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Uniform trade rules for uncleared markets

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Abstract We analyze markets in which the price of a traded commodity is such that the supply and the demand are unequal. Under standard assumptions, the agents then have single peaked preferences on their consumption or production choices. For such markets, we propose a class of *Uniform trade rules* each of which determines the volume of trade as the median of total demand, total supply, and an exogenous constant. Then these rules allocate this volume "uniformly" on either side of the market. We evaluate these "trade rules" on the basis of some standard axioms in the literature. We show that they uniquely satisfy *Pareto optimality, strategy proofness, no-envy*, and an informational simplicity axiom that we introduce. We also analyze the implications of *anonymity, renegotiation proofness*, and *voluntary trade* on this domain.

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1 Introduction

We analyze markets in which the price of a traded commodity is fixed at a level where the supply and the demand are unequal. This phenomenon is observed in many markets, either because the price adjustment process is slow, such as in the labor market, or because the prices are controlled from outside the market (e.g., by the state), such as in health, education, or agricultural markets. It is conceptualized in the idea of *market disequilibrium,* which has been particularly central in Keynesian economics after Clower (1965) and Leijonhufvud (1968) and starting in the early 1970's, which has led to the birth of a literature that enriches the rigorous market-clearing models of the Walrasian theory to encompass nonclearing markets and imperfect competition. For a review of this literature, see Bénassy (1993). For textbook presentations of such models, see Bénassy (1982, 2002).

A central component of these enriched models is an institution (hereafter, a *trade rule*) that specifies how transactions are made in a nonclearing market. In this paper, we axiomatically evaluate trade rules on the basis of some standard properties.[1]

In our model, a set of producers face demand from a set of consumers (who might be individuals as well as other producers that use the traded commodity as input). We assume that the individuals have strictly convex preferences on consumption bundles. They thus have single-peaked preferences on the boundary of their budget sets, and therefore, on their consumption of the commodity in question. Similarly, we assume that the producers have strictly convex production sets. Their profits are thus single-peaked in their output or input. Due to these observations, our paper is also related to earlier studies on single-peaked preferences.[2]

A trade rule, in our model, takes in the preferences of the buyers and the sellers and in turn, delivers (i) the volume of trade (i.e., the total trade that will be carried out between the buyers and the sellers) and (ii) how the volume of trade will be allocated among the agents on either side of the market. We introduce a class of *Uniform trade rules* each of which, in step (i), determines the volume of trade as the median of total demand, total supply, and an exogenous constant and in step (ii), allocates this volume "uniformly" among agents on either side of the market.

There are earlier papers related to either one of the above steps but not both. The second (allocation) step is related to the literature starting with Sprumont (1991) who analyzes the problem of allocating a fixed social endowment of a private commodity among agents with single-peaked preferences. The social endowment in those problems corresponds in our model to the volume of trade which, in the second step is treated as fixed, and is allocated as total supply among the buyers and total demand among the sellers. On Sprumont's domain, an allocation rule called the Uniform rule turns out to be central. It can be described as follows: if the sum of the agents' peaks is

[1] Though Bénassy (2002) discusses some properties a good trade rule should satisfy (such as *Pareto optimality*, *voluntary trade*, and *strategy proofness*), he refrains from an axiomatic analysis. Instead, he fixes a trade rule that uniformly rations the long side of the market and uses it throughout the rest of his analysis. For a characterization of this rule, please see Remark 3.

[2] For a firm s, the preference relation R_s is an ordinal representation of how it compares two production or input-consumption levels in terms of profits.

more (respectively, less) than the social endowment, each agent receives the minimum (respectively, the maximum) of his peak and a constant amount. The value of this constant is uniquely determined by the feasibility of the allocation. Sprumont (1991) shows that this rule uniquely satisfies (i) *Pareto optimality, strategy proofness*, and *anonymity* as well as (ii) *Pareto optimality, strategy proofness*, and *no-envy*. The Uniform rule satisfies many other desirable properties (e.g., see Ching 1992, 1994; Thomson 1994a,b). Thus it is no surprise that in our model, the aforementioned *Uniform trade rules* employ the Uniform rule to allocate the trade volume among agents on either side of the market.[3]

The first (trade-volume determination) step is intuitively (though not formally) related to Moulin (1980) who analyzes the determination of a one-dimensional policy issue among agents with single-peaked preferences.[4] This relation is particularly apparent (and formal) when there is a single buyer and a single seller. Then the volume of trade is exactly like a public good for these two agents. While this is no more true when there are multiple buyers or seller (who are sharing the trade volume among themselves), the mechanics of determining the trade volume as a function of the total demand and total supply still resemble Moulin (1980) model. This similarity becomes apparent in our results: parallel to the extended median rules proposed there, strategic considerations lead us to propose the determination of the volume of trade as the median of total demand, total supply and an exogenous constant.

Let us however note that our model is richer than a simple conjunction of the two models mentioned above. This is particularly due to the interaction between the determination of the agents' shares and the determination of the trade volume. For example, the agents can manipulate their allotments also by manipulating (possibly as a group) the volume of trade. Also, single-economy requirements like *Pareto optimality* or "fairness" become much more demanding as what is to be allocated becomes endogenous. Another important difference is the existence of two types of agents (buyers and sellers) in our model. This duality limits the implications of requirements like *anonymity* or *no-envy* and, for example in comparison to Moulin (1980), allows for a much larger class of median rules some of which discriminates between the buyers and the sellers.

Our model is also related to those of Barberà and Jackson (1995), Thomson (1995), and Klaus et al. (1997, 1998). Barberà and Jackson (1995) analyze a pure exchange economy with an arbitrary number of agents and commodities; they introduce and characterize a class of "fixed-proportion trading rules". Thomson (1995) and Klaus et al. (1997, 1998) alternatively analyze a single-commodity model where they consider the reallocation of an infinitely divisible good among agents with single-peaked preferences and individual endowments. In their models, the agents whose endowments are greater than their peaks (the suppliers) supply to those whose endowments are less then their peaks (the demanders). They show that a set of basic properties characterize a "Uniform reallocation rule".

[3] The Uniform rule comes up in other extensions of the Sprumont model as well (e.g. see Kıbrıs 2003; for an analysis of the allocation of pollution permits).

[4] Consider, for example, the determination of a tax rate, the budget of a project, or the provision of a public good.

The relation between these models and ours is quite similar to the one between pure exchange and production economies. In the pure exchange models, whether an agent is a supplier or a demander of the commodity in question depends on the relation between his preferences and his endowment. For example, by changing his preferences, a supplier can turn into a demander of the commodity in question and vice versa. In our production model, however, producers and consumers are exogenously distinct entities. This difference has significant implications on the analysis to be carried out. For example, fairness properties such as anonymity or no-envy compare all agents in the pure exchange version of the model whereas, in the production version, they can only compare agents on the same side of the market. Also, in our model, there are no exogenously set individual endowments. Only after the shares are determined, the production decisions are made.[5] These differences reflect to the results obtained in the two models as well. In the pure exchange model, basic properties imply that the short side of the market always clears whereas this is not the case in our model.[6] We thus interpret the exchange and production models (and their findings) as complements of each other in the aforementioned sense.

We look for trade rules that satisfy a set of standard properties such as *Pareto optimality*, *(coalitional) strategy proofness*, and *no-envy*. We also introduce a new property specific to this domain: *independence of trade volume* (*from in-group transfers)* requires the volume of trade only to depend on the total demand and supply but not on their individual components. For example, increasing agent i's demand and decreasing agent j's demand so as to keep total demand unchanged should have no effect on the volume of trade. Note that this change can still effect the shares of these two agents as well as others.

We observe that the above properties are logically independent and in Theorem 1, we show that they are uniquely satisfied by a class of *Uniform trade rules*. As noted above, these rules do not necessarily clear the short side of the market. Such practice might seem unrealistic at first glance. However, real life examples to it are in fact more common than one would initially expect, especially in markets with strong welfare implications for the society.[7]

Next, we analyze the implications of additional properties. In Corollary 1, we characterize the class of *Uniform trade rules* that respectively satisfy *in-group anonymity* and *between-group anonymity* in determining the volume of trade. We then observe, in Corollary 2, that among *Uniform trade rules*, *renegotiation proof* ones are those that clear exactly one side of the market in economies where there are less agents on

[5] Note that this is more than simply setting the endowments in Klaus et al. (1997, 1998) to zero since in that case all agents in their model would become demanders of the commodity.

[6] The *short side* of a market is where the aggregate volume of desired transaction is smallest. It is thus the demand side if there is excess supply and the supply side if there is excess demand. The other side is called the *long side*.

[7] In health or education sectors for example, it is not uncommon to observe excess demand due to price regulations and an overutilization of services (such as overfilled schools or hospitals). Similarly, there are many countries (such as that of the authors) where in response to an excess supply of labor, governments tend to over-employ in the public sector. Even in the private sector, since most labor contracts include restrictions on when and how the contract can be terminated, firms regularly experience periods in which they overemploy.

the short side of the market than there is on the long side. Interestingly enough, *renegotiation proofness* has no implications for societies with an equal number of buyers and sellers. We also observe that only the *Uniform trade rule* that clears the short side of the market satisfies a *voluntary trade* requirement that gives each agent the right to choose zero trade for himself [the term is introduced by Bénassy (1982), Chap. 6]. For this, we show in Proposition 2 that any *Pareto optimal* and *strategy proof* trade rule that satisfies *voluntary trade* has to clear the short side of the market.

The paper is organized as follows. In Sect. 2, we introduce the model and in Sect. 3, we introduce and discuss Uniform trade rules. Sect. 4 contains the main results. We conclude in Sect. 5.

2 The model

There is a (countable) universal set $\mathcal{B}$ of potential buyers and a (countable) universal set $\mathcal{S}$ of potential sellers. Let $\mathcal{B} \cap \mathcal{S} = \emptyset$. There is a perfectly divisible commodity that each seller produces and each buyer consumes. Let $\mathbb{R}_+$ be the consumption/production space for each agent. Each $i \in \mathcal{B} \cup \mathcal{S}$ is endowed with a (complete and transitive) preference relation R_i over $\mathbb{R}_+$. Let P_i denote the strict preference relation associated with R_i. The preference relation R_i is **single-peaked** if there is $p(R_i) \in \mathbb{R}_+$, called the **peak** of R_i, such that for all x_i, y_i in $\mathbb{R}_+$, $x_i < y_i \leq p(R_i)$ or $x_i > y_i \geq p(R_i)$ implies $y_i P_i x_i$. Let $\mathcal{R}$ denote the set of all single-peaked preference relations on $\mathbb{R}_+$.

Given a finite set $B \subset \mathcal{B}$ of buyers and a finite set $S \subset \mathcal{S}$ of sellers, let $N = B \cup S$ be a **society**. Let $\mathcal{N} = \{B \cup S \mid B \subset \mathcal{B}$ and $S \subset \mathcal{S}$ are finite sets$\}$ be the set of all societies.[8] A preference profile R_N for a society N is a list $(R_i)_{i \in N}$ such that for each $i \in N$, $R_i \in \mathcal{R}$. Let $\mathcal{R}^N$ denote the set of all profiles for the society N. Given $R_N \in \mathcal{R}^N$, let $p(R_N) = (p(R_i))_{i \in N}$. Given $M \subset N$ and $R_N \in \mathcal{R}^N$, let $R_M = (R_i)_{i \in M}$ denote the restriction of R_N to M. Also, for $M \subset N$, let $(R'_M, R_{N \setminus M})$ represent a preference profile where each $i \in M$ has preferences R'_i and each $j \in N \setminus M$ has preferences R_j. If $M = \{i\}$, with an abuse of notation, we will write $(R'_i, R_{N \setminus i})$.

A **market for society** $B \cup S$ is a profile of preferences for buyers and seller $(R_B, R_S) \in \mathcal{R}^{B \cup S}$. Let

$$\mathcal{M} = \bigcup_{(B \cup S) \in \mathcal{N}} \mathcal{R}^{B \cup S}$$

be the set of all markets.

A **(feasible) trade** for $(R_B, R_S) \in \mathcal{M}$ is a vector $z \in \mathbb{R}_+^{B \cup S}$ such that $\sum_{i \in B} z_i = \sum_{i \in S} z_i$. For each buyer (seller) i, z_i denotes how much he buys (sells). Let $Z(B \cup S)$ denote the set of all trades for (R_B, R_S). A *trade* $z \in Z(B \cup S)$ is **Pareto optimal with respect to** (R_B, R_S) if there is no $z' \in Z(B \cup S)$ such that for all $i \in B \cup S$, $z'_i R_i z_i$ and for some $j \in B \cup S$, $z'_j P_j z_j$. In our framework, *Pareto optimal* trades possess two properties: (i) *(same-sidedness)* agents on the same side of the market have consumption levels at the same side of their peaks and (ii) *(opposite-sidedness)*

8 Note that we allow the sets B and S to be empty.

agents on opposite sides of the market have consumption levels at opposite sides of their peaks.

Lemma 1 *For each* $(R_B, R_S) \in \mathcal{M}$, *the trade* $z \in Z(B \cup S)$ *is Pareto optimal with respect to* (R_B, R_S) *if and only if for* $K \in \{B, S\}$, $\sum_{k \in K} p(R_k) \leq \sum_{k \in N \setminus K} p(R_k)$ *implies* (i) $p(R_k) \leq z_k$ *for each* $k \in K$, (ii) $z_j \leq p(R_j)$ *for each* $j \in N \setminus K$, *and thus* (iii) $\sum_{k \in K} p(R_k) \leq \sum_{k \in K} z_k \leq \sum_{k \in N \setminus K} p(R_k)$.

Proof Let $(R_B, R_S) \in \mathcal{M}$ be such that $\sum_{k \in K} p(R_k) \leq \sum_{k \in N \setminus K} p(R_k)$.

Assume that $z \in Z(B \cup S)$ is Pareto optimal. First note that if there is $i \in K$ such that $z_i < p(R_i)$ and there is $j \in N \setminus K$ such that $z_j < p(R_j)$, then there is $\varepsilon > 0$ such that $z' \in Z(B \cup S)$ defined as for all $k \notin \{i, j\}$, $z'_k = z_k$, $z'_i = z_i + \varepsilon$, and $z'_j = z_j + \varepsilon$ Pareto dominates z. Similarly, if there is $i \in K$ such that $z_i > p(R_i)$ and there is $j \in N \setminus K$ such that $z_j > p(R_j)$, we obtain a similar contradiction.

Now note that if $\sum_{k \in K} z_k < \sum_{k \in K} p(R_k) \leq \sum_{k \in N \setminus K} p(R_k)$, then there is $i \in K$ such that $z_i < p(R_i)$ and there is $j \in N \setminus K$ such that $z_j < p(R_j)$. Similarly, if $\sum_{k \in K} p(R_k) \leq \sum_{k \in N \setminus K} p(R_k) < \sum_K z_k$, then there is $i \in K$ such that $z_i > p(R_i)$ and there is $j \in N \setminus K$ such that $z_j > p(R_j)$. Thus $\sum_{k \in K} p(R_k) \leq \sum_{k \in K} z_k \leq \sum_{k \in N \setminus K} p(R_k)$.

Finally, if there is $i, j \in K$ such that $z_i < p(R_i)$ and $z_j > p(R_j)$, there is $\varepsilon > 0$ such that $z'_i = z_i + \varepsilon$, $z'_j = z_j - \varepsilon$, and for all $k \in K \setminus \{i, j\}$, $z'_k = z_k$ is a Pareto improvement over z. This and $\sum_{k \in K} p(R_k) \leq \sum_{k \in K} z_k$ implies that for each $i, j \in K$, $z_i \geq p(R_i)$ and $z_j \geq p(R_j)$. A similar argument proves that for each $i, j \in N \setminus K$, $z_i \leq p(R_i)$ and $z_j \leq p(R_j)$.

For the converse, assume $p(R_k) \leq z_k$ for each $k \in K$ and $z_l \leq p(R_l)$ for each $l \in N \setminus K$. Let $z' \in Z(B \cup S)$ be such that for some $i \in K$, $z'_i P_i z_i$. Then $z'_i < z_i$. This implies that either there is $j \in K$ such that $z'_j > z_j \geq p(R_j)$ or there is $l \in N \setminus K$ such that $z'_l < z_l \leq p(R_l)$. Thus z' does not Pareto dominate z. A similar argument follows if there is $i \in N \setminus K$ such that $z'_i P_i z_i$. Thus z is Pareto optimal. □

A **trade rule** $F : \mathcal{M} \to \bigcup_{N \in \mathcal{N}} Z(N)$ associates each market (R_B, R_S) with a trade $z \in Z(B \cup S)$. For each $i \in \mathcal{B} \cup \mathcal{S}$, the function $F_i : \bigcup_{(B \cup S) \in \mathcal{N} \text{ s.t. } i \in B \cup S} \mathcal{R}^{B \cup S} \to \mathbb{R}_+$ gives agent i's share in each market, that is for $z = F(R_B, R_S)$, we have $F_i(R_B, R_S) = z_i$. Associated with each trade rule F, there is a function $\Omega_F : \mathcal{M} \to \mathbb{R}_+$, defined as $\Omega_F(\cdot) = \sum_{i \in B} F_i(\cdot)$, which determines the **volume of trade**. In what follows, we introduce properties that are related to the four main titles in axiomatic analysis: efficiency, nonmanipulability, fairness, and stability.

We start with efficiency. A trade rule F is **Pareto optimal** if for each $(R_B, R_S) \in \mathcal{M}$, the trade $F(R_B, R_S)$ is *Pareto optimal with respect to* (R_B, R_S).

We present two properties on nonmanipulability. A trade rule F is **strategy proof** if for each $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$, $F_i(R_i, R_{N \setminus i}) R_i F_i(R'_i, R_{N \setminus i})$. That is, regardless of the others' preferences, an agent is best-off with the trade associated with her true preferences. Strategy proof rules do not give the agents incentive for individual manipulation. They however are not immune to manipulation by groups. For this, a stronger property is necessary: a trade rule F is **coalitional strategy proof** if for each $N \in \mathcal{N}$, $R_N \in \mathcal{R}^N$, $M \subset N$, and $R'_M \in \mathcal{R}^M$, if there is

$i \in M$ such that $F_i(R'_M, R_{N\setminus M}) P_i F_i(R_M, R_{N\setminus M})$ then, there is $j \in M$ such that $F_j(R_M, R_{N\setminus M}) P_j F_j(R'_M, R_{N\setminus M})$.[9]

Our first fairness property is after Foley (1967). Since in our model the agents on different sides of the market are exogenously differentiated, our version of the property only compares agents on the same side of the market. A trade rule F is **envy free** (equivalently, satisfies **no-envy**) if for each $(R_B, R_S) \in \mathcal{M}$, $K \in \{B, S\}$, and $i, j \in K$, $F_i(R_B, R_S) R_i F_j(R_B, R_S)$. In an envy free trade, each buyer (respectively, seller) prefers his own consumption (respectively, production) to that of every other buyer (respectively, seller).

No-envy restricts the set of allocations a trade rule F can choose for each volume of trade. It however does not restrict the set of trade volumes that F can choose (since, for every positive volume of trade, there are envy free allocations as well as allocations that create envy). The following anonymity properties, on the other hand, regulate the way the trade volume is chosen.

A bijection $\pi : \mathcal{B} \cup \mathcal{S} \to \mathcal{B} \cup \mathcal{S}$ which satisfies $\pi(i) \in \mathcal{B}$ ($\pi(i) \in \mathcal{S}$) if and only if $i \in \mathcal{B}$ ($i \in \mathcal{S}$) is called an *in-group-permutation*. Let Π be the set of all in-group-permutations and let $R^{\pi}_{\pi(i)} = R_i$ for each $\pi \in \Pi$ and $i \in \mathcal{B} \cup \mathcal{S}$. A trade rule F satisfies **in-group anonymity of trade volume** if for each $(R_B, R_S) \in \mathcal{M}$ and each $\pi \in \Pi$, $\Omega_F(R_B, R_S) = \Omega_F(R^{\pi}_{\pi(B)}, R^{\pi}_{\pi(S)})$. This is a standard anonymity property which says that any two buyers (or, any two sellers) are similar in terms of how they affect the trade volume. That is, permuting their preferences has no effect on the trade volume, though it might affect the agents' shares. Note that the property does not compare a buyer to a seller.

A bijection $\phi : \mathcal{B} \cup \mathcal{S} \to \mathcal{B} \cup \mathcal{S}$ which satisfies $\phi(i) \in \mathcal{B}$ ($\phi(i) \in \mathcal{S}$) if and only if $i \in \mathcal{S}$ ($i \in \mathcal{B}$) is called a *between-group-permutation*.[10] Let Φ be the set of all between-group-permutations and let $R^{\phi}_{\phi(i)} = R_i$ for each $\phi \in \Phi$ and $i \in \mathcal{B} \cup \mathcal{S}$. A trade rule F satisfies **between-group anonymity of trade volume** if for each $(R_B, R_S) \in \mathcal{M}$ and each $\phi \in \Phi$, $\Omega_F(R_B, R_S) = \Omega_F(R^{\phi}_{\phi(S)}, R^{\phi}_{\phi(B)})$. Unlike *in-group anonymity*, this property compares two sides of the market in terms of how they affect the trade volume. It requires that permuting the supply and the demand data (that is, calling supply what used to be called demand and *vice versa*) has no effect on the trade volume. In this sense, a *between-group anonymous* rule satisfies a certain symmetry in terms of how it treats the two sides of the market (for more on this point, please see Corollary 1 and Remark 2).[11] For example, a trade rule that always picks the trade volume to be equal to the aggregate demand violates this property (even though it satisfies *in-group anonymity of trade volume*).

[9] Note that ours is the stronger formulation of the property. A weaker version considers only coalitional manipulations that make all agents in the coalition strictly better-off.

[10] For ϕ to be well-defined, one needs $|\mathcal{B}| = |\mathcal{S}|$. Since this assumption is not used elsewhere, it will be exclusively stated in results that use *between-group-permutations.*

[11] Note that desirability of every property depends on the specifics of the problem on which it is being used. In our opinion, between-group anonymity is particularly desirable if both sides of the market are comprised of firms. Then it requires symmetric treatment of two sectors.

It turns out that the two anonymity properties are logically related. This is because any *in-group permutation* can be written as a composition of two *between-group permutations*.

Lemma 2 *Let $|\mathcal{B}| = |\mathcal{S}|$. If a trade rule F satisfies between-group anonymity of trade volume, it also satisfies in-group anonymity of trade volume.*

Proof Let F satisfy *between-group anonymity of trade volume*. Let $(R_B, R_S) \in \mathcal{M}$ and $\pi \in \Pi$. We want to show $\Omega_F(R_B, R_S) = \Omega_F(R^{\pi}_{\pi(B)}, R^{\pi}_{\pi(S)})$.

Since $\mathcal{B}$ and $\mathcal{S}$ are countable sets, enumerate $\mathcal{B} = \{b_i\}_{i=1}^{|\mathcal{B}|}$ and $\mathcal{S} = \{s_i\}_{i=1}^{|\mathcal{S}|}$. Then, define $\phi : \mathcal{B} \cup \mathcal{S} \to \mathcal{B} \cup \mathcal{S}$ as $\phi(b_i) = s_i$ and $\phi(s_i) = b_i$ for all $i \in \{1, \dots, |\mathcal{B}|\}$. Finally, define $\phi' : \mathcal{B} \cup \mathcal{S} \to \mathcal{B} \cup \mathcal{S}$ as $\phi'(b_i) = \pi(s_i)$ and $\phi'(s_i) = \pi(b_i)$. Since $|\mathcal{B}| = |\mathcal{S}|$, $\phi, \phi' \in \Phi$ are well-defined *between-group permutations*.

Now $\phi'(\phi(b_i)) = \phi'(s_i) = \pi(b_i)$ and $\phi'(\phi(s_i)) = \phi'(b_i) = \pi(s_i)$ imply $\pi = \phi' \circ \phi$. Thus $\Omega_F(R^{\pi}_{\pi(B)}, R^{\pi}_{\pi(S)}) = \Omega_F(R^{\phi' \circ \phi}_{\phi'(\phi(B))}, R^{\phi' \circ \phi}_{\phi'(\phi(S))})$. Finally, applying *between-group anonymity of trade volume* twice gives $\Omega_F(R^{\phi' \circ \phi}_{\phi'(\phi(B))}, R^{\phi' \circ \phi}_{\phi'(\phi(S))}) = \Omega_F(R^{\phi}_{\phi(S)}, R^{\phi}_{\phi(B)}) = \Omega_F(R_B, R_S)$, the desired conclusion. □

Our fourth notion is stability. If a group of agents, by jointly deviating from an allocation (*i.e.* by blocking it), can all be better-off, we say that the allocation is not stable. Alternative stability properties are based on alternative assumptions on which coalitions can form and what they can achieve. We next introduce two alternative properties. The first property is for markets where a buyer–seller pair can renegotiate a deal among themselves. A trade rule F is **renegotiation proof** if for each $(R_B, R_S) \in \mathcal{M}$ there is no $i \in S$ and $j \in B$ such that for some $r \in \mathbb{R}_+$, $r P_i F_i(R_B, R_S)$ and $r P_j F_j(R_B, R_S)$.[12] Our second stability property is for markets where each agent is entitled to leaving the market, that is, buying or selling zero units. A trade rule F satisfies **voluntary trade** if for each $(R_B, R_S) \in \mathcal{M}$ and $i \in B \cup S$, $F_i(R_B, R_S) R_i 0$.

Lastly, we introduce the following informational simplicity property. It requires the volume of trade only to depend on the total demand and supply but not on their individual components. A trade rule F satisfies **independence of trade volume (from in-group transfers)** if for each $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S), (R'_B, R'_S) \in \mathcal{R}^{B \cup S}$, $\sum_{i \in B} p(R_i) = \sum_{i \in B} p(R'_i)$ and $\sum_{i \in S} p(R_i) = \sum_{i \in S} p(R'_i)$ implies $\Omega_F(R_B, R_S) = \Omega_F(R'_B, R'_S)$. Note that this property is not logically related to either *anonymity of trade volume* property since it does not make the determination of trade volume independent of the agents' identities. It merely relates two problems with the same set of agents.

It follows from Lemma 1 that verifying *Pareto optimality* only requires information about the agents' peaks. This is also true for *independence of trade volume*. Verification of all the other properties requires full preference information. Verifying *(coalitional) strategy proofness*, *no envy, renegotiation proofness,* and *voluntary trade* requires knowledge of how an agent compares two bundles at opposite sides

[12] We will later note that requiring a stronger version of the property that allows any coalition to form does not affect our results. Allowing some agents in a blocking-coalition to remain indifferent, on the other hand, has strong implications.

of his peak. The *anonymity* properties require verification of whether a profile is a permutation of another and thus also require full preference information.

We next introduce the class of Uniform trade rules and analyze the properties they all satisfy.

3 Uniform trade rules

Let $\beta : \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$ and $\sigma : \mathcal{N} \to \mathbb{R}_+ \cup \{\infty\}$ be two functions such that for each $B \cup S \in \mathcal{N}$, $B = \emptyset$ or $S = \emptyset$ implies $\beta(B \cup S) = \sigma(B \cup S) = 0$. The **Uniform trade rule with respect to** β **and** σ, $\mathbf{UT}^{\beta\sigma}$, is then defined as follows. We first determine the volume of trade: given $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S) \in \mathcal{R}^{B \cup S}$, let

$$\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \begin{cases} \text{median}\left\{\beta(B \cup S), \sum_B p(R_i), \sum_S p(R_i)\right\} & \text{if } \sum_B p(R_i) \leq \sum_S p(R_i)\}, \\ \text{median}\{\sigma(B \cup S), \sum_B p(R_i), \sum_S p(R_i)\} & \text{if } \sum_B p(R_i) \geq \sum_S p(R_i)\}. \end{cases}$$

That is, a median rule with the exogenous reference-point $\beta(B \cup S)$ is used when the buyers are the short side of the market. If, on the other hand, the sellers are the short side, then the reference point $\sigma(B \cup S)$ is used to calculate the median.

Next, we allocate the volume of trade among the agents: for $K \in \{B, S\}$, let

$$UT_K^{\beta\sigma}(R_B, R_S) = \begin{cases} (\min\{\lambda, p(R_i)\})_{i \in K} & \text{if } \sum_K p(R_i) \geq \Omega_{UT^{\beta\sigma}}(R_B, R_S), \\ (\max\{\lambda, p(R_i)\})_{i \in K} & \text{if } \sum_K p(R_i) \leq \Omega_{UT^{\beta\sigma}}(R_B, R_S). \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}_+$ satisfies

$$\sum_K \min\{\lambda, p(R_i)\} = \Omega_{UT^{\beta\sigma}}(R_B, R_S) \quad \text{if } \sum_K p(R_i) \geq \Omega_{UT^{\beta\sigma}}(R_B, R_S)$$

and

$$\sum_K \max\{\lambda, p(R_i)\} = \Omega_{UT^{\beta\sigma}}(R_B, R_S) \quad \text{if } \sum_K p(R_i) < \Omega_{UT^{\beta\sigma}}(R_B, R_S).$$

The class of *Uniform trade rules* is very rich. It contains rules that for example always favor the buyers ($\beta = 0$ and $\sigma = \infty$), rules that always favor the short side of the market ($\beta = \sigma = 0$), or rules that guarantee a fixed volume of trade unless both sides of the market wish to deviate from it ($\beta = \sigma = c \in \mathbb{R}_+$), as well as rules that mix between these and many other arbitration methods based on the identities of the agents and who constitutes the short side of the market.

We interpret the β and the σ functions as institutional parameters that are determined by the state through a political process and enforced via the legal system. For positive values of these parameters, there are markets in which some buyers (sellers) are required by law to buy (sell) more than their peak. As an example, consider countries with conscription requirements: each male (in some countries, also female) citizen of a certain age is required to supply a minimum amount of labor time (which can go up to two years) in the armed forces (some countries allow civil service as well). Since the pay during this service is typically below the market wage rate, there is excess demand in such labor markets. The amount of trade in a given period is a determinant of the size of a country's army which, in turn, is an important policy choice that is determined politically and not changed frequently. In a static model, it can be interpreted as a σ parameter for this labor market.

The following proposition analyzes the common properties that all *Uniform trade rules* satisfy.

Proposition 1 *All Uniform trade rules satisfy Pareto optimality, coalitional strategy proofness, no-envy, and independence of trade volume.*

Proof *Independence of trade volume* follows from the median definition of $\Omega_{UT^{\beta\sigma}}$. To show that $UT^{\beta\sigma}$ satisfies *Pareto optimality*, note that by the median definition of $\Omega_{UT^{\beta\sigma}}$, we have

$$\sum_{K} p(R_i) \leq \Omega_{UT^{\beta\sigma}}(R_B, R_S) \leq \sum_{N\setminus K} p(R_i)$$

for $K \in \{B, S\}$. Thus there is $\rho, \lambda \in \mathbb{R}_+$ such that

$$\sum_{K} \max\{\rho, p(R_i)\} = \Omega_{UT^{\beta\sigma}}(R_B, R_S) = \sum_{N\setminus K} \min\{\lambda, p(R_i)\}.$$

Thus for each $i \in K$, $UT_i^{\beta\sigma}(R_B, R_S) \geq p(R_i)$ and for each $i \in N\setminus K$, $UT_i^{\beta\sigma}(R_B, R_S) \leq p(R_i)$. This, by Lemma 1, implies the desired conclusion.

To show that $UT^{\beta\sigma}$ satisfies *no envy*,let $R_{B\cup S} \in \mathcal{M}$ and $i \in K \in \{B, S\}$. No envy trivially holds if $UT_i^{\beta\sigma}(R_N) = p(R_i)$. Alternatively $UT_i^{\beta\sigma}(R_N) < p(R_i)$ implies $UT_j^{\beta\sigma}(R_N) \leq UT_i^{\beta\sigma}(R_N)$ for each $j \in K$. Similarly $UT_i^{\beta\sigma}(R_N) > p(R_i)$ implies $UT_j^{\beta\sigma}(R_N) \geq UT_i^{\beta\sigma}(R_N)$ for each $j \in K$. Therefore, $UT_i^{\beta\sigma}(R_N) R_i UT_j^{\beta\sigma}(R_N)$ for each $j \in K$.

To show that $UT^{\beta\sigma}$ satisfies *coalitional strategy proofness*, take an arbitrary market $R_N = (R_B, R_S) \in \mathcal{M}$. Let $z = UT^{\beta\sigma}(R_N)$, $\omega = \Omega_{UT^{\beta\sigma}}(R_N)$, $M \subset N$, and $R'_M \in \mathcal{R}^M$. Let $R'_N = \left(R'_M, R_{N\setminus M}\right)$, $z' = UT^{\beta\sigma}(R'_N)$ and $\omega' = \Omega_{UT^{\beta\sigma}}(R'_N)$. Suppose there is $i \in M$ such that $z'_i P_i z_i$. This implies $z_i \neq p(R_i)$. Without loss of generality, let $i \in S$. Then, $\sum_S p(R_k) \neq \omega$. Without loss of generality, let $\sum_S p(R_k) > \omega$. Then, by the definition of $UT^{\beta\sigma}$, there is $\lambda \in \mathbb{R}_+$ such that $z_i = \lambda = \min\{\lambda, p(R_i)\} < z'_i$.

Case 1 $\omega' \leq \omega$ and $\sum_S p(R'_k) \geq \omega'$.

By the definition of $UT^{\beta\sigma}$, there is $\lambda' \in \mathbb{R}_+$ such that $z'_i = \min\{\lambda', p(R'_i)\} \leq \lambda'$. This implies $\lambda' > \lambda$. Since

$$\sum_{S} z'_k = \omega' \leq \omega = \sum_{S} z_k$$

there is $j \in S$ such that $z'_j < z_j$ which implies $z_j P_j z'_j$. Moreover, $j \in M$. To see this suppose $j \notin M$. Then, $R'_j = R_j$. This implies $z'_j = \min\{\lambda', p(R_j)\} \geq \min\{\lambda, p(R_j)\} = z_j$, a contradiction.

Case 2 $\omega' \leq \omega$ and $\sum_S p(R'_k) < \omega'$.

Then there is $\theta \in \mathbb{R}_+$ such that $z'_i = \max\{\theta, p(R'_i)\} > z_i = \min\{\lambda, p(R_i)\}$. Since $\omega' \leq \omega$, there is $j \in S$ such that $z'_j < z_j$ which implies $z_j P_j z'_j$. We claim that $j \in M$. To see this suppose $j \notin M$. Then, $z'_j = \max\{\theta, p(R_j)\} \geq p(R_j)$ and $z_j = \min\{\lambda, p(R_j)\} \leq p(R_j)$. This implies $z'_j \geq z_j$, a contradiction.

Case 3 $\omega' > \omega$.

Then, $\sum_B p(R'_k) \geq \omega'$. To see this, suppose $\sum_B p(R'_k) < \omega'$. But $\beta(B \cup S) \leq \omega < \omega'$ then contradicts

$$\omega' = \text{median}\left\{\beta(B \cup S), \sum_B p(R'_k), \sum_S p(R'_k)\right\}.$$

By the definition of $UT^{\beta\sigma}$, there are $\rho, \rho' \in \mathbb{R}_+$ such that $z_k = \max\{\rho, p(R_k)\}$ and $z'_k = \min\{\rho', p(R'_k)\}$ for each $k \in B$. Since $\omega = \sum_B z_k < \omega' = \sum_B z'_k$, there is $j \in B$ such that $z_j < z'_j$. Then $p(R_j) \leq z_j < z'_j$ which implies $z_j P_j z'_j$. We claim that $j \in M$. Suppose this is not the case. Then $R_j = R'_j$. So, $z_j = \max\{\rho, p(R_j)\} < z'_j = \min\{\rho', p(R_j)\}$, a contradiction. □

All Uniform trade rules satisfy a core-like property which requires that no coalition of agents can make all its members better-off by reallocating the shares (assigned by a trade rule) of its members among themselves. On the other hand, properties such as *anonymity of trade volume, renegotiation proofness,* and *voluntary trade* are not satisfied by all *Uniform trade rules*. In the next section, this is discussed in further detail.

4 Results

We first present two lemmas that are extensions of standard results by Ching (1994) to our domain.[13] They both are about the regularities that a *Pareto optimal* and *strategy proof* rule exhibits. The first result can be called a "monotonicity lemma" since it states that an increase (decrease) in an agent's peak moves his share in the same direction.

Lemma 3 *Let the trade rule F satisfy Pareto optimality and strategy proofness. Then for each $N \in \mathcal{N}, i \in N$, and $(R_i, R_{N\backslash i}), (R'_i, R_{N\backslash i}) \in \mathcal{R}^N$, $p(R_i) \leq p(R'_i)$, then $F_i(R_i, R_{N\backslash i}) \leq F_i(R'_i, R_{N\backslash i})$.*

Proof Suppose $F_i(R'_i, R_{N\backslash i}) < F_i(R_i, R_{N\backslash i})$. Then there are two possible cases. If $F_i(R_i, R_{N\backslash i}) \leq p(R'_i)$, then with preferences R'_i, agent i has an incentive to declare R_i. If $p(R'_i) < F_i(R_i, R_{N\backslash i})$, then let $K \in \{B, S\}$ be such that $i \in K$ and note

[13] Ching (1994) works on the Sprumont (1991) domain.

that $p(R_i') + \sum_{K\setminus\{i\}} p(R_k) \leq \Omega_F(R_{N\setminus K}, R_K) \leq \sum_{N\setminus K} p(R_k)$. Thus by *Pareto optimality*, $p(R_i') \leq F_i(R_i', R_{N\setminus i})$ and we have

$$p(R_i) \leq p(R_i') \leq F_i(R_i', R_{N\setminus i}) < F_i(R_i, R_{N\setminus i})$$

and then with preferences R_i, agent i has an incentive to declare R_i'. Since in both cases, *strategy proofness* is violated, the supposition is false. □

It follows from Lemma 3 that if $(R_i, R_{N\setminus i}), (R_i', R_{N\setminus i}) \in \mathcal{R}^N$ is such that $p(R_i) = p(R_i')$, then $F_i(R_i, R_{N\setminus i}) = F_i(R_i', R_{N\setminus i})$. That is, an agent who does not change his peak can not affect his share.

The following result can be called an "invariance lemma" since it states that even an agent changing his peak, if he does not cross to the other side of his share, can not affect it.

Lemma 4 *Let the trade rule F satisfy Pareto optimality and strategy proofness. Let $N \in \mathcal{N}$, $i \in N$, and $(R_i, R_{N\setminus i}), (R_i', R_{N\setminus i}) \in \mathcal{R}^N$. If $p(R_i) < F_i(R_i, R_{N\setminus i})$ and $p(R_i') \leq F_i(R_i, R_{N\setminus i})$, then $F_i(R_i', R_{N\setminus i}) = F_i(R_i, R_{N\setminus i})$. Similarly if $p(R_i) > F_i(R_i, R_{N\setminus i})$ and $p(R_i') \geq F_i(R_i, R_{N\setminus i})$, then $F_i(R_i', R_{N\setminus i}) = F_i(R_i, R_{N\setminus i})$.*

Proof To prove the first statement, suppose $p(R_i) < F_i(R_i, R_{N\setminus i})$, $p(R_i') \leq F_i(R_i, R_{N\setminus i})$, and $F_i(R_i', R_{N\setminus i}) \neq F_i(R_i, R_{N\setminus i})$. There are two possible cases. If $p(R_i) \leq p(R_i')$ then by Lemma 3, $F_i(R_i, R_{N\setminus i}) < F_i(R_i', R_{N\setminus i})$ and with preferences R_i', agent i has an incentive to declare R_i. Alternatively if $p(R_i') < p(R_i)$ then by Lemma 3, $F_i(R_i', R_{N\setminus i}) < F_i(R_i, R_{N\setminus i})$. Let $R_i'' \in \mathcal{R}$ be such that $p(R_i'') = p(R_i)$ and $0 P_i'' F_i(R_i, R_{N\setminus i})$. By Lemma 3, $F_i(R_i'', R_{N\setminus i}) = F_i(R_i, R_{N\setminus i})$. Thus $F_i(R_i', R_{N\setminus i}) < F_i(R_i'', R_{N\setminus i})$ and with preferences R_i'', agent i has an incentive to declare R_i'. Since in all cases, *strategy proofness* is violated, the supposition is false. The proof of the second statement is similar. □

4.1 Uniform trade rules

Our main result shows that only *Uniform trade rules* satisfy all of our four basic properties. Note that here, unlike in Proposition 1, we only state *strategy proofness*.

Theorem 1 *A trade rule F satisfies Pareto optimality, strategy proofness, no-envy, and independence of trade volume if and only if it is a Uniform trade rule.*

Proof We already showed that the *Uniform trade rules* satisfy these properties. Conversely, let F be a trade rule satisfying all properties. Let $N = B \cup S \in \mathcal{N}$.
Step 1 For each $K \in \{B, S\}$, $(R_{N\setminus K}, R_K), (R_{N\setminus K}, R_K') \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R_k)$ and $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R_k')$ implies $\Omega_F(R_{N\setminus K}, R_K') = \Omega_F(R_{N\setminus K}, R_K)$. Similarly, for each $K \in \{B, S\}$, $(R_{N\setminus K}, R_K), (R_{N\setminus K}, R_K') \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) > \sum_K p(R_k)$ and $\Omega_F(R_{N\setminus K}, R_K) > \sum_K p(R_k')$ implies $\Omega_F(R_{N\setminus K}, R_K') = \Omega_F(R_{N\setminus K}, R_K)$.

To prove the first statement, let $K \in \{B, S\}$, $(R_{N\setminus K}, R_K), (R_{N\setminus K}, R_K') \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R_k)$ and $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R_k')$.

Let $R^* \in \mathcal{R}$ be such that $p(R^*) = \frac{\sum_K p(R_k)}{|K|}$ and let $R^*_K \in \mathcal{R}^K$ be such that for each $k \in K$, $R_k = R^*$. By *independence of trade volume*, $\Omega_F(R_{N\setminus K}, R^*_K) = \Omega_F(R_{N\setminus K}, R_K)$. By *Pareto optimality* and *no-envy*, for each $k \in K$, $F_k(R_{N\setminus K}, R^*_K) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$. Note that $\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|} < p(R^*)$.

Now let $R^{**} \in \mathcal{R}$ be such that $p(R^{**}) = \frac{\sum_K p(R'_k)}{|K|}$ and $p(R^*) P^{**} \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$. Since $\Omega_F(R_{N\setminus K}, R_K) < \sum_K p(R'_k)$, we have $\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|} < p(R^{**})$.

Without loss of generality, let $K = \{1, \ldots, n\}$. For each $l \in \{0, 1, \ldots, n\}$, let $\widetilde{R}^l_K = \left(\widetilde{R}^l_1, \ldots, \widetilde{R}^l_n\right) \in \mathcal{R}^K$ be such that for $i \leq l$, $\widetilde{R}^l_i = R^{**}$ and for $j > l$, $\widetilde{R}^l_j = R^*$. We claim that for each $l \in \{0, 1, \ldots, n\}$ and $k \in K$

$$F_k(R_{N\setminus K}, \widetilde{R}^l_K) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}.$$

To prove the claim by induction, first note that for $l = 0$, $F_k(R_{N\setminus K}, \widetilde{R}^0_K) = F_K(R_{N\setminus K}, R^*_K) = (\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|})_{k\in K}$. Now let $i \in K$ and assume that the statement holds for $l = i - 1$. Thus for each $k \in K$,

$$F_k(R_{N\setminus K}, \widetilde{R}^{i-1}_K) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|} < \min\{p(R^*), p(R^{**})\}.$$

Then by Lemma 4, $F_i(R_{N\setminus K}, \widetilde{R}^i_K) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$. Let $j \in K\setminus\{i\}$. If $j < i$, then by *Pareto optimality* and *no-envy* $F_j(R_{N\setminus K}, \widetilde{R}^i_K) = F_i(R_{N\setminus K}, \widetilde{R}^i_K) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$. Alternatively assume $j > i$. If $F_j(R_{N\setminus K}, \widetilde{R}^i_K) < \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$, then j *envies* i and if $\frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|} < F_j(R_{N\setminus K}, \widetilde{R}^i_K)$, since by *Pareto optimality*, $F_j(R_{N\setminus K}, \widetilde{R}^i_K) \leq p(R^*)$, we have $F_j(R_{N\setminus K}, \widetilde{R}^i_K) P^{**} \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}$, that is, i *envies* j. Thus

$$F_j(R_{N\setminus K}, \widetilde{R}^i_K) = \frac{\Omega_F(R_{N\setminus K}, R_K)}{|K|}.$$

By this claim we have, for each $i \in K$,

$$\Omega_F(R_{N\setminus K}, \widetilde{R}^{i-1}_K) = \Omega_F(R_{N\setminus K}, \widetilde{R}^i_K).$$

This implies $\Omega_F(R_{N\setminus K}, R^{**}_K) = \Omega_F(R_{N\setminus K}, R_K)$. Finally note that $\sum_K p(R'_k) = |K|\, p(R^{**})$. This, by *independence of trade volume*, implies that $\Omega_F(R_{N\setminus K}, R'_K) = \Omega_F(R_{N\setminus K}, R_K)$.

The proof of the second statement of this step is similar.

Step 2 For each $(R_{N\setminus K}, R_K), (R_{N\setminus K}, R'_K) \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) \leq \sum_K p(R_k)$ and $\sum_{N\setminus K} p(R_k) \leq \sum_K p(R'_k) \leq \Omega_F(R_{N\setminus K}, R_K)$ implies $\Omega_F(R_{N\setminus K}, R'_K) = \sum_K p(R'_k)$. Similarly, for each $(R_{N\setminus K}, R_K), (R_{N\setminus K}, R'_K) \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\setminus K}, R_K) \geq \sum_K p(R_k)$ and $\sum_{N\setminus K} p(R_k) \geq \sum_K p(R'_k) \geq \Omega_F(R_{N\setminus K}, R_K)$ implies $\Omega_F(R_{N\setminus K}, R'_K) = \sum_K p(R'_k)$.

To prove the first statement, let $(R_{N\backslash K}, R_K)$, $(R_{N\backslash K}, R'_K) \in \mathcal{R}^{B\cup S}$, $\Omega_F(R_{N\backslash K}, R_K) \leq \sum_K p(R_k)$ and $\sum_{N\backslash K} p(R_k) \leq \sum_K p(R'_k) \leq \Omega_F(R_{N\backslash K}, R_K)$. Note that by *Pareto optimality* $\Omega_F(R_{N\backslash K}, R'_K) \leq \sum_K p(R'_k)$. Suppose $\Omega_F(R_{N\backslash K}, R'_K) < \sum_K p(R'_k)$. Then by Step 1, $\Omega_F(R_{N\backslash K}, R'_K) = \Omega_F(R_{N\backslash K}, R_K)$, a contradiction.

The proof of the second statement of this step is similar.

Step 3 Determining the functions β and σ.

Fix $B \in \mathcal{B}$ and $S \in \mathcal{S}$. We will next construct the values $\beta\,(B \cup S)$ and $\sigma\,(B \cup S)$. For $c \in \mathbb{R}_+$, let $R^c \in \mathcal{R}$ be such that $p(R^c) = c$ and for $K \in \{B, S\}$, let $R^c_K = (R^c)_{i\in K}$. Now for $d \in \mathbb{R}_+$, consider $(R^0_B, R^d_S) \in \mathcal{R}^{B\cup S}$ and

1. if there is $d^* \in \mathbb{R}_+$ such that $d^*|S| > \Omega_F(R^0_B, R^{d^*}_S)$, let $\beta(B\cup S) = \Omega_F(R^0_B, R^{d^*}_S)$,
2. if for each $d \in \mathbb{R}_+$, $d\,|S| = \Omega_F(R^0_B, R^d_S)$, let $\beta(B \cup S) = \infty$.

Similarly obtain $\sigma(B \cup S)$ by using the profiles $(R^{c^*}_B, R^0_S) \in \mathcal{R}^{B\cup S}$ for $c^* \in \mathbb{R}_+$. If no such c^* exists, set $\sigma(B \cup S) = \infty$.

Step 4 If $(R_B, R_S) \in \mathcal{R}^{B\cup S}$ satisfies $\sum_B p(R_k) \leq \sum_S p(R_k)$, then

$$\Omega_F(R_B, R_S) = \text{median}\left\{\beta(B \cup S), \sum_B p(R_k), \sum_S p(R_k)\right\}.$$

If $\sum_B p(R_k) = \sum_S p(R_k)$, the statement trivially follows from *Pareto optimality*. So let $\sum_B p(R_k) < \sum_S p(R_k)$.

First assume there is $d^* \in \mathbb{R}_+$ such that $d^*|S| > \Omega_F(R^0_B, R^{d^*}_S)$. Then by Step 3, $\beta(B \cup S) = \Omega_F(R^0_B, R^{d^*}_S)$.

There are three possible cases.

Case 1 $\sum_B p(R_k) < \beta(B \cup S) < \sum_S p(R_k)$.

Then since $0\,|B| < \beta(B \cup S) = \Omega_F(R^0_B, R^{d^*}_S) < d^*|S|$, applying Step 1 twice, we get $\Omega_F(R^0_B, R^{d^*}_S) = \Omega_F(R_B, R^{d^*}_S) = \Omega_F(R_B, R_S)$.

Case 2 $\beta(B \cup S) \leq \sum_B p(R_k) < \sum_S p(R_k)$.

Then since $0\,|B| \leq \beta(B \cup S) = \Omega_F(R^0_B, R^{d^*}_S) < d^*|S|$, applying Step 1 to S, we get $\Omega_F(R^0_B, R^{d^*}_S) = \Omega_F(R^0_B, R_S)$ and applying Step 2 to B, we get $\Omega_F(R_B, R_S) = \sum_B p(R_k)$.

Case 3 $\sum_B p(R_k) < \sum_S p(R_k) \leq \beta(B \cup S)$.

Then since $0\,|B| < \beta(B \cup S) = \Omega_F(R^0_B, R^{d^*}_S) < d^*|S|$, applying Step 1 to B, we get $\Omega_F(R^0_B, R^{d^*}_S) = \Omega_F(R_B, R^{d^*}_S)$ and applying Step 2 to S, we get $\Omega_F(R_B, R_S) = \sum_S p(R_k)$.

Next assume that for each $d \in \mathbb{R}_+$, $d\,|S| = \Omega_F(R^0_B, R^d_S)$. Then by Step 3, $\beta(B \cup S) = \infty$. Let $d > 0$ be such that $d\,|S| = \sum_S p(R_k)$. Then $\Omega_F(R^0_B, R^d_S) =$

$\sum_S p(R_k) > 0$. Thus by Step 1, $\Omega_F(R_B, R_S^d) = \sum_S p(R_k)$. Finally by Step 2 $\Omega_F(R_B, R_S) = \sum_S p(R_k)$.

Since in all cases $\Omega_F(R_B, R_S) = \text{median}\{\beta(B \cup S), \sum_B p(R_k), \sum_S p(R_k)\}$, the proof is complete.

Step 5 If $(R_B, R_S) \in \mathcal{R}^{B\cup S}$ satisfies $\sum_B p(R_k) \geq \sum_S p(R_k)$, then

$$\Omega_F(R_B, R_S) = \text{median}\left\{\sigma(B \cup S), \sum_B p(R_k), \sum_S p(R_k)\right\}.$$

The proof is similar to that of Step 4.

Step 6 $F = UT^{\beta\sigma}$

Suppose $F_K(R_N) \neq UT_K^{\beta\sigma}(R_N)$ for some $R_N \in \mathcal{R}^N$ and $K \in \{B, S\}$. By Steps 4 and 5, $\Omega_F(R_N) = \Omega_{UT^{\beta\sigma}}(R_N)$ and by our supposition, $\sum_K p(R_k) \neq \Omega_F(R_N)$.

First assume that $\sum_K p(R_k) > \Omega_F(R_N)$. Since $F_K(R_N) \neq UT_K^{\beta\sigma}(R_N)$, there is $i \in K$ such that

$$F_i(R_N) < UT_i^{\beta\sigma}(R_N) \leq p(R_i).$$

Let $R_i' \in \mathcal{R}$ be such that $p(R_i') = p(R_i)$ and for each $x > F_i(R_B, R_S)$, $x P_i' F_i(R_B, R_S)$. By Lemma 3,

$$F_i(R_i', R_{N\setminus i}) < UT_i^{\beta\sigma}(R_i', R_{N\setminus i}) \leq p(R_i').$$

Now since $\sum_K F_k(R_N) = \sum_K UT_k^{\beta\sigma}(R_N)$, there is $j \in K$ such that $UT_j^{\beta\sigma}(R_i', R_{N\setminus i}) < F_j(R_i', R_{N\setminus i})$. Thus $UT_j^{\beta\sigma}(R_i', R_{N\setminus i}) < p(R_j)$ and by definition of $UT^{\beta\sigma}$, $UT_i^{\beta\sigma}(R_i', R_{N\setminus i}) \leq UT_j^{\beta\sigma}(R_i', R_{N\setminus i})$. Then $F_i(R_i', R_{N\setminus i}) < F_j(R_i', R_{N\setminus i})$ and with preferences R_i', agent i envies agent j, a contradiction.

The proof of the second case where $\sum_K p(R_k) < \Omega_F(R_N)$ is similar. □

The properties of Theorem 1 are logically independent. First, the simple rule which always chooses zero trade satisfies all properties but *Pareto optimality*. Second, the rule which always clears the short side of the market and rations the long side by a priority order (according to which agents are served sequentially until the volume of trade is exhausted) satisfies all properties but *no-envy*. For the third and the fourth rules, let $N = \{1, 2, 3\}$ and $K = \{1, 2\}$. For this society, let the rule $\overline{F}$ determine the volume of trade as

$$\Omega_{\overline{F}}(R_1, R_2, R_3) = \text{median}\left\{p(R_1) + p(R_2), p(R_3), \frac{3\,(p(R_1) + p(R_2))}{2}\right\}.$$

For the same society, let the alternative rule $\widetilde{F}$ determine the volume of trade as

$$\Omega_{\widetilde{F}}(R_1, R_2, R_3) = \text{median}\{p(R_3), 2p(R_1), 2p(R_2)\}. \tag{2}$$

Let both rules determine the shares of agents 1 and 2 similar to the *Uniform trade rules* (Eq. 1). Finally, let each rule coincide with an arbitrary *Uniform trade rule* for every other society $(B \cup S) \in \mathcal{N}$. Then, the rule $\overline{F}$ satisfies all properties but *strategy proofness.* Also, the rule $\widetilde{F}$ satisfies all properties but *independence of trade volume*.[14]

We close this section with a discussion of the implications of dropping *independence of trade volume* from the list of properties in Theorem 1. First note that every trade rule $F : \mathcal{M} \to \bigcup_{N \in \mathcal{N}} Z(N)$ is a composition of (i) a rule $\Omega_F : \mathcal{M} \to \mathbb{R}_+$ that determines trade volume and (ii) a rule $f : \bigcup_{N \in \mathcal{N}} \mathcal{R}^N \times \mathbb{R}_+ \to \bigcup_{N \in \mathcal{N}} Z(N)$ that, given the trade volume, allocates it among buyers and sellers. While *independence of trade volume* is a property of Ω_F, it is not the only one. By Lemma 1, *Pareto optimality* requires that Ω_F is of the form

$$\Omega_F\left(R_N\right) = \text{median}\left\{\zeta\left(R_N\right), \sum_B p(R_i), \sum_S p(R_i)\right\}$$

where $\zeta : \mathcal{R}^N \to \mathbb{R}$ is an arbitrary function. For Uniform trade rules, ζ is restricted to choose one of two constants, $\beta\,(B \cup S)$ and $\sigma\,(B \cup S)$. In general however, ζ can be very sensitive to individual preference information. For example, consider the society $B = \{1, 2\}$, $S = \{3\}$. Without *independence of trade volume*, ζ functions like the following become admissible:

$$\begin{aligned}
\text{(i) } \zeta^{*}(R_N) &= p(R_1) \\
\text{(ii) } \zeta^{**}(R_N) &= 2\max\{p(R_1), p(R_2)\} \\
\text{(iii) } \zeta^{***}(R_N) &= \begin{cases} 3 & \text{if } p\left(R_1\right) = 1 \text{ and } 0 I_1 2, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

Adding *strategy proofness* to *Pareto optimality* restricts the class of admissible ζ functions in a way to satisfy Lemmas 3 and 4. But this restriction is not independent of the allocation rule f. For example, if f is the Uniform rule, both ζ^* and ζ^{***} lead to a violation of *strategy proofness,* while ζ^{**} does not. Alternatively, if f on B is dictatorship of agent 1, ζ^{**} now leads to a violation of *strategy proofness* while using either ζ^* or ζ^{***} satisfies the property. On Sprumont's model, Barberà et al. (1997) characterize all *Pareto optimal* and *strategy proof* allocation rules. Trade rules that satisfy the same properties have to be a composition of such an allocation rule f and an "appropriate" ζ function. Unfortunately, the characterization of these ζ function classes is currently an open question.

The function ζ^{***} shows us that, contrary to the public goods model (Moulin 1980), *Pareto optimal* and *strategy proof* trade rules need not have "peak-only" ζ functions (i.e., the volume of trade can depend on more information than the agents' peaks). However, every *Pareto optimal* and *strategy proof* trade rule is "own-peak only" (i.e., if an agent changes his preferences without moving his peak, his share remains unchanged).

[14] This rule is in fact *coalitional strategy proof.*

Adding *no-envy* to the above list of properties restricts the class of admissible allocation rules f and thus, as discussed above, restricts the class of admissible ζ functions. On Sprumont's domain, the Uniform rule uniquely satisfies *Pareto optimality*, *strategy proofness* and *no-envy.* Without *independence of trade volume,* it is not clear if a similar result holds for trade rules. The existing uniqueness proofs (of Sprumont 1991; Ching 1992) do not apply to our domain since they utilize the fact that the social endowment is constant.[15] However, if a trade rule F is additionally "nonbossy"[16], Step 5 of our proof shows that its associated allocation rule f must be the Uniform rule. While this, in turn, restricts the class of admissible ζ functions, examples such as Eq. 2 or ζ^{**} remain admissible.

In the following sections, we characterize those *Uniform trade rules* that satisfy additional properties.

4.2 Anonymity, renegotiation proofness, and voluntary trade

In this subsection, we analyze *Uniform trade rules* that satisfy additional properties. We first focus on two anonymity properties on the determination of the trade volume. We then analyze the implications of two stability properties that require an allocation not to be blocked either by a pair of agents (as in *renegotiation proofness*) or by a single agent (as in *voluntary trade*).

Corollary 1 *Let F be a trade rule that satisfies Pareto optimality, strategy proofness, no-envy, and independence of trade volume. Then*

(i) *F satisfies in-group anonymity of trade volume if and only if it is a Uniform trade rule $UT^{\beta\sigma}$ where for each $(B \cup S)$, $(B' \cup S') \in \mathcal{N}$ such that $|B| = |B'|$ and $|S| = |S'|$, $\beta(B \cup S) = \beta(B' \cup S')$ and $\sigma(B \cup S) = \sigma(B' \cup S')$,*

(ii) *assuming $|\mathcal{B}| = |\mathcal{S}|$, F satisfies between-group anonymity of trade volume if and only if it is a Uniform trade rule $UT^{\beta\sigma}$ where for each $(B \cup S)$, $(B' \cup S') \in \mathcal{N}$ such that $|B| = |S'|$ and $|S| = |B'|$, $\beta(B \cup S) = \sigma(B' \cup S')$ and $\sigma(B \cup S) = \beta(B' \cup S')$.*

Proof By Theorem 1, F is a *Uniform trade rule* $UT^{\beta\sigma}$. The proof of the first statement is trivial and omitted. For the second statement, first assume that $UT^{\beta\sigma}$ satisfies *between-group anonymity of trade volume*. Let $(B \cup S)$, $(B' \cup S') \in \mathcal{N}$ be such that $|B| = |S'|$ and $|S| = |B'|$. Let $(R_B, R_S) \in \mathcal{R}^{B \cup S}$ and $(R'_{B'}, R'_{S'}) \in \mathcal{R}^{B' \cup S'}$ be such that $R_B = R'_{S'}$, $R_S = R'_{B'}$, and $\sum_B p(R_k) \leq \beta(B \cup S) \leq \sum_S p(R_k)$.[17] Then $\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \beta(B \cup S)$ and $\Omega_{UT^{\beta\sigma}}(R'_{B'}, R'_{S'}) = \sigma(B' \cup S')$. By *between-group anonymity of trade volume* $\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \Omega_{UT^{\beta\sigma}}(R'_{B'}, R'_{S'})$. Thus $\beta(B \cup S) = \sigma(B' \cup S')$. One similarly obtains $\sigma(B \cup S) = \beta(B' \cup S')$.

[15] More precisely, these proofs do not explore the possibility that a change in agent i's preferences will affect what the other agents get in total (through the volume of trade), but will leave agent i's share unchanged.

[16] *Nonbossiness* requires that, if a change in agent i's preferences does not affect his own share, then it should also not affect the other agents' shares. If a trade rule is *Pareto optimal, strategy proof* and *nonbossy*, then it is *peak-only*.

[17] The construction in Step 3 of the proof of Theorem 1 can be used to obtain such profiles.

Now assume that β and σ satisfy the given property. Let $(R_B, R_S) \in \mathcal{R}^{B\cup S}$ and $\phi \in \Phi$. Without loss of generality assume $\sum_B p(R_k) \leq \sum_S p(R_k)$. Then,

$$\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \text{median}\left\{\beta(B\cup S), \sum_B p(R_k), \sum_S p(R_k)\right\}.$$

By the given property $\beta(B\cup S) = \sigma(\phi(S)\cup\phi(B))$. Also, $\sum_B p(R_k) = \sum_{\phi(B)} p(R_k^\phi)$ and $\sum_S p(R_k) = \sum_{\phi(S)} p(R_k^\phi)$. Thus $\sum_{\phi(B)} p(R_k^\phi) \leq \sum_{\phi(S)} p(R_k^\phi)$ and

$$\begin{aligned}\Omega_{UT^{\beta\sigma}}(R^\phi_{\phi(S)}, R^\phi_{\phi(B)}) &= \text{median}\left\{\sigma(\phi(S)\cup\phi(B)), \sum_{\phi(S)} p(R_k^\phi), \sum_{\phi(B)} p(R_k^\phi)\right\}\\ &= \Omega_{UT^{\beta\sigma}}(R_B, R_S).\end{aligned}$$

□

Since, assuming $|\mathcal{B}| = |\mathcal{S}|$, *between-group anonymity of trade volume* is stronger than *in-group anonymity of trade volume*, Property (ii) in Corollary 1 implies Property (i). The first part of this result states that *in-group anonymity of trade volume* makes β and σ only dependent on the number of buyers and sellers. According to the second part, *between-group anonymity of trade volume* additionally requires the treatment of buyers in a k-buyer, $l-$seller problem to be the same as the treatment of sellers in an l-buyer, k-seller problem.

Remark 1 A stronger version of *independence of trade volume* is also stronger than *in-group anonymity of trade volume*. It is defined as follows: a trade rule F satisfies *strong independence of trade volume* if for each $(B \cup S)$, $(B' \cup S') \in \mathcal{N}$, $(R_B, R_S) \in \mathcal{R}^{B\cup S}$, and $(R'_{B'}, R'_{S'}) \in \mathcal{R}^{B'\cup S'}$, $\sum_{i\in B} p(R_i) = \sum_{i\in B'} p(R'_i)$ and $\sum_{i\in S} p(R_i) = \sum_{i\in S'} p(R'_i)$ implies $\Omega_F(R_B, R_S) = \Omega_F(R'_{B'}, R'_{S'})$. A trade rule F satisfies *Pareto optimality, strategy proofness, no-envy,* and *strong independence of trade volume* if and only if it is a *Uniform trade rule* $UT^{\beta\sigma}$ where there is $c_\beta, c_\sigma \in \mathbb{R}_+ \cup \{\infty\}$ such that for all $(B \cup S) \in N$, $\beta(B \cup S) = c_\beta$ and $\sigma(B \cup S) = c_\sigma$. If, additional to the above properties (and thus, to *in-group anonymity*), F also satisfies *between-group anonymity,* then $c_\beta = c_\sigma$, that is, every problem is treated identically.

We next analyze the implications of *renegotiation proofness*.

Corollary 2 *A trade rule F satisfies Pareto optimality, strategy proofness, no-envy, independence of trade volume, and renegotiation proofness if and only if it is a Uniform trade rule $UT^{\beta\sigma}$ where for each $(B \cup S) \in \mathcal{N}$, $|B| < |S|$ implies $\beta(B \cup S) \in \{0, \infty\}$ and $|S| < |B|$ implies $\sigma(B \cup S) \in \{0, \infty\}$.*

Proof By Theorem 1, F is a *Uniform trade rule* $UT^{\beta\sigma}$. For the only if part suppose there is $(B \cup S) \in \mathcal{N}$ such that $|B| < |S|$ and $\beta(B \cup S) \in (0, \infty)$. Let $R^c \in \mathcal{R}$ be such that $p(R^c) = c \in (\frac{\beta(B\cup S)}{|S|}, \frac{\beta(B\cup S)}{|B|})$. Let $(R_B, R_S) \in \mathcal{R}^{B\cup S}$ be such that for each $i \in B \cup S$, $R_i = R^c$. Then, $\Omega_{UT^{\beta\sigma}}(R_B, R_S) = median\{\beta(B \cup S), c\,|B|, c\,|S|\} =$

$\beta(B \cup S)$. By *no-envy* and *Pareto optimality*, for each $i \in B, UT_i^{\beta\sigma}(R_B, R_S) = \frac{\beta(B\cup S)}{|B|}$ and for each $j \in S, UT_j^{m}(R_B, R_S) = \frac{\beta(B\cup S)}{|S|}$. This implies, there is $i \in B$ and $j \in S$ such that $cP_i UT_i^{\beta\sigma}(R_B, R_S)$ and $cP_j UT_j^{\beta\sigma}(R_B, R_S)$ and therefore that $UT^{\beta\sigma}$ is not renegotiation proof. Thus, $\beta(B \cup S) = 0$ or $\beta(B \cup S) = \infty$. A similar argument applies for the case $|S| < |B|$ and $\sigma(B \cup S)$.

The if part is as follows. If $(B \cup S) \in \mathcal{N}$ is such that $\beta(B \cup S), \sigma(B \cup S) \in \{0, \infty\}$, then for each $(R_B, R_S) \in \mathcal{R}^{B\cup S}$, there is $K \in \{B, S\}$ such that $\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \sum_{i \in K} p(R_i)$ and thus, $UT_i^{\beta\sigma}(R_B, R_S) = p(R_i)$ for each $i \in K$. In this case, no member of K is better-off by joining a blocking pair and therefore, renegotiation is not possible.

Next let $(B \cup S) \in \mathcal{N}$ be such that $|B| \geq |S|$ and $\beta(B \cup S) \in (0, \infty)$. Let $(R_B, R_S) \in \mathcal{R}^{B\cup S}$ be such that $\sum_{i\in B} p(R_i) < \beta(B \cup S) < \sum_{i\in S} p(R_i)$ (otherwise, one group gets its peak and has no incentive to renegotiate). Then, $\Omega_{UT^{\beta\sigma}}(R_B, R_S) = \beta(B \cup S)$ and for each $i \in B$, $UT_i^{\beta\sigma}(R_B, R_S) = \max\{\rho, p(R_i)\}$ where $\rho \in \mathbb{R}_+$ satisfies $\sum_B \max\{\rho, p(R_k)\} = \beta(B \cup S)$. Similarly for each $j \in S$, $UT_j^{\beta\sigma}(R_B, R_S) = \min\{\lambda, p(R_j)\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum_S \min\{\lambda, p(R_k)\} = \beta(B \cup S)$. This implies $\lambda \geq \frac{\beta(B\cup S)}{|S|}$, $\rho \leq \frac{\beta(B\cup S)}{|B|}$ and thus, $\rho \leq \lambda$. Now suppose there is a blocking pair $(i, j) \in B \times S$. Since neither i nor j can get his peak,

$$p(R_i) < UT_i^{\beta\sigma}(R_B, R_S) = \rho \leq \lambda = UT_j^{\beta\sigma}(R_B, R_S) < p(R_j).$$

For both agents to be strictly better off at some $r \in \mathbb{R}_+$, we must have $r < UT_i^{\beta\sigma}(R_B, R_S)$ and $r > UT_j^{\beta\sigma}(R_B, R_S)$. This implies $r < UT_i^{\beta\sigma}(R_B, R_S) \leq UT_j^{\beta\sigma}(R_B, R_S) < r$, a contradiction. Thus $UT^{\beta\sigma}$ is renegotiation proof. □

It is interesting to observe that *renegotiation proofness* has no implications on problems with an equal number of buyers and sellers while its implications on the remaining problems are quite strong. Let us also note that a stronger version of *renegotiation proofness* which allows blocking pairs where one agent is indifferent (while, of course the other is strictly better-off) is violated by all *Uniform trade rules*. On the other hand, strenghtening *renegotiation proofness* by allowing larger (than two-agent) coalitions to form has no effect on the conclusion of Corollary 2.[18]

We next analyze the implications of *voluntary trade*. We start with the much larger class of all *Pareto optimal* and *strategy proof* trade rules.

Proposition 2 *If a trade rule F satisfies Pareto optimality, strategy proofness,and voluntary trade, then for each $(R_B, R_S) \in \mathcal{M}, \Omega_F(R_B, R_S) = \min\{\sum_B p(R_k), \sum_S p(R_k)\}$.*

Proof Let $(R_B, R_S) \in \mathcal{M}$ and without loss of generality assume that $\sum_B p(R_k) \leq \sum_S p(R_k)$. By *Pareto optimality*, $\sum_B p(R_k) \leq \Omega_F(R_B, R_S) \leq \sum_S p(R_k)$. Suppose

[18] Formally, all *renegotiation proof* Uniform trade rules satisfy the following property: a trade rule F is *strong renegotiation proof* if for each $(R_B, R_S) \in \mathcal{M}$ there is no $S' \subset S$, $B' \subset B$, and $z \in Z(B' \cup S')$ such that $z_i P_i F_i(R_B, R_S)$ for each $i \in B' \cup S'$.

$\sum_B p(R_k) < \Omega_F(R_B, R_S)$. Then there is $i \in B$ such that $p(R_i) < F_i(R_B, R_S)$. Let $R_i' \in \mathcal{R}$ be such that $p(R_i') = p(R_i)$ and $0 P_i' F_i(R_B, R_S)$. By Lemma 3, $F_i(R_{B\setminus i}, R_i', R_S) = F_i(R_B, R_S)$ and thus $0 P_i' F_i(R_{B\setminus i}, R_i', R_S)$, violating *voluntary trade*. □

The following remark summarizes the implications of *voluntary trade* on *Uniform trade rules*. It trivially follows from Proposition 2 and Theorem 1.

Remark 2 A trade rule F satisfies Pareto optimality, strategy proofness, no-envy, and voluntary trade if and only if it is a Uniform trade rule $UT^{\beta\sigma}$ such that $\beta(B \cup S) = \sigma(B \cup S) = 0$ for all $(B \cup S) \in \mathcal{N}$.

5 Conclusions

In this section, we present and discuss some open questions. First, our model is motivated by a production economy. We pick a market there that is in disequilibrium, isolate it from other related markets, and then produce a trade vector for it. In doing this, our considerations are at the micro level. That is, our properties focus on a trade rule's performance at that particular market and not on its implications on say, related markets or on the overall competitiveness of the affected firms. In short, we do not analyze the implications of a trade rule on the overall economy. Such an analysis seems to be an important follow-up to our work.

Second, we do not consider population changes in this paper. Implications of properties such as consistency or population monotonicity (and in fact, good formulations of these ideas on this domain) remains an open question.

Finally, we analyze rules that satisfy *independence of trade volume*. We believe *independence* to be an intuitively desirable property and we obtain a very large class of rules that satisfy it. Nevertheless, there might be other interesting rules that violate this property. We hope that the discussion after Theorem 1 will be useful to the interested reader.

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