The *ordinal monotone path rule relative to the monotone path  $G$* ,  $M^G : \mathcal{B} \rightarrow \mathbb{R}^N$  is defined as follows: for each  $(S, d) \in \mathcal{B}_{\text{ord}}$ ,  $M^G(S, d)$  is the maximal point of  $S$  along  $G$ ; for an arbitrary problem  $(S, d) \in \mathcal{B}$ , let  $f \in \mathcal{F}$  be such that  $(f(S), f(d)) \in \mathcal{B}_{\text{ord}}$  and define  $M^G(S, d) = f^{-1}(M^G(f(S), f(d)))$ .

This class contains the dictatorial rules whose monotone paths are  $[0, e(i)]$  for some  $i \in N$ .<sup>6</sup> The Shapley–Shubik rule is also a member and its path is  $[0, \mathbf{1}]$ . Its original

<sup>5</sup> A bargaining rule  $F$  is independent on non-individually rational alternatives if for each  $(S, d) \in \mathcal{B}$ ,  $F(S, d) = F(I(S, d), d)$ .

<sup>6</sup> The dictatorial rule of agent  $i$ ,  $D^i$  is defined as  $D^i(S, d) = (m_i(S, d), d_{-i})$ .

definition in Shubik (1982) is as follows: for each  $(S, d) \in \mathcal{B}$ , the *Shapley–Shubik bargaining rule*,  $Sh$  selects the limit of the *brace* points (equivalently the limit of the sequence  $\{p^n(S, d)\}_{n \in \mathbb{N}}$ ) as the solution:  $Sh(S, d) = \lim_{n \rightarrow \infty} p^n(S, d)$ . Given this relation, it is no surprise that on  $\mathcal{B}_{\text{ord}}$ , the Shapley–Shubik rule coincides with the Egalitarian rule. Since  $\mathcal{B}_{\text{ord}} \subseteq \mathcal{B}_{0-1}$ , on this subclass it coincides with the Kalai–Smorodinsky rule as well.

**Proposition 2.** *On the subclass  $\mathcal{B}_{\text{ord}}$ , the Shapley–Shubik, Kalai–Smorodinsky, and Egalitarian rules coincide.*

Propositions 1 and 2 together imply that the Shapley–Shubik solution to any problem in  $\mathcal{B}$  is generated from the Egalitarian solution to an *ordinally equivalent* problem in  $\mathcal{B}_{\text{ord}}$ .

## 5. Implications of monotonicity

Monotonicity properties are based on the idea of solidarity. They essentially state that an expansion in the feasible set should not make any agent worse-off. The strongest formulation of this idea is *strong monotonicity* which holds if for each  $(S, d), (S', d) \in \mathcal{B}$ ,  $S \subseteq S'$  implies  $F(S, d) \leq F(S', d)$ . Among *Pareto optimal* and *symmetric* rules, the only one to satisfy *strong monotonicity* is the Egalitarian rule (Kalai, 1977). More specifically, no *scale invariant* (nondictatorial) rule satisfies the property. Such rules, however, satisfy a weaker version which restricts the comparison to “cardinally normalized” problem pairs:  $F$  is *cardinally monotonic* if for each  $(S, d), (S', d) \in \mathcal{B}_{0-1}$ ,  $S \subseteq S'$  implies  $F(S, d) \leq F(S', d)$ .<sup>7</sup> It is straightforward to check that *cardinal monotonicity* only compares “cardinally distinct” problems (that is, problems that can not be two alternative representations of the same physical problem). Among *scale invariant*, *Pareto optimal* and *symmetric* rules, the only *cardinally monotonic* one is the Kalai–Smorodinsky rule (Kalai and Smorodinsky, 1975; Roth, 1979).

Ordinally invariant (nondictatorial) rules violate both monotonicity properties. The next example demonstrates this on the Shapley–Shubik rule.

**Example 1.** Let  $d = 0$ . Let  $S = \text{comp}\{\text{conv}\{e(1), e(2), e(3)\} \mid d\}$ .  $S'$  is as shown in Fig. 6: let  $x^1 = (\frac{1}{4}, \frac{3}{4}, 0)$ ,  $x^2 = (\frac{1}{4}, 0, \frac{3}{4})$ , and  $x^3 = (0, \frac{3}{4}, \frac{3}{4})$ . Let  $Q^1 = \text{conv}\{e(1), x^1, x^2\}$ ,  $Q^2 = \text{conv}\{x^1, x^2, x^3\}$ ,  $Q^3 = \text{conv}\{x^1, e(2), x^3\}$ , and  $Q^4 = \text{conv}\{x^2, x^3, e(3)\}$ . Let  $S' = \text{comp}\{Q^1 \cup Q^2 \cup Q^3 \cup Q^4 \mid d^*\}$ . Now  $S \subseteq S'$  and  $m(S, d) = m(S', d)$ . But  $Sh(S, d) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  and  $Sh(S', d) = (\frac{1}{6}, \frac{1}{2}, \frac{1}{2})$ . Therefore,  $Sh(S, d) \not\leq Sh(S', d)$ .

We next introduce a monotonicity property that, like cardinal monotonicity, restricts the comparison to normalized problem pairs: a bargaining rule  $F$  is *ordinally monotonic* if for each  $(S, d), (S', d) \in \mathcal{B}_{\text{ord}}$ ,  $S \subseteq S'$  implies  $F(S, d) \leq F(S', d)$ . Restriction to  $\mathcal{B}_{\text{ord}}$

<sup>7</sup> In the literature, an alternative formulation which is equivalent for cardinal rules is better known:  $F$  is *restricted monotonic* if for each  $(S, d), (S', d') \in \mathcal{B}$ ,  $S \subseteq S'$ ,  $d = d'$ , and  $m(S, d) = m(S', d')$  imply  $F(S, d) \leq F(S', d')$ .



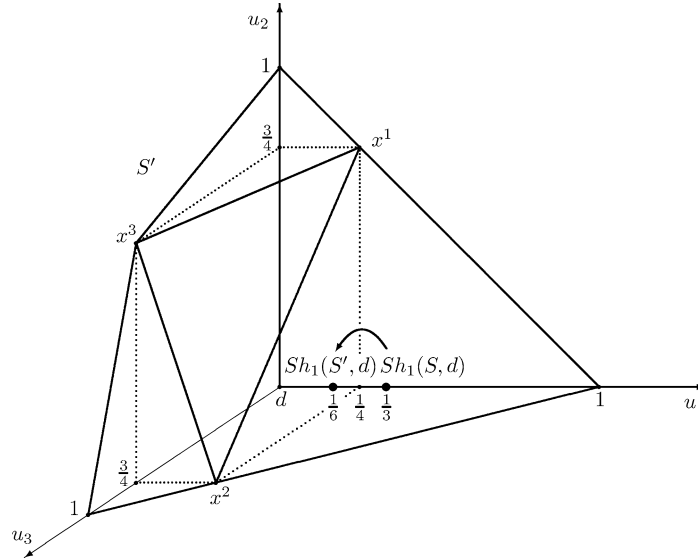


Fig. 6. Agent 1 is worse-off as the feasible set extends.

effectively rules out the possibility of comparing the solutions to two alternative representations of the same physical problem.<sup>8</sup> Ordinal monotone path rules, the Egalitarian and the Kalai–Smorodinsky rules all satisfy this property. Furthermore, among *Pareto optimal* and *ordinally invariant* rules, the ordinal monotone path rules are the only ones to satisfy *ordinal monotonicity*.

**Theorem 3.** *A bargaining rule is Pareto optimal, ordinally invariant, and ordinally monotonic if and only if it is an ordinal monotone path rule.*

Shapley–Shubik rule is the only *symmetric* member of this class.

**Theorem 4.** *The Shapley–Shubik rule is the only bargaining rule that is Pareto optimal, symmetric, ordinally invariant, and ordinally monotonic.*

Ordinal monotonicity essentially concerns the *extended brace* of a problem. Alternatively, one can require that all agents weakly gain from an expansion of the feasible set that leaves the *brace* Pareto optimal (and therefore, unchanged). Some ordinal monotone path rules including the Shapley–Shubik rule do satisfy this stronger monotonicity property. If, however, the property is strengthened further by allowing expansions at which

<sup>8</sup> More precisely, unless  $IP(S, d) \neq IP(S', d')$ , two problems in  $\mathcal{B}_{\text{ord}}$  can be ordinally equivalent. Thus, it is the following weakening of ordinal monotonicity that only compares ordinally distinct problems:  $F$  is *weakly ordinally monotonic* if for each  $(S, d), (S', d') \in \mathcal{B}_{\text{ord}}$ ,  $IP(S, d) \neq IP(S', d')$  and  $S \subseteq S'$  imply  $F(S, d) \leq F(S', d')$ . However, all results that we state for ordinal monotonicity also hold for weak ordinal monotonicity. We use the stronger version simply for presentation purposes.

some brace points cease to remain Pareto optimal, a negative result follows. Call  $F$  *partial brace-monotonic* if for all  $(S, d), (S', d) \in \mathcal{B}$ ,  $S \subseteq S'$  and  $\text{br}(S, d) \setminus \{b^{n,i}(S, d)\} \subset P(S', d)$  for some  $i \in N$  and  $n \in \mathbb{N}$  imply  $F(S, d) \leq F(S', d)$ . This property, together with *ordinal invariance*, puts severe restrictions on the solution to the “unit” problem.

**Lemma 5.** *Let  $d^* = 0$  and  $S^* = \text{comp}\{\text{conv}\{e(1), e(2), e(3)\} \mid d^*\}$ . If  $F$  is an ordinally invariant and partial brace-monotonic bargaining rule,  $F(S^*, d^*) \in \{d^*, e(1), e(2), e(3)\}$ .*

The following result follows from this lemma.

**Theorem 6.** *A bargaining rule satisfies Pareto optimality, ordinal invariance, and partial brace-monotonicity if and only if it is a dictatorial rule.*

A direct corollary of this theorem is that no ordinally invariant and strongly individually rational bargaining rule is partial brace-monotonic.

## 6. Conclusion

We demonstrate that Sprumont’s (2000) result can be used to obtain a class of problems that *ordinally spans* all three-agent strictly comprehensive problems. However, this class is not an *ordinal basis*, even though it is a first step in obtaining one. At this point, the question of how to refine it to obtain a *basis* remains open. Also, if one restricts the analysis to problems in which every payoff profile is individually rational, ordinally normalized problems become a basis. This is not a severe restriction since many desirable bargaining rules only take into account individually rational payoffs.

Since a sufficiently symmetric class of normalized problems does not yet exist for more than three agents, our results are restricted to this case and thus makes use of the Shapley–Shubik idea. A generalization of the Shapley–Shubik formula to an arbitrary number of agents is recently proposed (Safra and Samet, 2001b). It is an open question whether it can be used to construct a normalized class of problems for more agents.

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## Appendix A

Sprumont (2000) introduced a class of Pareto surfaces that form an ordinal basis for a class of “0–1 normalized” Pareto surfaces. His main result can be rephrased as follows.

**Theorem 7** (Sprumont, 2000). *Let  $R = IP(S, d)$  for some  $(S, d) \in \mathcal{B}$ . There exists a unique  $R^*$  satisfying  $R^* = IP(S^*, d^*)$  for some  $(S^*, d^*) \in \mathcal{B}_{\text{ord}}$  and to which  $R$  is ordinally equivalent. Moreover, for each  $i \in N$ , there is a unique continuous and increasing function  $h_i : [d_i, m_i(S, d)] \rightarrow [0, 1]$  such that for  $h = (h_1, h_2, h_3)$ ,  $h(R) = R^*$ .*

The following lemma states that the  $IP$  set is ordinally defined.

**Lemma 8.** *For each  $(S, d) \in \mathcal{B}$  and  $f \in \mathcal{F}$ ,  $IP(f(S), f(d)) = f(IP(S, d))$ .*

**Proof.** Let  $w \in f(IP(S, d))$  and let  $x = f^{-1}(w)$ . Since  $x \in IP(S, d)$ ,  $x \in S$ ,  $x \geq d$ , and for each  $y \geq x$ ,  $y \notin S$ . Note that  $x \in S$  implies  $w \in f(S)$  and  $x \geq d$  implies  $w \geq f(d)$ . Let  $z \geq w$ . Then  $f^{-1}(z) \geq x$  and therefore,  $f^{-1}(z) \notin S$ . By definition of  $f(S)$ , this implies that  $f(f^{-1}(z)) = z \notin f(S)$ . Therefore,  $w \in IP(f(S), f(d))$ . Since  $f$  is invertible, similarly if  $v \in IP(f(S), f(d))$ , then  $v \in f(IP(S, d))$ .  $\square$

We conclude this section with the proofs of the results in the text.

**Proof of Proposition 1.** Let  $(S, d) \in \mathcal{B}$ . We will first construct an  $f \in \mathcal{F}$  such that the transformed problem,  $(f(S), f(d)) \in \mathcal{B}_{\text{ord}}$ . It follows from Theorem 7 that for each  $i \in N$  there is a unique continuous and increasing  $h_i : [d_i, m_i(S, d)] \rightarrow [0, 1]$  such that  $h_i(d_i) = 0$ ,  $h_i(m_i(S, d)) = 1$  and for  $h = (h_1, h_2, h_3)$ ,  $h(IP(S, d))$  is the  $IP$  set of at least one ordinally normalized problem. Therefore,  $A^* \subset h(IP(S, d))$ . Let  $f \in \mathcal{F}$  be defined as follows: for each  $i \in N$

$$f_i(x_i) = \begin{cases} x_i - d_i & \text{if } x_i < d_i, \\ h_i(x_i) & \text{if } d_i \leq x_i \leq m_i(S, d), \\ x_i - m_i(S, d) + 1 & \text{if } m_i(S, d) < x_i. \end{cases}$$

Let  $f(S) = S'$  and  $f(d) = d'$ . Note that  $d' = (0, 0, 0)$  and  $m(S', d') = (1, 1, 1)$ . Finally, by Lemma 12,  $IP(S', d') = f(IP(S, d)) = h(IP(S, d))$ . Therefore,  $A^* \subset IP(S', d')$ . By the uniqueness of the brace,  $A(S', d') = A^*$ . Therefore,  $(S', d') \in \mathcal{B}_{\text{ord}}$ . Thus  $\mathcal{B}_{\text{ord}}$  ordinally spans  $\mathcal{B}$ . For the second claim, a similar argument proves that  $\mathcal{B}_{\text{ord}} \cap \mathcal{B}_I$  ordinally spans  $\mathcal{B}_I$ . Its being a basis follows from the uniqueness of  $h$  (on  $[d, m(S, d)]$ ) and that for each  $(S, d) \in \mathcal{B}_I$ ,  $x \in S$  implies  $x \in [d, m(S, d)]$ .  $\square$

**Proof of Proposition 2.** Let  $E$  and  $K$  denote the Egalitarian and the Kalai–Smorodinsky rules, respectively. Let  $(S, d) \in \mathcal{B}_{\text{ord}}$ . Note that then  $d = 0$  and  $m(S, d) = (1, 1, 1)$ . Therefore,  $E(S, d) = K(S, d)$ . Also  $\lim_{n \rightarrow \infty} p^n(S, d) = Sh(S, d)$ . Thus for each  $i, j \in N$ ,  $Sh_i(S, d) - d_i = Sh_j(S, d) - d_j$ . This and  $Sh(S, d) \in P(S, d)$  implies that  $Sh(S, d) = E(S, d)$ .  $\square$

**Proof of Theorem 3.** The if part is straightforward. Conversely, let  $F$  satisfy these properties. By *ordinal invariance*, it is sufficient to show that  $F$  coincides with an ordinally normalized monotone path rule,  $M^G$  on  $\mathcal{B}_{\text{ord}}$ .

**Step 1** (Forming the path  $G$ ). Note that  $d = (0, 0, 0)$  is fixed. Let  $S^0 = \text{comp}\{A^* \mid d\}$ . Note that  $S^0$  uniquely satisfies the property that for each  $(S, d) \in \mathcal{B}_{\text{ord}}$ ,  $S^0 \subset S$ . Also,  $(S^0, d) \notin \mathcal{B}$  since  $S^0$  is not strictly  $d$ -comprehensive.

Similarly, let  $S^1 \subset \mathbb{R}_+^N$  be such that  $A^*$  is weakly Pareto optimal in  $S^1$  and for each  $(S, d) \in \mathcal{B}_{\text{ord}}$ ,  $S \subset S^1$ . Note that  $S^1$  is uniquely defined and can be described by the following equation: for each  $k \in \{-1\} \cup \mathbb{N}$  and  $i \in N$ , let  $B^{k,i} = \text{comp}\{(b_i^{k,i}, b_{-i}^{k+1,i-1}), b^{k+1,i-2} \mid d\}$  and  $C^{k,i} = \text{comp}\{(b_{-i}^{k+1,i}, b_i^{k+3,i}), (b_{-i}^{k,i-1}, b_i^{k+3,i}) \mid d\}$  with the convention that if, say,  $i = 1, i - 1 = 3$ . Then,

$$S^1 = \bigcup_{k=-1}^{\infty} \bigcup_{i=1}^3 (B^{k,i} \cup C^{k,i}).$$

Since  $A^*$  is not Pareto optimal (even though weakly Pareto optimal) in  $S^1$ ,  $(S^1, d) \notin \mathcal{B}$  as well. Now, for each  $\alpha \in (0, 1)$ , fix an  $S^\alpha \subset \mathbb{R}_+^N$  such that

- (i)  $(S^\alpha, d) \in \mathcal{B}_{\text{ord}}$ ,
- (ii) if  $\alpha < \alpha'$ ,  $S^\alpha \subset S^{\alpha'}$  and  $IP(S^\alpha, d) \cap IP(S^{\alpha'}, d) = A^*$ ,
- (iii)  $\lim_{\alpha \rightarrow 0} S^\alpha = S^0$ , and
- (iv)  $\lim_{\alpha \rightarrow 1} S^\alpha = S^1$ .

Note that for each  $\alpha \in (0, 1)$ ,  $S^0 \subset S^\alpha \subset S^1$ . Let  $x^\alpha = F(S^\alpha, d)$ ,  $x^0 = \lim_{\alpha \rightarrow 0} x^\alpha$ ,  $x^1 = \lim_{\alpha \rightarrow 1} x^\alpha$ , and

$$G = \{x^\alpha : \alpha \in (0, 1)\} \cup [(0, 0, 0), x^0] \cup [x^1, (1, 1, 1)].$$

By *Pareto optimality* of  $F$ , for each  $\alpha \in (0, 1)$   $x^\alpha \in IP(S^\alpha, d)$ . By *ordinal monotonicity* of  $F$ , the limit points  $x^0$  and  $x^1$  are well-defined and  $\alpha < \alpha'$  implies  $x^\alpha \leq x^{\alpha'}$ . Therefore,  $G$  is a monotone and continuous path.

**Step 2** (For each  $(S, d) \in \mathcal{B}_{\text{ord}}$ ,  $F(S, d) = M^G(S, d)$ ). Let  $x = M^G(S, d)$ . Since  $x \in G$ , there is  $\alpha \in (0, 1)$  such that  $x = F(S^\alpha, d) = M^G(S^\alpha, d) \in IP(S^\alpha, d)$ . Let  $T = S \cap S^\alpha$ . Note that  $(T, d) \in \mathcal{B}_{\text{ord}}$ .

Now  $T \subset S^\alpha$  and  $T \subset S$ . By *ordinal monotonicity* applied to the pair  $\{(S^\alpha, d), (T, d)\}$ ,  $x \geq F(T, d)$ . Since  $x \in IP(T, d)$ ,  $x = F(T, d)$ . Also, by *ordinal monotonicity* applied to the pair  $\{(T, d), (S, d)\}$ ,  $x \leq F(S, d)$ . Since  $x \in IP(S, d)$ ,  $x = F(S, d)$ .  $\square$

**Proof of Theorem 4.** It is straightforward to show that the Shapley–Shubik rule,  $Sh$  satisfies these properties. Conversely, let  $F$  satisfy the four properties. Note that by Theorem 4,  $F$  is an ordinally normalized monotone path rule. Let  $(S, d) \in \mathcal{B}$ . We will show that  $F(S, d) = Sh(S, d)$ . By *ordinal invariance* of the two rules, we can assume that  $(S, d) \in \mathcal{B}_{\text{ord}}$ . Let  $x = Sh(S, d) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ . Note that  $x \in P(S, d)$ .

Let  $S' = \text{comp}\{\text{conv}\{e(1), e(2), e(3)\} \mid d\}$ . Note that  $(S', d) \in \mathcal{B}_{\text{ord}}$ . Since  $(S', d)$  is a symmetric problem, by *Pareto optimality* and *symmetry*,  $F(S', d) = x$ . Then,  $F$  being an

ordinally normalized monotone path rule,  $F(S', d) = x$ , and  $x \in P(S, d)$  together imply that  $F(S, d) = x$ . □

**Proof of Lemma 5.** Let  $d = 0$  and  $S = d - \text{comp} \circ \text{conv}\{e(1), e(2), e(3)\}$ . Note that  $IP(S, d) = P(S, d)$  is the two-dimensional unit simplex and  $(S, d) \in \mathcal{B}_{\text{ord}}$ . Let  $F(S, d) = x$ . Suppose  $x \notin \{d, e(1), e(2), e(3)\}$ . Then, there is  $i \in N$  such that  $x_i \notin \{0, 1\}$ . Without loss of generality, let  $i = 1$ . That is, let  $x_1 \in (0, 1)$ . Now let  $S'$  be defined as in Example 1 (see Fig. 6). Define  $f \in \mathcal{F}_{[0,1]}$  as

$$f_1(t) = \begin{cases} \frac{1}{2}t & \text{if } t \in [0, \frac{1}{2}] \\ -\frac{1}{2} + \frac{3}{2}t & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad f_2(t) = f_3(t) = \begin{cases} \frac{3}{2}t & \text{if } t \in [0, \frac{1}{2}] \\ \frac{1}{2} + \frac{1}{2}t & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$$

Note that  $f(S) = S'$  and thus  $(S, d)$  and  $(S', d)$  are ordinally equivalent. Next, we will show that the pair  $\{(S, d), (S', d)\}$  satisfies the conditions of *partial brace-monotonicity*: for each  $k \in \mathbb{R}$  define  $\mathbb{R}_k^N = \{x \in \mathbb{R}^N : x_1 \geq k\}$ . Note that  $P(S, d) \cap \mathbb{R}_{1/4}^N = P(S', d) \cap \mathbb{R}_{1/4}^N$ . Also note that, for each  $n \in \mathbb{N} \setminus \{0\}$  and  $i \in N$ ,  $b^{n,i}(S, d) \in P(S, d) \cap \mathbb{R}_{1/4}^N$ . Finally,  $b^{0,2}(S, d), b^{0,3}(S, d) \in P(S', d)$ . Therefore,  $\text{br}(S, d) \setminus \{b^{0,1}(S, d)\} \subset P(S', d)$ . Also,  $S \subset S'$ . Therefore, by *partial brace-monotonicity* applied to the pair  $\{(S, d), (S', d)\}$ ,  $x \leq f(x)$ . By *ordinal invariance*,  $F(S', d) = f(x)$ . However,  $x_1 \in (0, 1)$  implies  $x_1 > f_1(x_1)$ , a contradiction. □

**Proof of Theorem 6.** It is straightforward to show that the dictatorial rules satisfy these properties. Conversely, let  $F$  be a bargaining rule that satisfies the given properties. Note that *partial brace-monotonicity* is a stronger property than *weak ordinal monotonicity*. Therefore, by Theorem 4,  $F$  is an ordinally normalized monotone path rule. Let  $d = 0$  and  $S = d - \text{comp} \circ \text{conv}\{e(1), e(2), e(3)\}$ . Note that  $(S, d) \in \mathcal{B}_{\text{ord}}$ . Let  $F(S, d) = x$ . By Lemma 5,  $x \in \{d, e(1), e(2), e(3)\}$ . By Pareto optimality,  $x \neq d$ . Therefore,  $x = D^i(S, d)$  for some  $i \in N$ . Let  $(T, d) \in \mathcal{B}_{\text{ord}}$ . Then,  $x \in IP(T, d)$  and, since  $F$  is an ordinally normalized monotone path rule,  $F(T, d) = x = D^i(T, d)$ . Since both  $F$  and  $D^i$  are *ordinally invariant*, this implies  $F = D^i$ . □

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