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Ordinal invariance in multicoalitional bargaining

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Abstract

A multicoalitional bargaining problem is a non-transferable utility game and for each coalition, a bargaining rule. We look for ordinally invariant solutions to such problems and discover a subrule of Bennett's (1997, Games Econ. Behav. 19, 151–179) that satisfies the property. On a subclass of problems that is closely related to standard bargaining problems and allocation problems with majority decision-making, the two rules coincide. Therefore, Bennett solutions to such problems are immune to misrepresentation of cardinal utility information. We also show that Shapley–Shubik solution to any bargaining problem is the limit of a sequence of unique Bennett solutions to associated multicoalitional problems.

JEL classification: C71; C78; D30

Keywords: Ordinal invariance; Multicoalitional bargaining; Bennett rule; Extreme-Bennett rule; Allocation problems; Shapley–Shubik rule

1. Introduction

In an ordinal theory of bargaining, it is essential that the physical bargaining outcome remains invariant under equivalent utility representations of the agents' preferences. A property called ordinal invariance formulates this idea for bargaining rules. It is much stronger than the scale invariance property of Nash (1950) and is violated by all of the well-known bargaining rules.

In this paper, we look for ordinally invariant solutions to *multicoalitional bargaining problems*: these are environments where several interrelated bargaining processes simultaneously take place. Formation of a coalitional government in a parliamentary system, trade

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in an exchange economy, formation of jurisdictions, and production of public goods in a local public goods economy are a few of the many examples to multicoalitional bargaining.

Formally, a multicoalitional bargaining problem is a non-transferable utility game and for each coalition, a bargaining rule which summarizes how bargaining takes place and how it is resolved in that coalition. Unlike in standard bargaining theory, no exogenous disagreement outcome is specified. Rather, the disagreement payoffs for the members of a coalition are endogenously determined by their best potential agreement in other coalitions.

The literature on multicoalitional bargaining starts with Binmore (1985) who analyzes a particular class of three-agent bargaining problems. A different analysis of the same class is given by Bennett and Houba (1992). Bennett (1987, 1997) presents a general formulation of multicoalitional bargaining problems and proposes a solution rule (hereafter, the *Bennett rule*). She also provides a very appealing noncooperative interpretation for this rule.

If, in a multicoalitional problem, the bargaining rules used by individual coalitions violate *ordinal invariance*, one might predict the "multicoalitional solution" to violate the property as well. We, however, show that even then, a subset of Bennett solutions (hereafter, the *extreme-Bennett solutions*) remains ordinally invariant. At extreme-Bennett solutions, the agents' conjectures on what they can achieve in alternative coalitions are so high that no formed coalition can offer its members more than these conjectured disagreement payoffs. It is this feature that generates their ordinal invariance and makes them "extreme" among all Bennett solutions of the problem.

Unfortunately, there are problems for which the extreme-Bennett rule is empty-valued.[1] Nevertheless, we discover a subclass of problems for which the extreme-Bennett rule is not only nonempty-valued, but it also coincides with the Bennett rule. That is, on this subclass, the Bennett rule itself is ordinally invariant. This subclass turns out to be quite interesting since

(i) each problem in this class has a direct link to bargaining problems; and
(ii) any allocation problem with majority decision-making corresponds to a multicoalitional bargaining problem in this class.

It then follows from ordinal invariance that Bennett solutions to such allocation problems are immune to manipulation via cardinal utility information: competition to be a member of the winning coalition rules out any possibility of manipulation.

The literature on ordinal invariance starts with Shapley (1969) who shows that for two agents, no strongly individually rational bargaining rule satisfies this property. Later, a three-agent bargaining rule (hereafter, the *Shapley–Shubik rule*) satisfying both properties appears in Shubik (1982). Kıbrıs (2001, 2002) presents two characterizations which suggest this rule to be the ordinal counterpart of both the Nash (1950) and the Kalai and Smorodinsky (1975) bargaining rules.[2] While the Shapley–Shubik rule has not been

[1] This, however, is solely due to the nature of the coalitions' cooperative opportunities and not the bargaining rules they use to resolve conflicts.

[2] This rule uniquely satisfies *ordinal invariance, Pareto optimality, symmetry*, and a monotonicity property (similar to that of Kalai and Smorodinsky). Replacing the latter with an independence property (similar to that of Nash) also characterizes the same rule.

previously discussed in relation to multicoalitional bargaining, it turns out to be intimately related to the Bennett rule as well: we show that the Shapley–Shubik solution to a bargaining problem is the limit of an iteratively defined sequence of unique Bennett solutions to associated multicoalitional problems.

2. Model

Let $N = \{1, \ldots, n\}$ be the set of agents and $\mathcal{N} = 2^N \setminus \{\emptyset\}$ be the set of coalitions. For each $S \in \mathcal{N}$, the *feasible set of coalition* S, $V(S) \subset \mathbb{R}^S_+$ is compact and strictly comprehensive.[3] Assume that for each $i \in N$, $V(\{i\}) = \{0\}$. The interior (boundary) of $V(S)$ relative to $\mathbb{R}^S_+$ is denoted by $\mathrm{int}(V(S))[\partial(V(S))]$.[4] Since $V(S)$ is strictly comprehensive, $\partial(V(S))$ is equal to the set of Pareto optimal payoff profiles in $V(S)$. Note that

$$V : \mathcal{N} \to \bigcup_{S \in \mathcal{N}} \mathbb{R}^S_+$$

is *a non-transferable utility* (*NTU*) *game*. Let $\mathcal{V}^n$ be the set of all such games. For each $m \in N$, let $\mathcal{V}^n_m \subset \mathcal{V}^n$ be the set of NTU games satisfying the following property: for each $S \in \mathcal{N}$, if $|S| \geqslant m$,

$$V(S) = \left\{p_S \in \mathbb{R}^S_+ \;\middle|\; (p_S, 0_{N\setminus S}) \in V(N)\right\};$$

otherwise, $V(S) = \{0\}$. For games in $\mathcal{V}^n_m$ only coalitions of size at least m are decisive. Also note that $\mathcal{V}^n_n$ is the class of n-agent bargaining problems.

Let $p \in \mathbb{R}^N_+$ be a *potential agreement*. For each $T \in \mathcal{N}$ such that $i \in T$, the *best agreement* i *can reach at* T is

$$a^T_i(V, p) = \begin{cases} p_i & \text{if } p_T \in V(T), \\ \max\left\{0, t_i \;\middle|\; (p_{-i}, t_i) \in V(T)\right\} & \text{otherwise.} \end{cases}$$

Let $S \in \mathcal{N}$ and $i \in S$. The *outside option of* i *in* S is

$$d^S_i(V, p) = \max\left\{a^T_i(V, p) \;\middle|\; i \in T \text{ and } T \in \mathcal{N} \setminus \{S\}\right\},$$

the best agreement he can reach outside S. The *outside option vector of* S is $d^S(V, p) = (d^S_i(V, p))_{i \in S}$. For each $S \in \mathcal{N}$, $f^S : \mathcal{V}^{|S|}_{|S|} \times \mathbb{R}^S_+ \to \mathbb{R}^S_+$ is the *bargaining rule* of coalition S. For each $S \in \mathcal{N}$, f^S satisfies

(i) (*Pareto optimality*) if $d^S \in V(S)$, $f^S(V(S), d^S) \in \partial(V(S))$,
(ii) (*strong individual rationality*) if $d^S \in \mathrm{int}(V(S))$, $f^S(V(S), d^S) \gg d^S$ and if $d^S \in \partial(V(S))$, $f^S(V(S), d^S) \geqslant d^S$,
(iii) (*continuity*) f^S is a continuous function of d^S, and
(iv) (*agreeing to disagree*) if $d^S \notin V(S)$, $f^S(V(S), d^S) = d^S$.

[3] $V(S)$ is strictly comprehensive if for each $x \in V(S)$ and $y \in \mathbb{R}^S_+$ such that $y \leqslant x$, $y \in V(S)$ and there is $z \in V(S)$ such that $z \gg y$.

[4] Note that the intersection of $V(S)$ with the boundary of $\mathbb{R}^S_+$ belongs to $\mathrm{int}(V(S))$.

Let $f = \{f^S \mid S \in \mathcal{N}\}$ and let $\mathcal{F}$ be the set all such f.

A *multicoalitional bargaining problem* is a couple $(V, f) \in \mathcal{V}^n \times \mathcal{F}$. Bennett (1997) defines a solution to such problems to be a fixed point of the process of determining potential agreements and corresponding outside options. The *Bennett rule* $B : \mathcal{V}^n \times \mathcal{F} \to \mathbb{R}^N_+$ is defined as follows: for each $(V, f) \in \mathcal{V}^n \times \mathcal{F}$ and $p \in \mathbb{R}^N_+$, $p \in B(V, f)$ if for each $S \in \mathcal{N}$, $p_S = f^S(V(S), d^S(V, p))$. The *extreme-Bennett rule* $EB : \mathcal{V}^n \times \mathcal{F} \to \mathbb{R}^N_+$ is defined as follows: for each $(V, f) \in \mathcal{V}^n \times \mathcal{F}$ and $p \in \mathbb{R}^N_+$, $p \in EB(V, f)$ if for each $S \in \mathcal{N}$, $p_S = f^S(V(S), d^S(V, p))$ and $p_S = d^S(V, p)$. Note that $EB \subset B$.

For each $i \in N$, let $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing continuous function such that $\lambda_i(0) = 0$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$. Let Λ be the set of all such function tuples. For each $\lambda \in \Lambda$ and $V \in \mathcal{V}^n$, $\lambda(V)$ is defined as follows: for each $S \in \mathcal{N}$, $\lambda_S(V(S)) = \{\lambda(p) \mid p \in V(S)\}$. A multicoalitional bargaining rule $F : \mathcal{V}^n \times \mathcal{F} \to \mathbb{R}^N_+$ satisfies *ordinal invariance* if for each $(V, f) \in \mathcal{V}^n \times \mathcal{F}$ and $\lambda \in \Lambda$, $\lambda(F(V, f)) = F(\lambda(V), f)$.

3. Extreme-Bennett rule

We start with a review of Bennett's (1997) main theorem which, in several aspects, is more general than this version. Bennett originally discusses payoff configurations and shows that at any *Bennett solution* they reduce to payoff profiles. For expositional purposes, we use this property in our definition.

Theorem 1 (Bennett, 1997). *For each $(V, f) \in \mathcal{V}^n \times \mathcal{F}$, $B(V, f) \neq \emptyset$. Moreover, for each $p \in B(V, f)$ and $i \in N$, there is $S \in \mathcal{N}$ such that $i \in S$ and $p_S \in V(S)$.*

We first observe that in a *Bennett solution*, a coalition cannot improve upon the outside options of its members if and only if the members' outside options are equal to their Bennett payoffs.

Lemma 2. *Let $(V, f) \in \mathcal{V}^n \times \mathcal{F}$. Let $p \in B(V, f)$ and $S \in \mathcal{N}$. Then, $d^S(V, p) \notin \operatorname{int}(V(S))$ if and only if $p_S = d^S(V, p)$.*

Proof. Since $p \in B(V, f)$, for each $S \in \mathcal{N}$ $p_s = f^S(V(S), d^S(V, p))$. Let $S \in \mathcal{N}$ and $d^S(V, p) \notin \operatorname{int}(V(S))$. If $d^S(V, p) \in V(S)$, by strong individual rationality of f^s and by strict comprehensiveness of $V(S) p_S = d^S(V, p)$. If $d^S(V, p) \notin V(S)$, by agreeing to disagree $p_S = d^S(V, p)$.

Now let $S \in \mathcal{N}$ and $p_S = d^S(V, p)$. Suppose $d^S(V, p) \in \operatorname{int}(V(S))$. Then, by Pareto optimality $p_S = f^S(V(S), d^S(V, p)) \neq d^S(V, p)$, a contradiction. □

This observation comes handy in the proof of the following result.

Theorem 3. *The extreme-Bennett rule, EB, is* ordinally invariant.

Proof. Let $(V, f) \in \mathcal{V}^n \times \mathcal{F}$ and $\lambda \in \Lambda$. Let $p \in EB(V, f)$. Then, for each $S \in \mathcal{N}$,

$$p_S = f^S\big(V(S), d^S(V, p)\big) = d^S(V, p).$$

Step 1. For each $T \in \mathcal{N}$ such that $i \in T$, $a_i(\lambda(V), \lambda(p)) = \lambda_i(a_i^T(V, p))$.

Case 1 *($p_T \in V(T)$).* Then $\lambda_T(p_T) \in \lambda_T(V(T))$ and $a_i^T(V, p) = p_i$. The former implies $a_i^T(\lambda(V), \lambda(p)) = \lambda_i(p_i)$. Therefore, $a_i^T(\lambda(V), \lambda(p)) = \lambda_i(a_i^T(V, p))$.

Case 2 *($p_T \notin V(T)$).* Then $\lambda_T(p_T) \notin \lambda_T(V(T))$. In this case $a_i^T(V, p) = \max\{0, q_i \mid (q_i, p_{T\setminus i}) \in V(T)\}$. Therefore, $a_i^T(V, p) \geqslant q_i$ for each $q_i \in \mathbb{R}_+$ such that $(q_i, p_{T\setminus i}) \in V(T)$. Since λ_i is increasing, $\lambda_i(a_i^T(V, p)) \geqslant \lambda_i(q_i)$. Now $(q_i, p_{T\setminus i}) \in V(T)$ if and only if $(\lambda_i(q_i), \lambda_{T\setminus i}(p_{T\setminus i})) \in \lambda_T(V(T))$. Thus,

$$\lambda_i\big(a_i^T(V, p)\big) = \max\big\{0, t_i \bigm| \big(t_i, \lambda_{T\setminus i}(p_{T\setminus i})\big) \in \lambda_T\big(V(T)\big)\big\} = a_i^T\big(\lambda(V), \lambda(p)\big).$$

Step 2. For each $S \in \mathcal{N}$ such that $i \in S$, $d_i^S(\lambda(V), \lambda(p)) = \lambda_i(d_i^S(V, p))$.

$$\begin{aligned} d_i^S\big(\lambda(V), \lambda(p)\big) &= \max\big\{a_i^T\big(\lambda(V), \lambda(p)\big) \bigm| i \in T \text{ and } T \neq S\big\} \\ &= \lambda_i\big(\max\big\{a_i^T(V, p) \bigm| i \in T \text{ and } T \neq S\big\}\big) = \lambda_i\big(d_i^S(V, p)\big). \end{aligned}$$

Step 3. For each $S \in \mathcal{N}$, $\lambda_S(p_S) = d^S(\lambda(V), \lambda(p))$.

For $i \in S$, $p_i = d_i^S(V, p)$. By Step 2, $\lambda_i(p_i) = \lambda_i(d_i^S(V, p)) = d_i^S(\lambda(V), \lambda(p))$.

Step 4. For each $S \in \mathcal{N}$, $\lambda_S(p_S) \notin \mathrm{int}(\lambda_S(V(S)))$.

Case 1 *($p_S \notin V(S)$).* Then, $\lambda_S(p_S) \notin \lambda_S(V(S))$. Thus, $\lambda_S(p_S) \notin \mathrm{int}(\lambda_S(V(S)))$.

Case 2 *($p_S \in V(S)$).* Then, $p_S \in \partial V(S)$. Thus, there is no $q_S \in V(S)$ such that $q_S > p_S$. Since for each $i \in S$, λ_i is increasing, there is no $q_S \in V(S)$ such that $\lambda_S(q_S) > \lambda_S(p_S)$. Since $q_S \in V(S)$ if and only if $\lambda_S(q_S) \in \lambda_S(V(S))$, there is no $t_S \in \lambda_S(V(S))$ such that $t_S > \lambda_S(p_S)$. Thus, $\lambda_S(p_S) \notin \mathrm{int}(V(S))$.

Step 5. $\lambda(p) \in EB(\lambda(V), f)$.

By Steps 3 and 4, for each $S \in \mathcal{N}$, $d^S(\lambda(V), \lambda(p)) \notin \mathrm{int}(\lambda_S(V(S)))$. Thus, by Lemma 2, $f^S(\lambda_S(V(S)), d^S(\lambda(V), \lambda(p)) = d^S(\lambda(V), \lambda(p)) = \lambda_S(p_S)$. This, by definition implies that $\lambda(p) \in EB(\lambda(V), f)$. □

We next demonstrate that the *extreme-Bennett rule* may be empty-valued.

Example 1. Let $N = \{1, 2, 3\}$. Let $v(12) = 3$ and $v(13) = v(23) = 1$. For each $S \in \mathcal{N}$ such that $|S| \neq 2$, let $v(S) = 0$. For each $S \in \mathcal{N}$, let

$$V(S) = \left\{p_S \in \mathbb{R}_+^S \Big/ \sum_S p_i \leqslant v(S)\right\}.$$

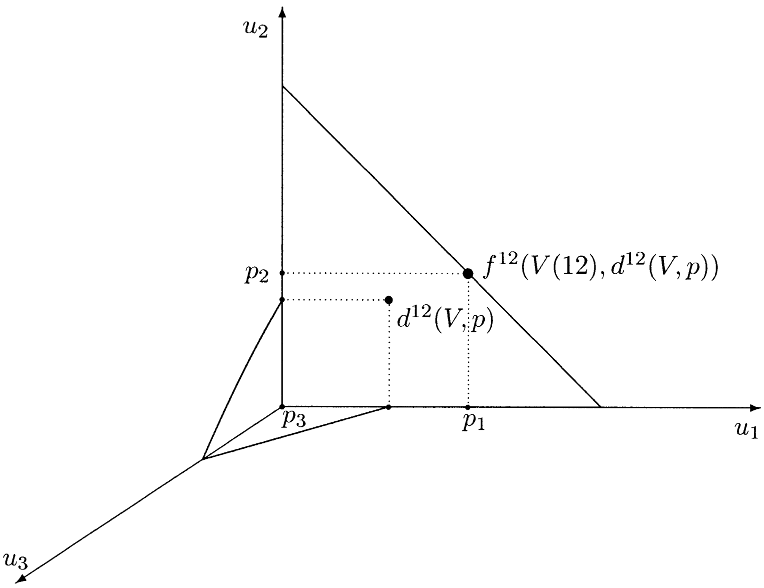

Fig. 1. Construction of Example 1.

The unique Bennett solution to this problem is $p = (f^{12}(V(12), (1,1)), 0)$ (see Fig. 1). Since $(1,1) \in \text{int}(V(12))$ and f^{12} is strongly individually rational, p is not an extreme-Bennett solution to this problem.

However, for an interesting class of problems, the *extreme-Bennett rule* is not only nonempty-valued, but also coincides with the *Bennett rule*. Therefore, on this subclass, the *Bennett rule* itself is *ordinally invariant*.

Theorem 4. *Let $m \in \mathbb{N}$ satisfy $n/2 < m < n$. If $(V, f) \in \mathcal{V}_m^n \times \mathcal{F}$, then $EB(V, f) = B(V, f)$.*

Proof. Let $(V, f) \in \mathcal{V}_m^n \times \mathcal{F}$. Note that by definition, $EB(V, f) \subseteq B(V, f)$. For the opposite inclusion, let $p \in B(V, f)$. If for each $S \in \mathcal{N}$ $p_S = d^S(V, p)$, $p \in EB(V, f)$. Let $|S| < m$. If $|S| < m$, $V(S) = \{0\}$ and the result follows from Lemma 2. Now let $|S| \geqslant m$. Suppose $p_S \neq d^S(V, p)$. Then by Lemma 2, $d^S(V, p) \in \text{int}(V(S))$ and by Pareto optimality $p_S \in \partial(V(S))$.

First let $|S| > m$. Let $T \subset S$ with $|T| = m$. Then, for each $i \in T$ $d_i^T(V, p) \geqslant p_i$. Since $p_T = f^T(V(T), d^T(V, p))$, strong individual rationality implies that for each $i \in T$ $p_i \geqslant d_i^T(V, p)$. Therefore, $p_T = d^T(V, p)$. Then by Lemma 2, $p_T \notin \text{int}(V(T))$. Since $p_S \in V(S)$, $T \subset S$, and $|T| = m$, $p_T \in V(T)$. Hence, $p_T \in \partial(V(T))$. Therefore, for each $i \in S \setminus T$ $p_i = 0$. Note that for each $i \in S$ there is a $T \subseteq S \setminus \{i\}$ such that $|T| = m$. This implies that for each $i \in S$ $p_i = 0$, contradicting $V(S) \neq \{0\}$ and $p_S \in \partial(V(S))$.

Now let $|S| = m$. Then, by strong individual rationality, for each $i \in S$ $p_i > d_i^S(V, p)$. Let $j \notin S$. By Theorem 1, there is $T \in \mathcal{N}$ such that $j \in T$ and $p_T \in V(T)$. If $T \cap S \neq \emptyset$, since for each $i \in T \cap S$ $p_i > d_i^S(V, p)$ $p_T \notin V(T)$. So $T \cap S = \emptyset$. But then $|T| < m$ and therefore $V(T) = \{0\}$. Thus, for each $j \notin S$ $p_j = 0$. Let $T \neq S$ be such that $|T| = m$. Since $p_T \notin V(T)$ and for each $j \in T \setminus S$, $p_j = 0$, $(p_{T\cap S}; 0_{T\setminus S}) \notin V(T)$. Since $V \in \mathcal{V}_m^n$, this implies that $(p_{T\cap S}, 0_{N\setminus(T\cap S)}) \notin V(N)$. By the same reason $(p_{T\cap S}, 0_{S\setminus T}) \notin V(S)$. By comprehensiveness of $V(S)$, $p_S \notin V(S)$, a contradiction. □

Theorem 4 assumes decisive coalitions to be majorities (i.e., $m > n/2$). This, however, is a natural assumption since, otherwise, multiple decisive coalitions can simultaneously form and contradict to each other. Also, *strong individual rationality* of individual coalitions' bargaining rules is essential for Theorem 4. We next demonstrate that when it is replaced with weak individual rationality (i.e., $f^S(V(S), d^S) \geqslant d^S$), the two rules do not coincide.

Example 2. Let $N = \{1, 2, 3\}$. For each $S \in \mathcal{N}$, if $|S| < 2$, let $V(S) = \{0\}$; otherwise, let $V(S) = \{p_S \in \mathbb{R}_+^S \mid \sum_S p_i = 1\}$. Let $p = (1, 1, 0)$. The payoff vector p is a Bennett solution for weakly individually rational bargaining rules. However, for coalition $S = \{2, 3\}$ the outside option vector is $d^S(V, p) = (0, 0)$ is not equal to the agreement point $p = (1, 0)$.

4. Bennett and the Shapley–Shubik rules

We first define the (three-agent) *Shapley–Shubik rule*. Let $N = \{1, 2, 3\}$ and $(V(N), \mathbf{0})$ be a bargaining problem. Let $x^0 \in \mathbb{R}_+^N$ satisfy $\{(x_1^0, x_2^0, 0), (x_1^0, 0, x_3^0), (0, x_2^0, x_3^0)\} \subset \partial(V(N))$. Next, let $0 \leqslant y^0 \leqslant x^0$ satisfy $\{(y_1^0, y_2^0, x_3^0), (y_1^0, x_2^0, y_3^0), (x_1^0, y_2^0, y_3^0)\} \subset \partial(V(N))$ (see Fig. 2). Iterating in this manner, for each $k \in \mathbb{N}$ let x^k and y^k satisfy

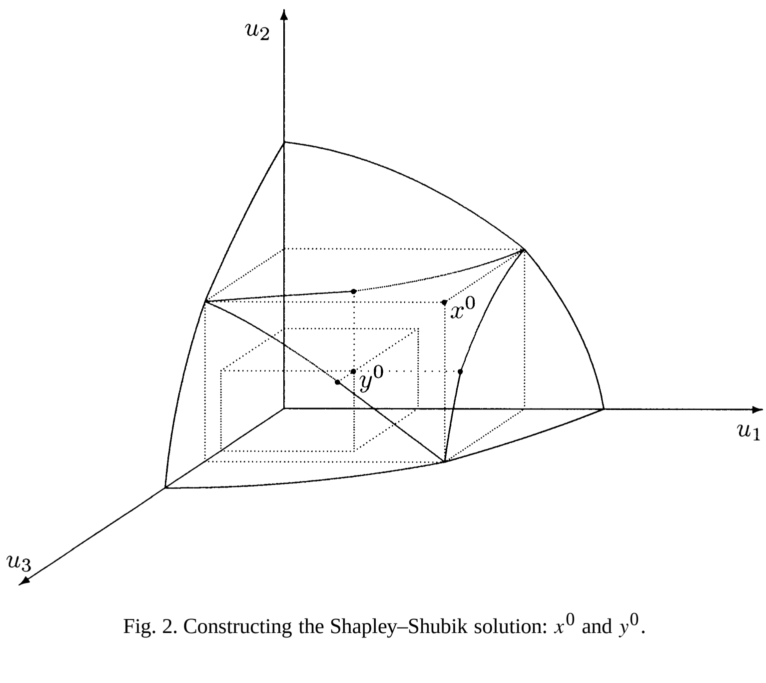

Fig. 2. Constructing the Shapley–Shubik solution: x^0 and y^0.

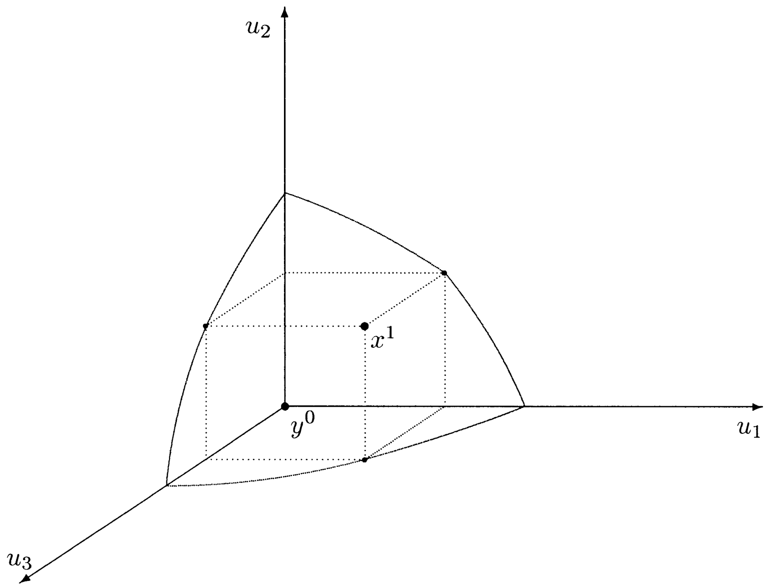

Fig. 3. Constructing the Shapley–Shubik solution: x^1.

$$\big\{\big(x_1^k, x_2^k, y_3^{k-1}\big), \big(x_1^k, y_2^{k-1}, x_3^k\big), \big(y_1^{k-1}, x_2^k, x_3^k\big), \big(y_1^k, y_2^k, x_3^k\big), \big(y_1^k, x_2^k, y_3^k\big), \big(x_1^k, y_2^k, y_3^k\big)\big\} \subset \partial\big(V(N)\big)$$

(see Fig. 3). For each $k \in \mathbb{N}$, x^k and y^k are uniquely defined. The *Shapley–Shubik solution* to $V(N)$ is the unique limit of both sequences $\{x^k\}$ and $\{y^k\}$.

As previously discussed, each bargaining problem, $(V(N), 0)$ is associated with a game V in $\mathcal{V}_2^3$. Multicoalitional problems obtained from such games have a unique *Bennett solution* that satisfies certain properties:

Lemma 5. *Let* $|N| = 3$ *and* $(V, f) \in \mathcal{V}_2^3 \times \mathcal{F}$. *Then* $B(V, f) = \{p\}$ *is such that* $(p_1, p_2, 0)$, $(p_1, 0, p_3)$, *and* $(0, p_2, p_3)$ *are all Pareto optimal in* $V(N)$.

Proof. Let $p \in B(V, f)$.

Step 1. For each $i, j \in N p_{\{i,j\}} \in \partial(V(\{i, j\}))$.

Suppose $p_{\{i,j\}} \notin \partial(V(\{i, j\}))$. By Pareto optimality of $f^{\{i,j\}}$, $p_{\{i,j\}} \notin V(\{i, j\})$. Since $V \in \mathcal{V}_2^3$, $(p_{\{i,j\}}, 0) \notin V(N)$. Therefore, $p \notin V(N)$. Moreover, by agreeing to disagree, $d^{\{i,j\}}(V, p) = p_{\{i,j\}}$. Now suppose $p_i = 0$. Then $p_j > 0$. This implies $p_j \notin V(j)$. Since both $V(\{i, j\})$ and $V(\{j, k\}\}$ are projections of $V(N)$, $(0, p_j) \notin V\{i, j\})$ implies $(p_j, 0) \notin V(\{j, k\})$. Therefore, $p_{\{j,k\}} \notin V(\{j, k\})$. But then there is no feasible coalition in which agent j gets p_j. This contradicts p being a *Bennett solution* to (V, f). Therefore, $p_{\{i,j\}} \gg 0$.

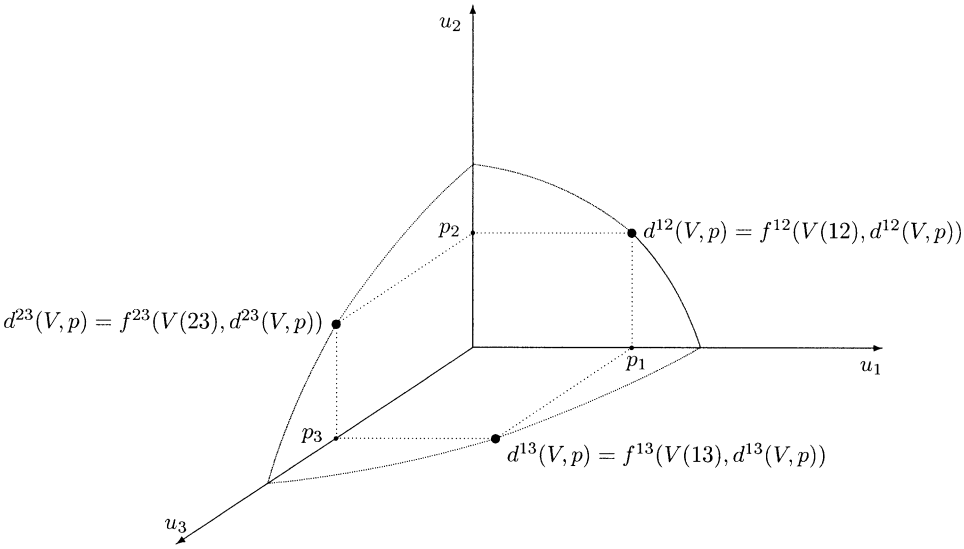

Fig. 4. In Lemma 5, $\{p\} = B(V, f) = OB(V, f)$.

Since $p_{\{i,j\}} \notin V(\{i, j\})$, $p \notin V(N)$, and $p_{\{i,j\}} \gg 0$ we have $p_{\{i,k\}} \in V(\{i,k\})$ and $p_{\{j,k\}} \in V(\{j,k\})$. Since $\{i,k\}$ is the only feasible coalition that contains i, $d_i^{\{i,k\}}(V, p) < p_i$. Since k gets p_k in both $\{i,k\}$ and $\{j,k\}$, $d_k^{\{i,k\}}(u, p) = p_k$. But these contradict strong individual rationality of $f^{\{i,k\}}$.

Step 2. $|B(V, f)| = 1$.

By Step 1, $p \in B(V, f)$ implies $\{(p_1, p_2, 0),\ (p_1, 0, p_3),\ (0, p_2, p_3)\} \subset \partial(V(N))$. For uniqueness, let $b^* = \max\{b_i \in \mathbb{R}_+ \mid (b_i, 0_{-i}) \in V(N)\}$ be the highest utility agent i can get in $V(N)$. For $i \in \{1, 2\}$ let $f_{i+1} : [0, b_i^*] \to \mathbb{R}_+$ be defined as $(x_i, f_{i+1}(x_i)) \in \partial(V(i, i+1))$. Let $f_1 : [0, b_3^*] \to \mathbb{R}_+$ be defined as $(f_1(x_3), x_3) \in \partial(V(1, 3))$. By strict comprehensiveness of $V(N)$, all these functions are well-defined. Moreover they are continuous an decreasing. Now let $f = f_1 \circ f_3 \circ f_2$. Then, f is a continuous, decreasing function on $[0, b_1^*]$ and it satisfies $f(0) = b_1^*$ and $f(b_1^*) = 0$. Therefore, it has a unique fixed point, p_1. Then $p_2 = f_2(p_1)$ and $p_3 = f_3(p_2)$ are also uniquely defined. □

Next, we present the main result of this section.

Proposition 6. *Let* $|N| = 3$ *and* $(V, f) \in \mathcal{V}_2^3 \times \mathcal{F}$. *The Shapley–Shubik solution* x *of* $(V(N), 0)$ *is the limit of a sequence of unique Bennett solutions.*

Proof. Let $\{x^k\}$ and $\{y^k\}$ be the sequences defining the Shapley–Shubik solution to $(V(N), 0)$. Let (V, f) be the multicoalitional bargaining problem associated with $(V(N), 0)$. Let $V^0 = V$. By Lemma 5, the problem (V^0, f) has a unique *Bennett solu-*

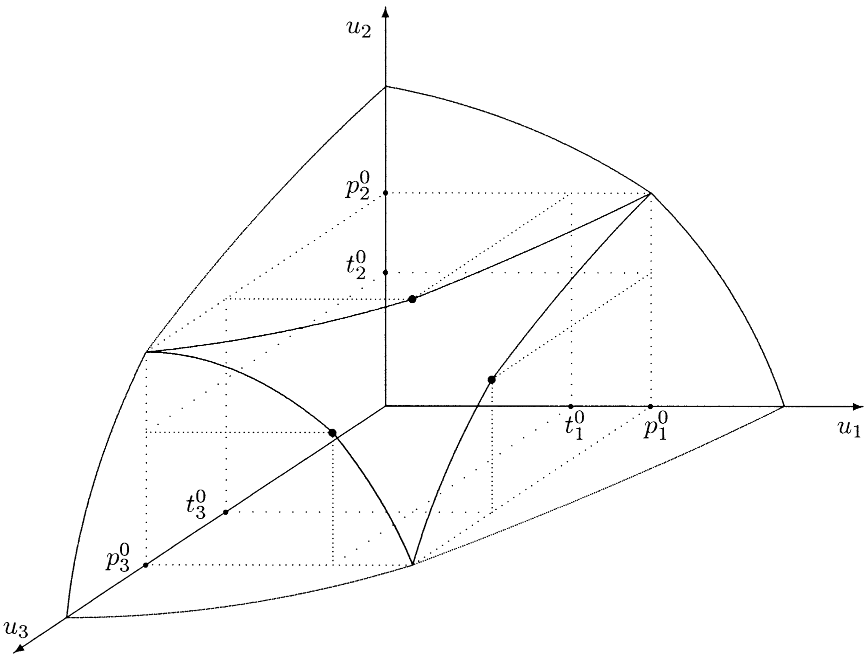

Fig. 5. Constructing p^0 and t^0 in the proof of Proposition 6.

tion. Let $\{p^0\} = B(V^0, f)$. Note that $\{(p_1^0, p_2^0, 0), (p_1^0, 0, p_3^0), (0, p_2^0, p_3^0)\} \subset \partial(V^0(N))$. Therefore, $p_0 = x^0$.

Let U^0 be defined as follows. For $S \in \mathcal{N}$ such that $|S| = 2$, let $U^0(S) = \{s \in \mathbb{R}_+^S \mid$ for $i \notin S$, $(s, p_i^0) \in V^0(N)\}$. Let $U^0(N) = \{0\}$ and for each $i \in N$, let $U^0(\{i\}) = \{0\}$. The proof of Lemma 5 can be used to show that the unique $\{t^0\} = B(U^0, f)$ satisfies $\{(t_1^0, t_2^0, p_3^0), (t_1^0, p_2^0, t_3^0), (p_1^0, t_2^0, t_3^0)\} \subset \partial(V(N))$. Therefore, $t^0 = y^0$ (see Fig. 5).

Given t^0, let the problem $(V^1(N), 0)$ be defined as $V^1(N) = \{x \in \mathbb{R}_+^N \mid x + t_S^0 \in V^0(N)\}$ (see Fig. 6). Let (V^1, f) be the multicoalitional problem associated with $(V^1(N), 0)$. By Lemma 5, (V^1, f) has a unique *Bennett solution*. Let $\{q^1\} = B(V^1)(u), f)$. Then, $\{(q_1^1, q_2^1, 0), (q_1^1, 0, q_3^1), (0, q_2^1, q_3^1)\} \subset \partial(V^1(N))$. Let $p^1 = q^1 + t^0$. Then, $\{(p_1^1, p_2^1, t_3^0), (p_1^1, t_2^0, p_3^1), (t_1^0, p_2^1, p_3^1)\} \subset \partial(V^0(N))$. Thus $p^1 = x^1$. Continuing in this manner, for each $k \in \mathbb{N}$ $p^k = x^k$ and $t^k = y^k$, the desired conclusion. $\square$

5. Conclusion

Standard bargaining solutions to allocation problems are manipulable via cardinal utility information. However, such solutions require a unanimous agreement. If, on the other hand the agreement of a majority is sufficient to implement an allocation, our results imply that the *Bennett solutions* to the resulting multicoalitional problem are immune to such manipulation.

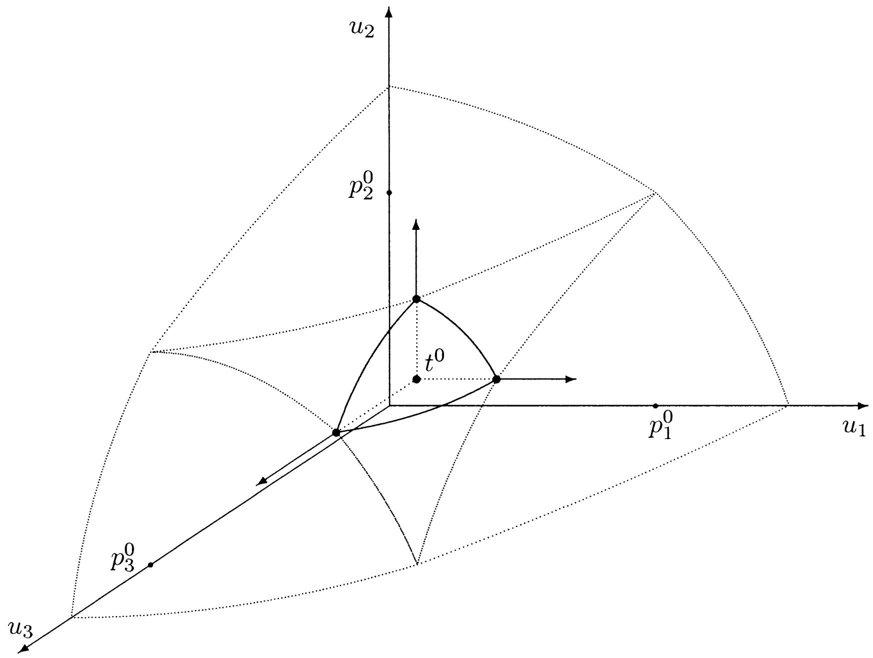

Fig. 6. The reduced problem is defined by t^0.

The *Bennett rule* is descriptive rather than normative: it does not rule out any outcome on the basis of desirability. The *extreme-Bennett rule*, since it is *ordinally invariant*, is a more desirable refinement. However, further axiomatic study is needed to analyze the implications of other desirable properties. Also, the relation between the *Bennett* and *Shapley–Shubik* rules may be useful in extending the latter's definition to more than three agents. Finally, the noncooperative interpretation of the *Bennett rule* may be useful in constructing a game that implements the *Shapley–Shubik rule.*

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