Rev. Econ. Design 8, 449–464 (2004)
DOI: 10.1007/s10058-004-0114-6

Review of Economic Design

Bargaining power in stationary parallelogram games

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Received: 15 January 2001 / Accepted: 4 December 2003

Abstract. A stationary variant of the repeated prisoners' dilemma in which the game frontier is a parallelogram is analyzed. By using the probabilistic cheap talk concept of [3], the discount factor becomes fungible, and for a critical value of the discount factor a unique Pareto-optimal and Pareto-dominant solution can be found. The relative bargaining power of the players can be quantified in terms of the shape of the parallelogram. If the parallelogram is asymmetric, the solution results in an asymmetric allocation of payoffs. Players with more bargaining power receive a greater share of the allocation. The solution satisfies some standard bargaining axioms within the class of parallelogram games. A characterization is provided in terms of these axioms and one new axiom, weak-monotonicity, which is in the spirit of, but different from, the Kalai-Smorodinsky restricted-monotonicity axiom.

1 Introduction

Consider a standard symmetric prisoners' dilemma game. In the plane formed by the payoffs for the two players, the feasible set is a diamond. With a typical set of payoffs, the joint cooperative point is on the 45 degree line, and the cooperate-defect points are symmetrically located. (In all that follows, the noncooperative point is assumed to be at the origin.) These four points are the apices of a symmetric parallelogram-shaped equilibrium allocation set that is attainable via mixed strategies. In this note the scope of these games is widened to include 2-player games with asymmetric payoffs, in which the diamond shape of the feasible set is expanded to include arbitrary parallelograms.

A cousin of the repeated game will be considered, namely games with random deadlines, as treated in [3], [2], and [1]. These games are called "Probabilistic Cheap Talk" (PCT) games. The games work as follows: each player chooses a strategy each

We thank Wayne Shafer for interlocution and helpful comments.

period. There is an invariant probability δ of the game ending in each period. If the game ends, the strategies, which are potentially mixed strategies, are implemented and the one-shot payoffs are distributed. If the game does not end, which happens with probability $1-\delta$, the strategies are not implemented, but they are publicly revealed; hence the name PCT. A subsequent round is then played.

The PCT games replicate repeated games, and the folk theorem applies with its surfeit of cooperative equilibria. This surfeit can be winnowed with a selection criterion, and a natural one emerges from PCT games. The structure of the game makes it possible to add two elements to the standard repeated game for this purpose. First, the probability of ending, δ, is equivalent to a discount factor (actually it is $1-\delta$ that corresponds to the discount factor), but it is much more fungible than the discount factor of a repeated game. Second, it is possible to focus on stationary play – that is, the game in which players' strategies are chosen once and for all at the beginning of play.

Stationarity can be imposed by assumption, but it is also the outcome of a simple mechanism, deviation-induced termination (DIT). In this mechanism the probability of termination δ becomes unity if either player is observed deviating from a stationary mixed strategy. This mechanism does not require that an enforcer view strategy outcomes or even the (mixed) strategy itself. Only the fact that a deviation from the initial mixed strategy has occurred must be detectable. If such a deviation does occur, the one-shot prisoners' dilemma is played in the final period. Since it is one-shot, the non-cooperative outcome is the only Nash equilibrium. This serves as a punishment for the deviating player, thus supporting the cooperative equilibria.[1]

In this paper, we analyze a subclass of PCT games. In these stage-games, a player's payoff change from switching strategies is independent of his opponents' strategies. Note that this is a property of the agents' von Neumann-Morgenstern preferences, not of their utility functions. However, the property implies the resulting sets of feasible payoff profiles to be parallelograms. We will hence call these games stationary parallelogram games.

It will be demonstrated below that stationary parallelogram games (SPG) – games whose game frontiers are parallelograms, even ones with asymmetric payoffs, have cooperative equilibria that are cones bounded by straight line segments emanating from $(0,0)$. There is a threshold value δ^* of the probability δ of the deadline occurring. For higher δ, the set of cooperative equilibria disappears altogether and only the noncooperative outcome of the one-shot game is an equilbrium. For lower δ, the cone widens until it encompasses all the individually rational payoffs in the payoff set – the classic folk theorem equilibria. At the threshold or critical value of δ, namely δ^*, the equilibrium set is a ray, $\rho(\delta^*)$, emanating from the origin. The ray has

[1] The DIT can be practically implemented. We can think of the players setting randomizing devices at the beginning of each round. Concretely, this could mean choosing a probability in a random number generator. Re-setting the random number generator, which would be done by a player deviating from stationarity, would be detectable.
The DIT mechanism is the implementation of a bargaining solution. The players must commit to the mechanism, and to stationarity in particular, in order to implement the solution.

an extremely simple algebraic characterization. There is a unique equilibrium point $\pi(\delta^*)$ on the Pareto frontier, and moreover it Pareto dominates all other equilibria.

Because the Pareto-dominant equilibrium point is unique, its comparative statics properties can be explored. The critical property for characterizing the equilibrium point is each player's marginal gain from defection relative to the cost of that defection imposed on the other player. We interpret this as a measure of each player's relative strength and we refer to it as that player's bargaining power.[2] Asymmetric bargaining power can be represented within the class of parallelogram games, and the player with the greater bargaining power receives a higher payoff. The details will be set out below.

In the symmetric parallelogram (diamond) game, analyzed in [3], the $\rho(\delta^*)$ ray is superimposed on the 45 degree line and so encompasses the full-cooperation payoff point. However when the parallelogram is not symmetric this is no longer the case: the minimal equilibrium set for the threshold value of δ does not encompass the full cooperation point. Its deviation from that point depends on the bargaining power of the players. The full cooperation outcome is almost incidental in the diamond game – the symmetry of the payoffs really reflects the equal bargaining power of the two players.

Again because the Pareto-dominant equilibrium is unique, we are able to follow the Nash [7] program of relating noncooperative and cooperative solution concepts, and interpret it as the outcome of a cooperative bargaining solution. We then axiomatically analyze this bargaining solution.

The bargaining power measure has two main properties. First, the slope of the δ^* ray is invariant with respect to parallel transformations of the payoff set of the game. This reflects a kind of independence property. Therefore the solution satisfies weaker versions of both Nash's [7] IIA axiom and Kalai and Smorodinsky's [5] monotonicity axiom. Monotonicity, along with other standard axioms – invariance, symmetry and Pareto-optimality – characterize our bargaining solution. Second, the bargaining power of the players also influences the critical δ which controls the expected duration of the game. Increasing the bargaining power of either player decreases the value of the critical δ, and hence increases the expected waiting time of the game. The DIT mechanism as a bargaining solution effectively discounts the payoffs of players with a high degree of bargaining power.

The next section of the paper sets out the model and the solutions are derived. Section 3 explores the solution properties in technical detail. Section 4 demonstrates that at the critical value of δ the Pareto-dominant equilibrium of the class of SPG games satisfies standard bargaining axioms. The characterization is set out in Sect. 5. The conclusion discusses some literature and empirical ramifications of the model.

The equilibria of rhomboid non-parallelogram games have more nonlinearity, resulting in curvature of the boundary of the equilibrium set. This nonlinearity arises from the asymmetry between the threats players can make in deviating from full cooperation and the threats they can make in punishing deviation by choosing full

[2] Many well-known bargaining solutions, such as the Nash [7] solution and the Perles-Maschler [10] solution, make use of this measure of bargaining power.

noncooperation. The symmetric case is treated in [3]; we leave the more general case for a separate paper.

2 The parallelogram game

There are two players, 1 and 2. Each has one of two actions C (cooperate) and D (defect) that can be chosen via a mixed strategy. The first argument of the payoff function $A_i(a_i, a_{\tilde{i}})$ for player i denotes player i's action, and the second argument is the action of the opposing player. The payoffs for player i at the apices of the game are : $A_i(D,D) = 0$; $A_i(C,D)$, $A_i(D,C)$, and $A_i(C,C)$. The prisoner's dilemma property requires

$$A_i(C,D) < 0 \qquad A_i(D,C) > A_i(C,C)$$

The parallelogram relationship is established by imposing the condition

$$A_i(C,C) - A_i(D,C) = A_i(C,D) - A_i(D,D) \tag{1}$$

The apices define line segments that form the game frontier F, and the payoff set S, the convex hull and interior defined by the apices, that is attainable with mixed strategies. The game will be denoted by $\textsc{spg}(S, \delta)$.

The payoff for player 1 in a mixed strategy equilibrium is

$$\begin{aligned} x_1 &= p_1 p_2 A_1(C,C) + p_1(1-p_2)A_1(C,D) \\ &\quad + (1-p_1)p_2 A_1(D,C) + (1-p_1)(1-p_2)A_1(D,D) \\ &= p_1p_2(A_1(C,C) - A_1(D,C) - A_1(C,D)) + p_1A_1(C,D) + p_2A_1(D,C) \\ &\quad -(p_1+p_2)A_1(D,D) + A_1(D,D). \end{aligned}$$

where p_i is the probability that player i cooperates in the mixed strategy. Since $A_i(D,D) = 0$ by assumption, and since the parallelogram relation holds, this simplifies to

$$x_1 = p_1 A_1(C,D) + p_2 A_1(D,C).$$

The parallelogram property eliminates the nonlinear products p_1p_2. Writing this equation and the corresponding one for player 2 and solving for the probabilities yields

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} A_1(C,D) & A_1(D,C) \\ A_2(D,C) & A_2(C,D) \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with solutions

$$\begin{aligned} p_1 &= \Delta^{-1}(A_1(D,C)x_2 - A_2(C,D)x_1); \\ p_2 &= \Delta^{-1}A_2(D,C)x_1 - A_1(C,D)x_2; \\ \Delta &= A_1(D,C)A_2(D,C) - A_1(C,D)A_2(C,D); \end{aligned}$$

These solutions will enter into the non-defection constraints.

Nondefection constraints. Player 1's defection is blocked if the payoff from defection from the mixed strategy equilibrium is dominated by the equilibrium payoff. That is, player 1 will not defect if the following condition holds:

$$\delta p_2 A_1(D, C) \leq x_1 \tag{2}$$

The quantity on the left is the payoff from defecting, weighted by the probability of the game ending. If the game does end, the defector receives that payoff. If the game does not end, he receives the non-cooperation payoff $A_1(D, D) = 0$ in the continuation game as a punishment.

Care must be taken in interpreting the quantity on the right. It is the value of purely stationary play with no current deviation but also no future deviation. In the general repeated game, the right hand side would implicitly include the option value of future defection. When the inequality holds with equality however, since defection and nondefection are the same, that option value is effectively zero; the inequality is therefore valid.

Stationarity also matters in the calculation of the right hand side. There are only two possible equilibria in each period: (i) stay with the stationary strategy or (ii) defect and revert to the one-shot game. If the decision to stay is made it will be made in every period, and so it is valid to calculate x_1 as the mixed strategy payoff.

Substituting from the solution for p_2 into (2), the condition becomes

$$\delta \Delta^{-1}(A_2(D, C)x_1 - A_1(C, D)x_2)A_1(D, C) \leq x_1.$$

After algebraic manipulation this becomes

$$\frac{x_2}{x_1} \leq \frac{A_2(D, C)}{A_1(C, D)} - \delta^{-1} \frac{\Delta}{A_1(D, C)A_1(C, D)}. \tag{3}$$

If equality holds, this is the equation for a ray from the origin that is the upper boundary of a cone.

The corresponding equation for player 2 is symmetric:

$$\frac{x_1}{x_2} \leq \frac{A_1(D, C)}{A_2(C, D)} - \delta^{-1} \frac{\Delta}{A_2(D, C)A_2(C, D)}.$$

If both the player 1 and player 2 equations hold with equality, the minimal nontrivial equilibrium is determined: it is a ray from the origin. The threshold value δ^* can be found by simultaneously solving the two equations for the cone edge when equality holds in both conditions. Multiplying the two equations together yields

$$1 = \left[\frac{A_2(D, C)}{A_1(C, D)} - \delta^{-1} \frac{\Delta}{A_1(D, C)A_1(C, D)}\right] \times \left[\frac{A_1(D, C)}{A_2(C, D)} - \delta^{-1} \frac{\Delta}{A_2(D, C)A_2(C, D)}\right].$$

After performing some algebraic manipulation, this can be expressed in intermediate form as a quadratic in δ:

$$\delta^2 \left(1 - \frac{A_2(D, C)A_1(D, C)}{A_1(C, D)A_2(C, D)}\right)$$

$$+\delta\left(\frac{A_2(D,C)}{A_1(C,D)}\frac{\Delta}{A_2(D,C)A_2(C,D)}+\frac{A_1(D,C)}{A_2(C,D)}\frac{\Delta}{A_1(D,C)A_1(C,D)}\right)$$
$$-\frac{\Delta^2}{A_1(D,C)A_1(C,D)A_2(D,C)A_2(C,D)}=0$$

After some more algebra, this reduces further to

$$(\delta-1)^2-\frac{A_1(C,D)A_2(C,D)}{A_2(D,C)A_1(D,C)}=0.$$

Plotting this as a function of δ shows that there will be a fractional root (or in extreme cases a negative root) and a root greater than unity; choose the fractional root. Taking the square root of both sides of the equation above yields the solution for the fractional root,

$$\delta^*=1-\left(\frac{A_2(C,D)}{A_1(D,C)}\frac{A_1(C,D)}{A_2(D,C)}\right)^{1/2} \tag{4}$$

The expression is in terms of slopes: the first ratio is the slope of the upper and lower faces of the game frontier from the perspective of player 1, and the second is the slope of the left and right faces of the game frontier, from the perspective of player 2.[3]

Recall that the payoff set is denoted S; implicitly δ^* is a mapping from sets to the reals, $\delta^*(S)$.

The folk theorem. Inspecting Eq. (3), as δ is lowered below δ^*, the equilibrium set becomes a cone; the cone widens and in the limit as δ approaches zero, the cone encompasses the entire positive quadrant. This is the set of equilibria generated by the folk theorem. Shrinking δ effectively reduces discounting by making the realization of the deadline less and less likely. This has the effect of increasing the set of equilibria. This is because punishment becomes less and less effectual as δ shrinks, since the punishment has an effect that is deferred to the future. The standard folk theorem is in this sense a negative result because it reflects the failure of the players to have sufficient bargaining power over each other.

[3] Observe that as the parallelogram becomes a rectangle, δ^* approaches zero: the outcome is attained with no delay. Beyond this point, namely if the (C,D) payoffs exceed the (D,C) payoffs, no fractional δ^* exists. Conversely, as the parallelogram becomes a flattened diamond and degenerates into a line, δ^* approaches unity, so that it takes an indefinite amount of time to attain the outcome. This extreme corresponds to a zero-sum game. Finally, for parameters such that the cooperation point is in the negative quadrant, the parallelogram becomes everted and it is then no longer a prisoners' dilemma.
Departures from the parallelogram payoff structure change the shape of the equilibrium set. If we hold the outside apices of the game frontier fixed – that is if we hold $A_i(D,C)$ and $A_i(C,D)$ fixed – and move the cooperation point inward toward the $(0,0)$ non-cooperation point, the equilibrium set becomes lens-shaped. In the extreme, Pareto dominance of the Pareto-optimal equilibrium is lost. Going in the other direction, if we move the cooperation point outward, the interior equilibria disappear en tirely, leaving only the noncooperation point and the Pareto-optimal point as equilibria. The value of δ^* changes in corresponding ways: as the noncooperation point moves inward, δ^* decreases and the expected duration of the game increases, and vice versa. These points are discussed in technical detail in [3].

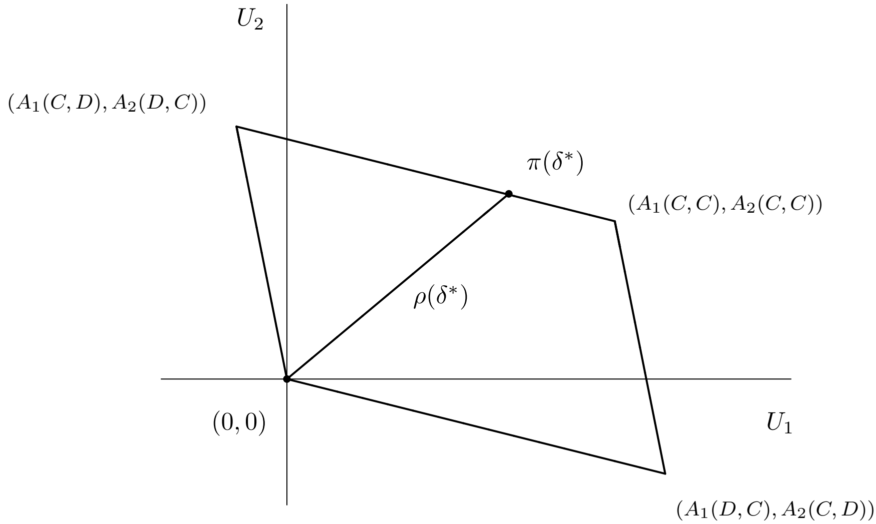

Fig. 1. An asymmetric parallelogram game

The minimal equilibrium set. Substituting the solution for δ^* from Eq. (4) into Eq. (3) yields the equation for the slope of the ray $\rho(\delta^*)$ of the minimal equilibrium set:

$$\frac{x_2}{x_1} = \left(\frac{A_2(D,C)/A_1(C,D)}{A_1(D,C)/A_2(C,D)}\right)^{1/2}. \tag{5}$$

The intersection of $\rho(\delta^*)$ with the game frontier is defined to be $\pi(\delta^*)$. The ratio in the numerator of (5) is the slope of the left and right parts of the game frontier; the bottom ratio is the inverse of the slope of the upper and lower segments of the game frontier.

The properties of the ratio in (5) are as follows. As the top ratio becomes large, the slope of the minimal set becomes steeper. This corresponds to player 2 receiving a larger and larger share of the payoffs. In the limit as the slope of the left and right segments tends to infinity, player 2 collects all the surplus. It should be kept in mind that the focus is only on the threshold δ^*, and that as the slopes change this threshold changes.

The reason for this limiting result is that player 1's bargaining power is decreasing. (See Fig. 1.) Player 1's bargaining power is inversely related to the slope of the $(A_1(C,C), A_2(C,C)) - (A_1(D,C), A_2(C,D))$ facet; the steeper the slope, the lower the marginal cost of providing cooperation to player 2 by player 1. That is, to move the equilibrium point along the steep facet of the game frontier requires a smaller sacrifice by player 1 than by player 2. Consequently player 2 must receive a higher marginal compensation.

Because only the slopes of the frontier matter, it is an incidental consequence of symmetry that full cooperation emerges as the boundary of the minimal equilibrium set. What is really going on in this case is an equal division rule induced by equal bargaining power. This can be verified by shifting one of the faces of the parallelogram parallel to itself, starting from the symmetric diamond case; because the slopes of the faces do not change, the equal division rule persists.

In the next section we develop the bargaining power measure more formally and characterize outcomes as algebraic functions of bargaining power.

3 Properties of SPG$(\delta^*(S))$

Some properties of asymmetric SPG's at the critical δ^* will now be stated, commencing with a definition of bargaining power.

Definition 1. *The bargaining power β_i of a player i is i's marginal gain from switching from cooperation to defection, relative to the the loss this change induces in the opponent's payoff, holding the action of i's opponent fixed.*

We provide a cooperative-game interpretation of this definition in the next section.

By this definition,

$$\beta_1 = \frac{A_1(D,C) - A_1(C,C)}{A_2(C,C) - A_2(C,D)} \qquad \beta_2 = \frac{A_2(D,C) - A_2(C,C)}{A_1(C,C) - A_1(C,D)} \tag{6}$$

The basic formulas of the SPG$(S, \delta^*(S))$ game can be expressed in terms of bargaining power:

$$\delta^* = 1 - (\beta_1 \beta_2)^{1/2} \tag{7}$$

$$\frac{x_1}{x_2} = \left(\frac{\beta_1}{\beta_2}\right)^{1/2} \tag{8}$$

Proposition 1. *$\rho(\delta^*)$ and δ^* are invariant with respect to parallel shifts of the game frontier.*

Proof. The result follows from algebraic calculation in Eqs. (4) and (5). □

The result reflects an independence property that will be discussed in detail in the next section. One consequence of the independence property is that by shifting one of the game frontiers in a parallel fashion, the invariant ray is induced to intersect the full-cooperation point of the game, that is, the cusp of the game frontier.

Proposition 2. *Consider the set of SPG games with fixed β_1 and β_2. Then there is a subset of these games such that $\pi(\delta^*)$ lies on the full-cooperation point of the game frontier.*

Proof. The result is a corollary of Proposition 2: one of the game frontiers can be shifted parallel until the intersection with the other frontier coincides with the ray $\rho(\delta^*)$. □

Duration. The expected duration of the game is ${\delta^*}^{-1}$. The next result shows how bargaining power influences the expected duration of the game.

Proposition 3. *The expected duration of the game is increasing in the bargaining power of the players.*

Proof. Substituting from Eq. (6) into the definition of the expected duration,

$$\frac{1}{\delta^*} = \frac{1}{1-(\beta_1\beta_2)^{1/2}} \qquad \square$$

Proposition 4. *The payoff for player i is increasing in his bargaining power.*

Proof. The result is a direct consequence of the formula (8). □

4 Cooperative solution concepts: definitions

We now extend our definitions to encompass cooperative bargaining. Recall that S is the set of ordered pairs of payoffs that are attainable with mixed strategies. Let $\mathcal{S}$ be the set of all such S which also have the parallelogram property (1).

A bargaining rule B is a mapping which assigns each $S \in \mathcal{S}$ to a $B(S) \in S$. Let $B^*(S)$ be the bargaining rule such that for each $S \in \mathcal{S}$, $B^*(S) = \pi(\delta^*(S))$, that is, $B^*(S)$ is the maximal point in S satisfying

$$\frac{B_2^*(S)}{B_1^*(S)} = \left(\frac{\beta_2}{\beta_1}\right)^{1/2} \tag{9}$$

Because B^* determines a unique Pareto-dominant equilibrium point, a comparison with axiomatic bargaining solutions with two players seems warranted.

The most well-known bargaining rules in the literature are the Nash, Kalai-Smorodinsky, Kalai-Rosenthal, Egalitarian, and the Utilitarian rules. On our domain Kalai-Smorodinsky and Kalai-Rosenthal rules coincide. If we normalize the payoffs so that $A_i(C,D) = -1$, the ratio of the agents' payoffs at the Kalai-Smorodinsky rule is

$$\frac{A_2(D,C) - \frac{1}{A_1(D,C)}}{A_1(D,C) - \frac{1}{A_2(D,C)}}$$

and it easily simplifies to $A_2(D,C)/A_1(D,C)$, which is the payoff-ratio at the Kalai-Rosenthal rule.[4]

Because the payoffs in B^* satisfy a ratio property as well, our solution seems related to these rules. Under the normalization $A_i(C,D) = -1$, the K-S payoffs are defined by the ratio

$$\frac{K_2(S)}{K_1(S)} = \frac{A_2(D,C)}{A_1(D,C)} \tag{10}$$

[4] Recall that the Kalai-Rosenthal rule chooses the maximal point with the ratio equal to the ratio of players' maximal payoffs, while the Kalai-Smorodinsky rule does the same with the ratio of the players' maximal individually rational payoffs.

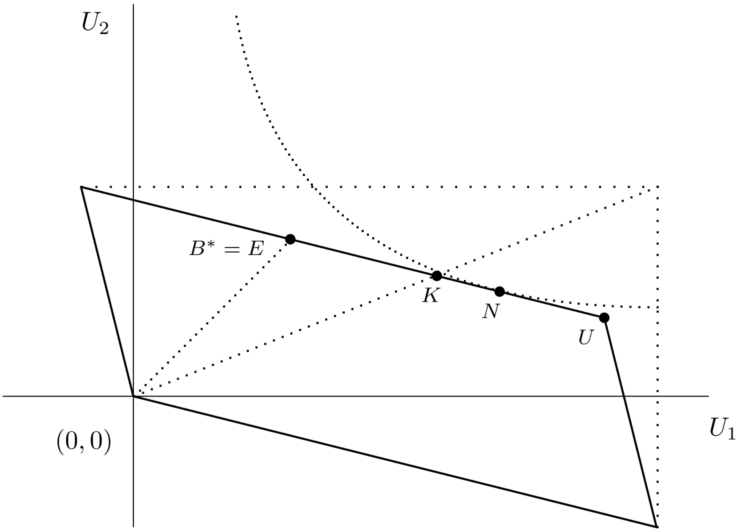

Fig. 2. Comparison of bargaining solutions

whereas the B^* payoffs are defined by the ratio

$$\frac{B_2^*(S)}{B_1^*(S)} = \left(\frac{A_2(D, C)}{A_1(D, C)}\right)^{1/2}. \tag{11}$$

The following figure presents a problem for which each of the rules proposes a different solution.

Note that K stands for both the Kalai-Smorodinsky and the Kalai-Rosenthal solutions to the problem. Because in Fig. 2 $\beta_1 = \beta_2$, the Egalitarian and the B^* rules coincide. However, a slight change in β_1 would move the B^* solution in either direction without affecting any of the other solutions.

We continue by setting out some standard definitions.

A bargaining rule $B : \mathcal{S} \to R_+^2$ is *Pareto optimal* (PO) if for each $S \in \mathcal{S}$ and $y \in \mathbb{R}_+^2$, $y \geq B(S)$ and $y \neq B(S)$ implies $y \notin S$. Nash, in his 1950 paper, assumes the players to be sufficiently rational not to pick a Pareto dominated profile. To motivate Pareto optimality, he writes "each individual wishes to maximize the utility to himself of the ultimate bargain". Very much like in bargaining, changing a Pareto dominated equilibrium (or agreement) requires the consent of both players in our domain (since both have to update their strategies).

The next property requires a bargaining rule to treat the agents identically if the bargaining problem is completely symmetric. It is based on the principle of anonymity but is considerably weaker. A bargaining rule B is *symmetric* (SYM) if for each $S \in \mathcal{S}$ such that $S = \{(a, b) \mid (b, a) \in S\}$ we have $B_1(S) = B_2(S)$.

Our next property requires the bargaining solution to be covariant to increasing linear transformations of the agents' utilities. If this is the case, the physical bargaining outcome is invariant under utility changes as long as the underlying cardinal preferences are unchanged. A bargaining rule B is *scale invariant* (SINV) if for each $S \in \mathcal{S}$ and $\alpha \in \mathbb{R}^2_{++}$, if $S' = \{(a,b) \mid a = \alpha_1 a' \text{ and } b = \alpha_2 b' \text{ for some } (a',b') \in S\}$then $B(S') = (\alpha_i B_i(S))_{i \in N}$.

Nash's *independence of irrelevant alternatives* (IIA) property requires that the solution to a bargaining problem should not change as some of the alternatives (other than the "relevant" ones: solution and the disagreement points) cease to be feasible: for each S, $S' \in \mathcal{S}$, $B(S') \in S \subseteq S'$ implies $B(S) = B(S')$. It has been frequently criticized on the basis that it requires the bargaining rule to be too insensitive to changes in the set of feasible utility profiles (e.g. [6] and [11]).

Monotonicity properties are based on the idea that gains should be mutual. They essentially state that an expansion in the set of feasible profiles should not make any agent worse off. The strongest formulation of this idea is as follows: a bargaining rule F is *strongly monotonic* (MON) if for each S, $S' \in \mathcal{S}$, $S \subseteq S'$ implies $B(S) \leq B(S')$. Among *Pareto optimal* and *symmetric* rules, the only one to satisfy strong monotonicity is the Egalitarian rule [4]. More specifically, no scale invariant (nondictatorial) rule satisfies the property. Such rules, however, satisfy a weaker version which restricts the comparison to problem pairs with identical ideal points. We will propose an alternative weakening that requires the players' bargaining powers to remain unchanged.

Now we can establish which of these properties are satisfied by B^*. It is immediately apparent from the formulas (4) and (5) that symmetry is satisfied. Although no mechanism has been included here for inducing Pareto optimality, the Pareto-dominance feature of the equilibrium set generates the Pareto-optimal solution if the selection mechanism includes Pareto dominance as a criterion, as noted in [3]. This will be interpreted here as satisfying PAR.

It can be demonstrated that INV is satisfied within the class SPG.

Proposition 5. *B^* satisfies INV.*

Proof. The preferences of the players are simply the payoffs that are the functions of the mixed strategy probability simplex. Without loss of generality consider an affine transformation of the payoffs for player 1:

$$A_1(C,C)' = a_1 + b_1 A_1(C,C) \qquad A_1(C,D)' = a_1 + b_1 A_1(C,D)$$
$$A_1(D,C)' = a_1 + b_1 A_1(D,C) \qquad A_1(D,D)' = a_1 + b_1 A_1(D,D)$$

The additive term a_1 laterally shifts the payoff set S, preserving parallelism, so it is possible to focus on the $a_1 = 0$ case without loss of generality. Parallelism is preserved by the linear transformation $b_1 A_1(i,j)$: from Eq. (1), parallelism holds if

$$A_1(C,C) - A_1(D,C) - A_1(C,D) = 0$$

The transformed preferences therefore also satisfy this property:

$$b_1 A_1(C,C) - b_1 A_1(D,C) - b_1 A_1(C,D) = 0$$

The critical δ solution of the SPG is

$$\delta^*(S') = 1 - (\beta_1 \beta_2)^{1/2}$$

Examining the formulas for β_i in (6) shows that the following formulas hold for bargaining power under the affine transformation:

$$\beta_1' = b_1\beta_1 \qquad \beta_2' = \frac{1}{b_1}\beta_2$$

Therefore $\delta^*(S') = \delta^*(S)$.

Finally, the ray $\rho(\delta^*)$ must be such that player 1's payoff increases in proportion to b:

$$\frac{x_1'}{x_2'} = \left(\frac{\beta_1'}{\beta_2'}\right)^{1/2} = \left(\frac{b_1\beta_1}{\frac{1}{b_1}\beta_2}\right)^{1/2} = b_1\frac{x_1}{x_2} \qquad \square$$

While B^* does not satisfy IIA, a weaker version of IIA is satisfied by B^*.

Definition 2. *A bargaining rule is weak IIA (WIIA) if for each S, $S' \in \mathcal{S}$ such that $\beta_1(S) = \beta_1(S')$, $\beta_2(S) = \beta_2(S')$ and $B(S') \in S \subseteq S'$, $B(S') = B(S)$.*

Proposition 6. *B^* satisfies WIIA.*

Proof. This is a direct corollary of Proposition 1. $\square$

The content of the proposition is that if a parallel shift of the game frontier F occurs such that the solution $\pi(\delta^*)$ lies on the new frontier as well, the solution remains fixed. Because the solution is, except for Pareto dominance, a non-cooperative one, this is an instance of the "Nash program" [8] linking non-cooperative and cooperative behavior. It is clear that the driving force of this correspondence is the stationarity assumption.

The result is illustrated in Fig. 3:

In the next section we demonstrate that a limited version of monotonicity holds, but B^* does not satisfy full monotonicity, as illustrated by the following proposition.

Proposition 7. *Let S be a parallelogram payoff set and let $P(S)$ be the Pareto set of S. Let $S \subset S'$, where S' is a parallelogram payoff set such that $\beta(S') \neq \beta(S)$ and such that $B^*(S) \in P(S')$. Then $B^*(S') \neq B^*(S)$.*

Proof. $\beta(S') \neq \beta(S)$ implies that $\rho(\delta^*(S')) \neq \rho(\delta^*(S))$. $\square$

The result is illustrated in Fig. 4.

The dashed lines in Fig. 4 show an increase in player 1's bargaining power while holding player 2's bargaining power fixed, and while preserving the relevant portion of the Pareto frontier. The result is that the ratio of payoffs is affected: player 1's payoff increases relative to player 2's payoff.

We next introduce a weaker monotonicity axiom:

Definition 3. *A bargaining rule is weakly-monotonic (WMON) if for each S, $S' \in \mathcal{S}$ such that $\beta_1(S) = \beta_1(S')$, $\beta_2(S) = \beta_2(S')$ and $S \subseteq S'$, $B(S) \leq B(S')$.*

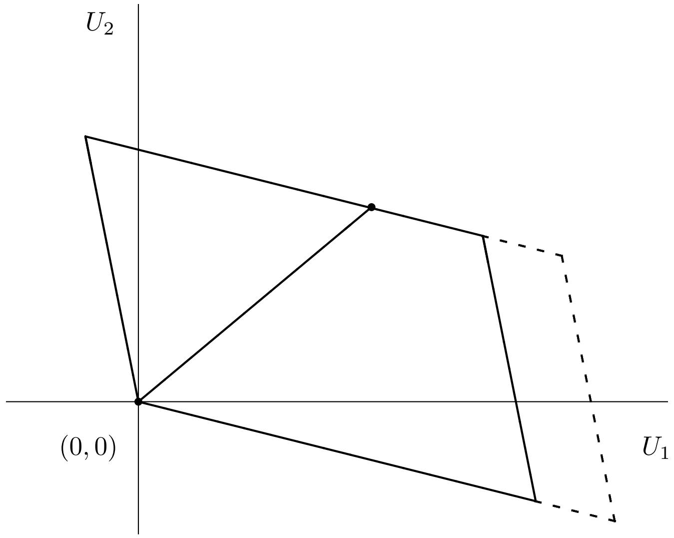

Fig. 3. Parallel expansion

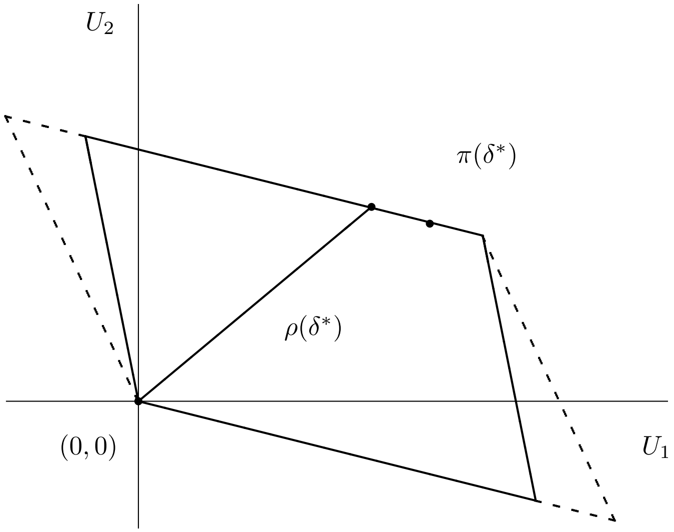

Fig. 4. B^* violates monotonicity

This axiom is in the same spirit as the restricted monotonicity characterizing the Kalai-Smorodinsky rule.[5] We next explore the implications of WMON.

[5] Restricted monotonicity says that if the feasible set expands in a way such that the agents' best individually rational alternative (i.e. their dictatorial alternatives) remain unchanged, then they should both weakly gain.

5 A characterization

We first establish that restricted monotonicity (WMON) and weak independence (WIIA) are independent axioms.

Claim 1. WIIA does not imply WMON.

Proof. The Nash bargaining rule, because it satisfies IIA, also satisfies WIIA. It, however, violates WMON. □

Claim 2. WMON does not imply WIIA.

Proof. Let $\widetilde{B}$ be the rule which, for each problem S, chooses the intersection of the ray $\rho(S)$ and the line segment connecting (C, D) and (D, C). This rule satisfies WMON. It, however, violates WIIA. □

Claim 3. WMON and PO imply WIIA.

Proof. Let $B(S') \in S \subseteq S'$ and $\beta(S) = \beta(S')$. Then by WMON, $B(S) \leq B(S')$. Since $B(S') \in S$ however, by Pareto optimality, $B(S) = B(S')$. □

Claim 4. PO, SYM, SINV, WMON are independent axioms.

Proof. The rule B_2^* satisfies all properties but Pareto optimality.

The weighted versions of B^* satisfy all properties but symmetry. For example, for any fixed $w \in \mathbb{R}^2_{++}$, consider B^w which chooses the maximal point in S satisfying

$$\frac{B_2^w(S)}{B_1^w(S)} = \left(\frac{w_2\beta_2}{w_1\beta_1}\right)^{1/2}.$$

The Egalitarian rule [4] satisfies all properties but scale invariance. The Nash rule [7] satisfies all properties but WMON. □

We now demonstrate that this weak monotonicity axiom characterizes B^*.

Proposition 8. *B^* is the only bargaining rule that satisfies PAR, SYM, INV, and WMON.*

Proof. It has already been demonstrated that SPG satisfies PAR, SYM and INV. To demonstrate WMON, note that if the β_i are kept constant, the ratio x_2/x_1 remains unchanged as well. Therefore, an expansion cannot make an agent worse off.

Uniqueness will now be demonstrated. Let B be a bargaining rule satisfying the given axioms. Let $S \in \mathcal{S}$. By scale invariance (INV), let S be such that $B^*(S) = (1, 1)$. Then $\beta_1 = \beta_2$. Just as in Proposition 3.3, a parallel shift of the game frontier can be undertaken, and as a result there is a unique symmetric problem $T \in \mathcal{S}$ such that $(1, 1)$ is Pareto optimal in T and for each $i \in N$ $\beta_i(T) = \beta_i(S)$. By Pareto optimality and symmetry, $B(T) = (1, 1) = B^*(T)$. Now observe that $T \subseteq S$. Therefore, $B(T) \leq B(S)$ by weak-monotonicity. But $B^*(S) = (1, 1)$ is Pareto-optimal in S. Therefore $(1, 1) = B(S)$. □

6 Conclusion

The solution set out here bears some resemblance to the alternating offer game of Rubinstein [12], and with additional assumptions that game generates a unique solution that is equivalent to the Nash [7] bargaining solution. Like SPG, Rubinstein's model also displays fixed discounting which results in a kind of stationarity of play. In addition, it admits asymmetric preferences, but in the of heterogeneous discount factors.[6] This asymmetry results in a kind of bargaining power that is positively related to the discount factor; players with higher discount factors get a larger share of the equilibrium allocation [9], p. 49].

There are differences as well. Rubinstein has a notion of the breakdown of play that superficially resembles the probability of ending in PCT games. The effect of this probability is to induce equilibrium in the initial period, in contrast to PCT, which results in actual delay in the equilibrium allocation. Additionally, equivalence of the Rubinstein game with the Nash bargaining solution occurs as the effective discount factor approaches unity; the equivalence of the $\pi(\delta)$ outcome of the SPG game with a bargaining solution (not the Nash solution) occurs at δ^*, and in general $\delta^* < 1$, entailing delay.

The B^* solution is not equivalent to the Nash solution: it cannot be described by the maximization of a Cobb-Douglas utility function over the feasible set. This can be seen by considering the contrapositive. If such a solution attains the cusp, then one of the facets of the game frontier can be expanded parallel, and the new Nash tangency will attain the new cusp, changing the ratio of payoffs; this contradicts the constant ratio of the payoffs in SPG games.

The SPG game, while not equivalent to the Nash bargaining outcome, does resemble the Kalai-Smorodinsky solution in that it can be represented by the maximization of Leontief preferences over payoffs. Starting from a parallelogram bargaining game and its solution B^*, we can expand the payoff set parallel to the original payoff set, so that the bargaining power β is preserved. Then the new solution lies on an extension of the ray $\rho(\delta^*)$. The same Leontief preferences then character ize the new solution as well. The slope of the ray characterizing the Leontief preferences is different from the Kalai-Smorodinsky ray however, due to the square-root aspect of the solution.

The papers [3], [2] and [1] noted that probabilistic cheap talk and the additional imposition of stationarity by a DIT mechanism was a simple and realistic mechanism describing many actual bargaining situations. The model here characterizes these games to the point that empirical tests are possible: if payoffs are in parallelogram form, a bargaining solution can be implemented. Moreover, unlike the Nash bargaining model, the SPG model also provides a prediction of the average duration of bargaining, which can be measured empirically. Both the bargaining solution allocations and the expected duration are functions of bargaining power, which, because it is a function of the payoff structure, is also empirically measurable.

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[6] The asymmetry can also appear in heterogeneity of the von Neumann-Morgenstern utility functions.

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