Strict Nash equilibrium:

Is (T,L) more likely in one of the tables?

		COLUMN		
		L	C	R
ROW	T	1, 1	1, 0	0, 1
	B	1, 0	0, 1	1, 0

		COLUMN		
		L	C	R
ROW	T	1, 1	1, 0	0, 0
	B	0, 0	0, 1	1, 0

Definition: a strict Nash Equilibrium of the game

$$G = (N , (S_1,\ldots,S_n) , (u_1,\ldots,u_n))$$

is a strategy profile $(s_1^*,\ldots,s_n^*)$ such that for **every** player i in N,

$$u_i(s_i^* , s_{-i}^*) > u_i(s_i , s_{-i}^*)$$

for **every** s_i in S_i different than s_i^* .

Symmetric two-player games:

A two-player game is symmetric if

1. $S_1 = S_2$
2. *for every a, b in* $S_1 = S_2$, $u_1(a, b) = u_2(b, a)$

Symmetric Nash equilibrium:

A Nash equilibrium $(s_1^*, \ldots, s_n^*)$ where $s_1^* = s_2^* = \ldots = s_n^*$

Game tables with three players:

TALIA chooses:

Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	5, 5, 5	3, 6, 3
	Don't	6, 3, 3	4, 4, 1

Don't Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	3, 3, 6	1, 4, 4
	Don't	4, 1, 4	2, 2, 2

1. **Find all Nash equilibria.**
2. **Find all strict Nash equilibria.**
3. **Is this game symmetric? Is it competitive?**

		COLUMN		
		A	B	C
ROW	A	1, 1	3, 1	1, -1
	B	1, 3	2, 2	7, 4
	C	–1, 1	4, 7	3, 3

EXERCISE 4.4

Minimax Method (only for zero-sum games)

		DEFENSE			
		Run	Pass	Blitz	
OFFENSE	Run	2	5	13	min = 2
	Short Pass	6	5.6	10.5	min = 5.6
	Medium Pass	6	4.5	1	min = 1
	Long Pass	10	3	–2	min = –2
		max = 10	max = 5.6	max = 13	

The equality test: Increase 5 to 6?

Best Response Analysis

The best response correspondence of agent i: $B_i(.)$

for each strategy profile of the other agents, s_{-i} ,

$B_i(s_{-i})$ is the set of agent i's strategies that **maximize** his payoff

i.e. the set of agent i's best responses to s_{-i}

$$B_i(s_{-i}) = \{ s_i^* \text{ in } S_i \text{ such that}$$

$$u_i (s_i^* , s_{-i}) \geq u_i (s_i , s_{-i})$$

$$\text{for all } s_i \text{ in } S_i \}$$

Typically, there are several best responses and $B_i(s_{-i})$ is a set.

In some games however, there will always be a single best response.

Only for such games, the best response function of agent i: $b_i(.)$

for each strategy profile of the other agents, s_{-i},

$b_i(s_{-i})$ is the strategy of agent i that **uniquely maximizes** his payoff

i.e. the unique best response of agent i to s_{-i}

$$b_i(s_{-i}) = s_i^* \text{ in } S_i \text{ such that}$$

$$u_i (s_i^* , s_{-i}) > u_i (s_i , s_{-i})$$

$$\text{for all } s_i \text{ in } S_i$$

Construct the best response correspondences.

		Student 2	
		Goof off (Defect)	Work hard (Cooperate)
Student 1	Goof off (Defect)	1, 1	3, 0
	Work hard (Cooperate)	0, 3	2, 2

Battle of the Sexes

		Wife	
		Football	Soap opera
Husband	Football	2, 1	0, 0
	Soap opera	0, 0	1, 2

The Tennis Game

		HINGIS	
		DL	CC
SELES	DL	50	80
	CC	90	20

Matching Pennies

		Veli	
		Head	Tail
Ali	Head	1, -1	-1, 1
	Tail	-1, 1	1, -1

The Stag hunt

		Hunter 2: Stag	Hunter 2: Hare
Hunter 1	Stag	2, 2	0, 1
Hunter 1	Hare	1, 0	1, 1

The Chicken Game

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	–1, 1
	Straight (Tough)	1, –1	–2, –2

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7

EXERCISE 4.7

		B		
		1	2	3
A	1	10, 10	0, 0	0, 0
	2	0, 0	15, 15	0, 0
	3	0, 0	0, 0	15, 15

EXERCISE 4.9

		B		
		L	C	R
A	U	0, 0	0, 5	0, 0
	M	5, 0	0, 0	–5, 0
	D	0, 0	0, –5	–5, –5

FIGURE 4.13 Lottery

Remember the Definition: a Nash Equilibrium of the game

$$G = (N, (S_1, \ldots, S_n), (u_1, \ldots, u_n))$$

is a strategy profile $(s_1^*, \ldots, s_n^*)$ such that for **every** player *i* in N,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

for **every** s_i in S_i.

New (equivalent) Definition: a Nash Equilibrium of the game

$$G = (N, (S_1, \ldots, S_n), (u_1, \ldots, u_n))$$

is a strategy profile $(s_1^*, \ldots, s_n^*)$ such that for **every** player *i* in N,

$$s_i^* \text{ is in } B_i(s_{-i}^*)$$

If all agents' best responses are functions,

the definition can be written as: a Nash Equilibrium of the game

$$G = (N, (S_1, \ldots, S_n), (u_1, \ldots, u_n))$$

is a strategy profile $(s_1^*, \ldots, s_n^*)$ such that for **every** player i in N,

$$s_i^* = b_i(s_{-i}^*)$$

This Nash equilibrium is in fact strict!

Go back to find Nash equilibria by using these new definitions.

Application: dividing money

Players: 1 and 2

Strategies: declare how much of 4$ you want

(Agent 1 declares x and Agent 2 declares y from {0,1,2,3,4})

Payoffs: if $x + y \leq 4$, (x , y)

if $x + y > 4$, and $x < y$, $(x , 4-x)$

if $x + y > 4$, and $x > y$, $(4-y , y)$

if $x + y > 4$, and $x = y$, $(2 , 2)$

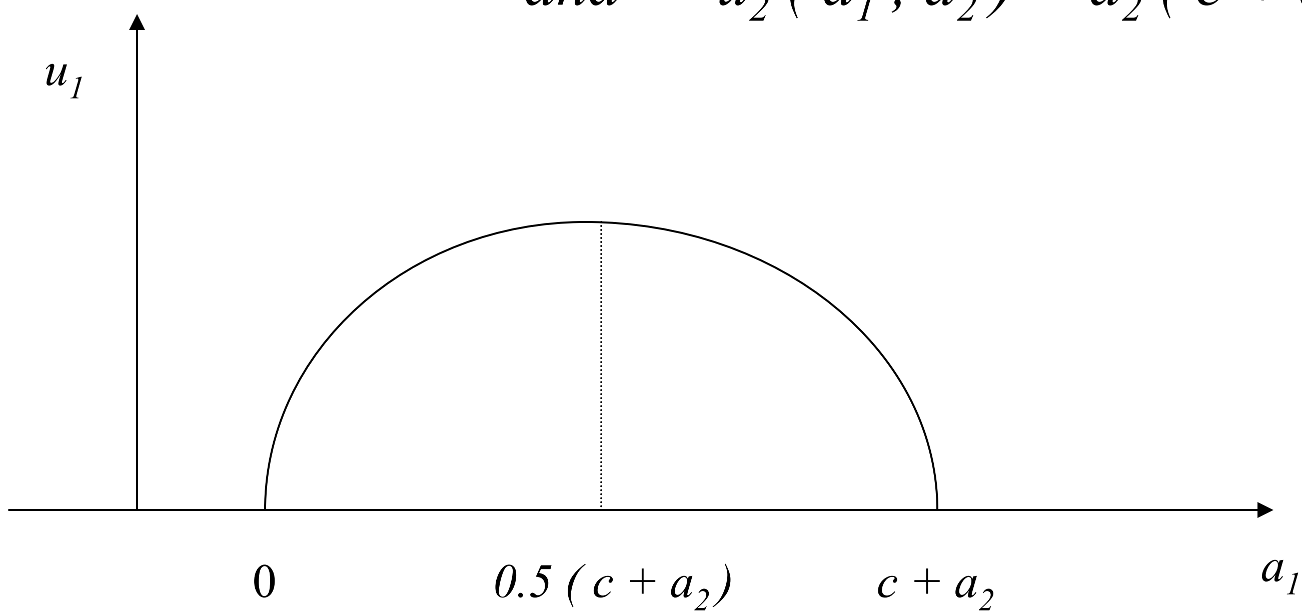
Application: A synergistic relationship

Players: 1 and 2

Strategies: choose effort level a_i (any nonnegative real number)

Payoffs: $u_1(a_1, a_2) = a_1(c + a_2 - a_1)$

and $u_2(a_1, a_2) = a_2(c + a_1 - a_2)$



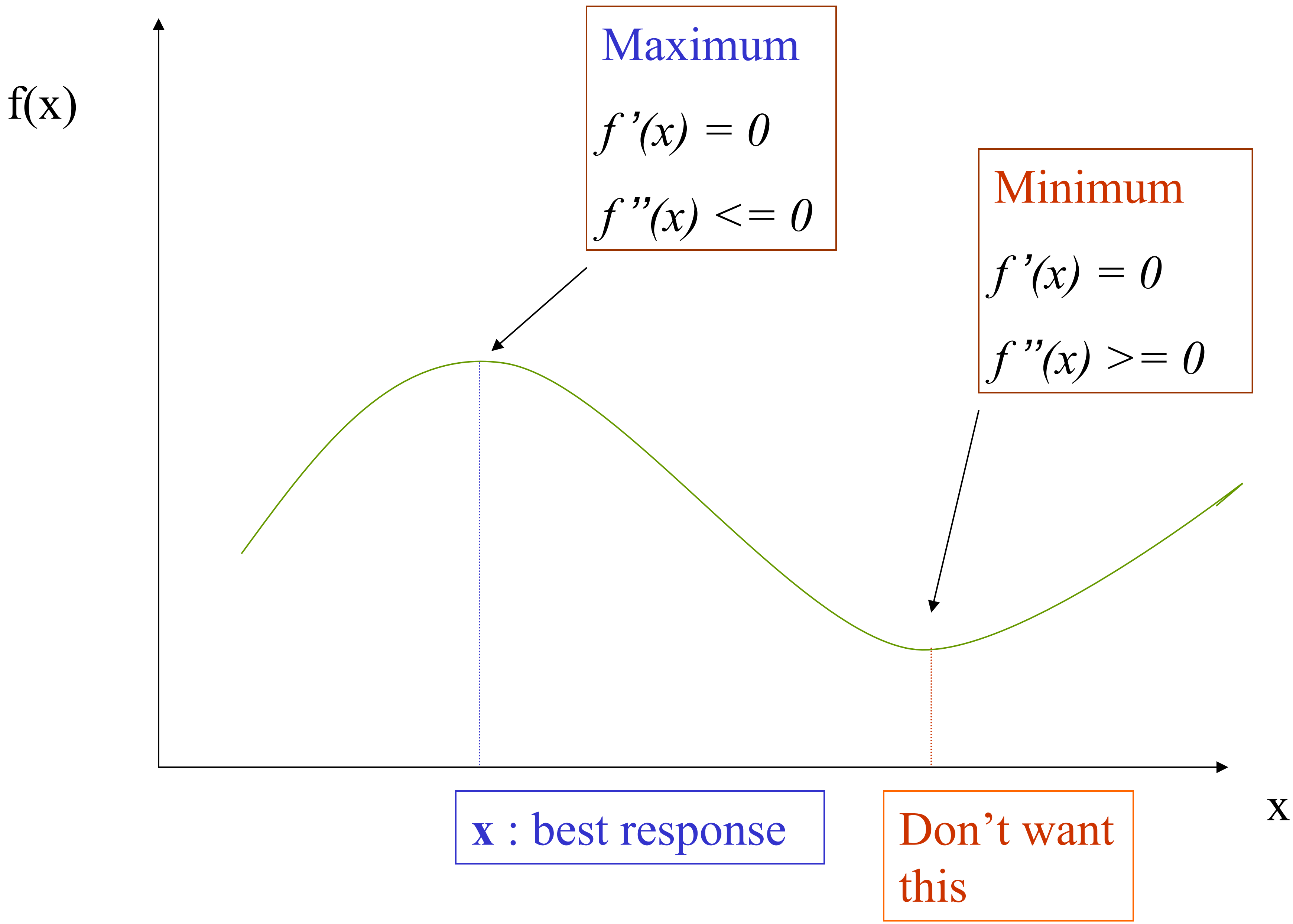
Or take derivatives (both first and second)

How to obtain the best-response functions from the players' payoff functions ?

1. Need to take derivatives of the payoff (profit) functions and equate them to ZERO

2. Need to check the second derivatives to make sure that they are maximized

3. Check the boundaries of the strategy set as well


f(x)
Maximum
f '(x) = 0
f ''(x) <= 0
Minimum
f '(x) = 0
f ''(x) >= 0
x : best response
Don't want this
x

In our example

$$u_1 (a_1 , a_2) = a_1 (c + a_2 - a_1)$$

The first derivative is equated to zero:

$$(c + a_2 - a_1) + a_1 (- 1) = 0$$

$$a_1^* = 0.5 (c + a_2)$$

Note that the second derivative is -2

So a_1^* maximizes i's payoff in response to a_2

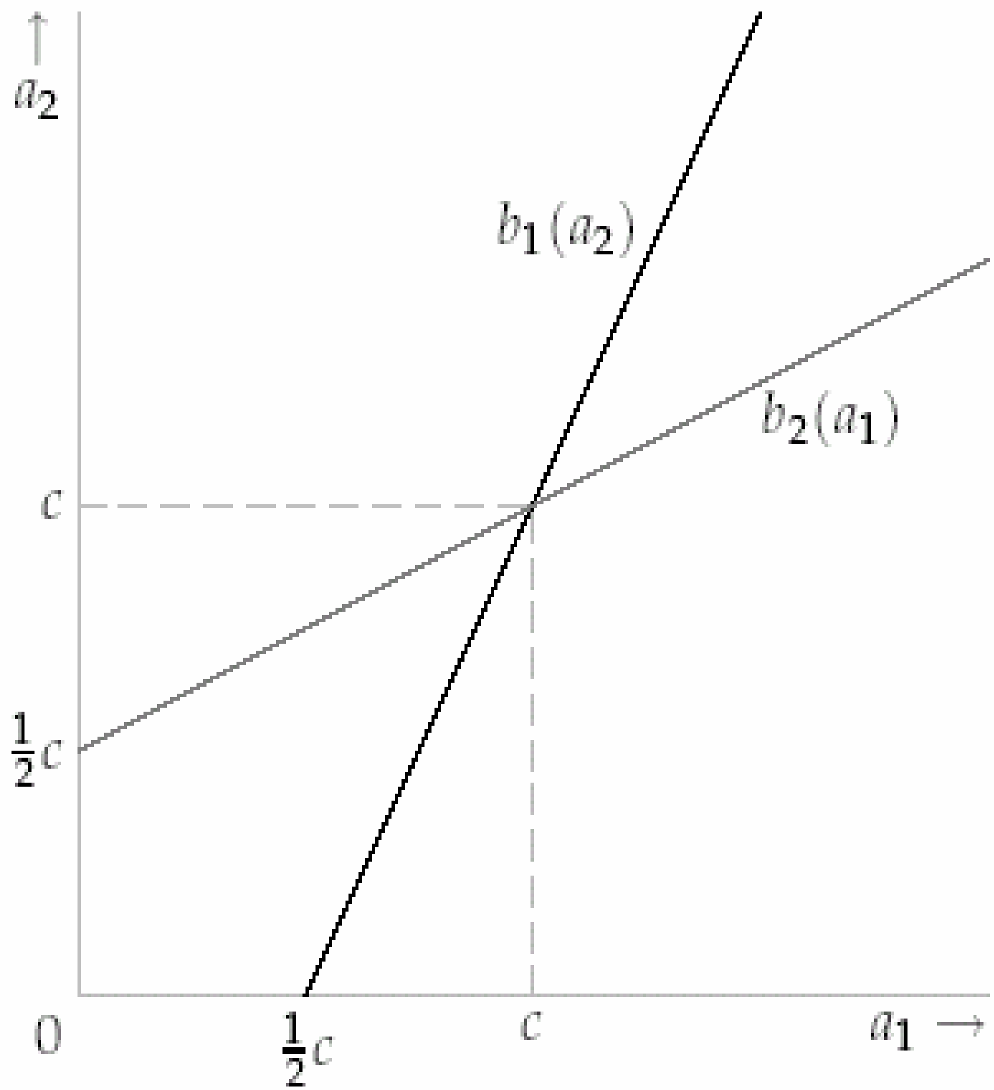
So a_1^* is a best response to a_2

The best response functions:

$$b_1(a_2) = 0.5(c + a_2)$$

Nash equilibrium (from the graph):

Draw the graphs of the two best response functions and check the intersection.



↑
a_2
$b_1(a_2)$
$b_2(a_1)$
c
$\frac{1}{2}c$
0
$\frac{1}{2}c$
c
a_1 →

The Nash equilibrium (from calculation):

(a_1, a_2) such that $a_1 = b_1(a_2) = 0.5(c + a_2)$

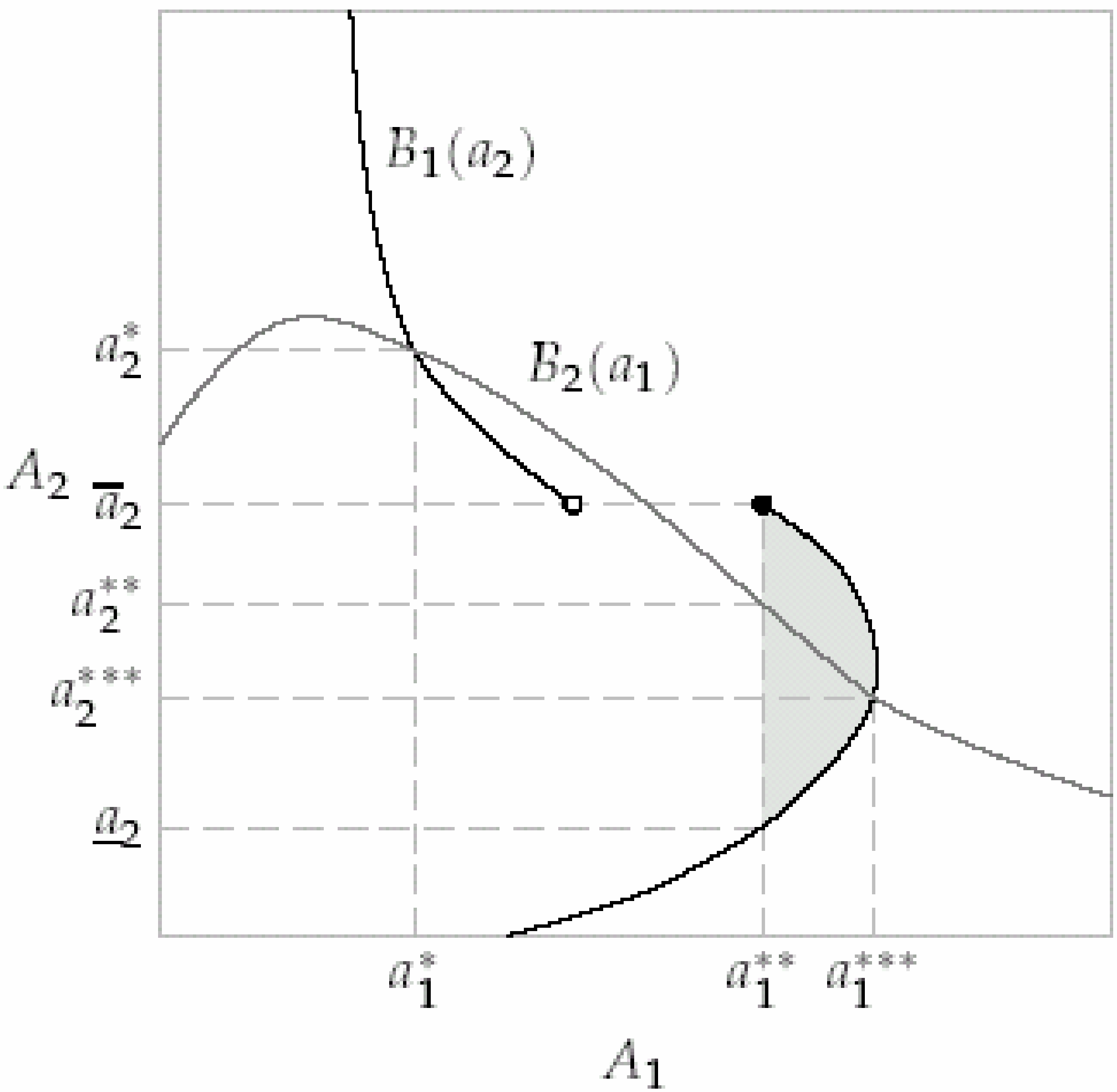
$a_2 = b_2(a_1) = 0.5(c + a_1)$

$$a_1 = 0.5(c + a_2) = 0.5(c + 0.5(c + a_1))$$

$$a_1 = c$$

$$(a_1, a_2) = (c, c)$$

Discontinuous and **thick** best response curves:



- Take the price-setting game between
  Donna's Deep Dish: chooses P_{Donna}
  Pierce's Pizza Pies: chooses P_{Pierce}

Market surveys show that given the prices each sells (in 1000 pizzas per week):

$$Q_{Donna} = 12 - P_{Donna} + 0.5P_{Pierce}$$

$$Q_{Pierce} = 12 - P_{Pierce} + 0.5P_{Donna}$$

Note: If Pierce increases his price, his sales go down and Donna's sales go up

- Cost of each pizza: 3 USD

- Pierce's profit (i.e. his **payoff**) (in 1000 USD) is then

$$\begin{aligned} Y_{Pierce} &= P_{Pierce}Q_{Pierce} - 3Q_{Pierce} \\ &= (P_{Pierce} - 3)Q_{Pierce} \\ &= (P_{Pierce} - 3)(12 - P_{Pierce} + 0.5P_{Donna}) \\ &= (15 + 0.5P_{Donna})P_{Pierce} - P_{Pierce}^2 - 36 - 1.5P_{Donna} \end{aligned}$$

- Given P_{Donna}, Pierce will choose his price to maximize his payoff

  Taking the derivative of Y_{Pierce} with respect to P_{Pierce}

$$\frac{dY_{Pierce}}{dP_{Pierce}} = 15 + 0.5P_{Donna} - 2P_{Pierce}$$

  When Y_{Pierce} is maximized, this derivative is equal to 0

$$\frac{dY_{Pierce}}{dP_{Pierce}} = 15 + 0.5P_{Donna} - 2P_{Pierce} = 0$$

Solving for P_{Pierce} we have

$$P^*_{Pierce} = \frac{15 + 0.5P_{Donna}}{2} = 7.5 + 0.25P_{Donna}$$

This is the **best-response function** of Pierce.

NOTE: We have to verify that what we found by equating the derivative to 0 is a maximum (it can also be a minimum or a saddle-point). For this, we must check if the second derivative at P^*_{Pierce} is negative:

$$\frac{d^2Y_{Pierce}}{dP^2_{Pierce}} = -2$$

So it's O.K.. We have maximized Y_{Pierce} at P^*_{Pierce} and therefore, we have a best-ersponse function.

Doing the same things for Donna, we obtain her best response function as

$$P^*_{Donna} = 7.5 + 0.25 P_{Pierce}$$

Note that the game is symmetric, so the best response function of Donna is similar to Pierce's

- Now we

  1. draw the two players' best response functions and
  2. find the intersection points of their graphs.
  3. The intersection points correspond to Nash equilibria of our game.

- Analytically, this is equivalent to solving the two best-response functions together:

$$
\begin{aligned}
P^*_{Pierce} &= 7.5 + 0.25P^*_{Donna} \\
&= 7.5 + 0.25(7.5 + 0.25P^*_{Pierce}) \\
&= 7.5 + 1.875 + 0.0625P^*_{Pierce}
\end{aligned}
$$

  This simplifies to

$$P^*_{Pierce} = 10$$

  By symmetry of the game, we also have

$$P^*_{Donna} = 10.$$

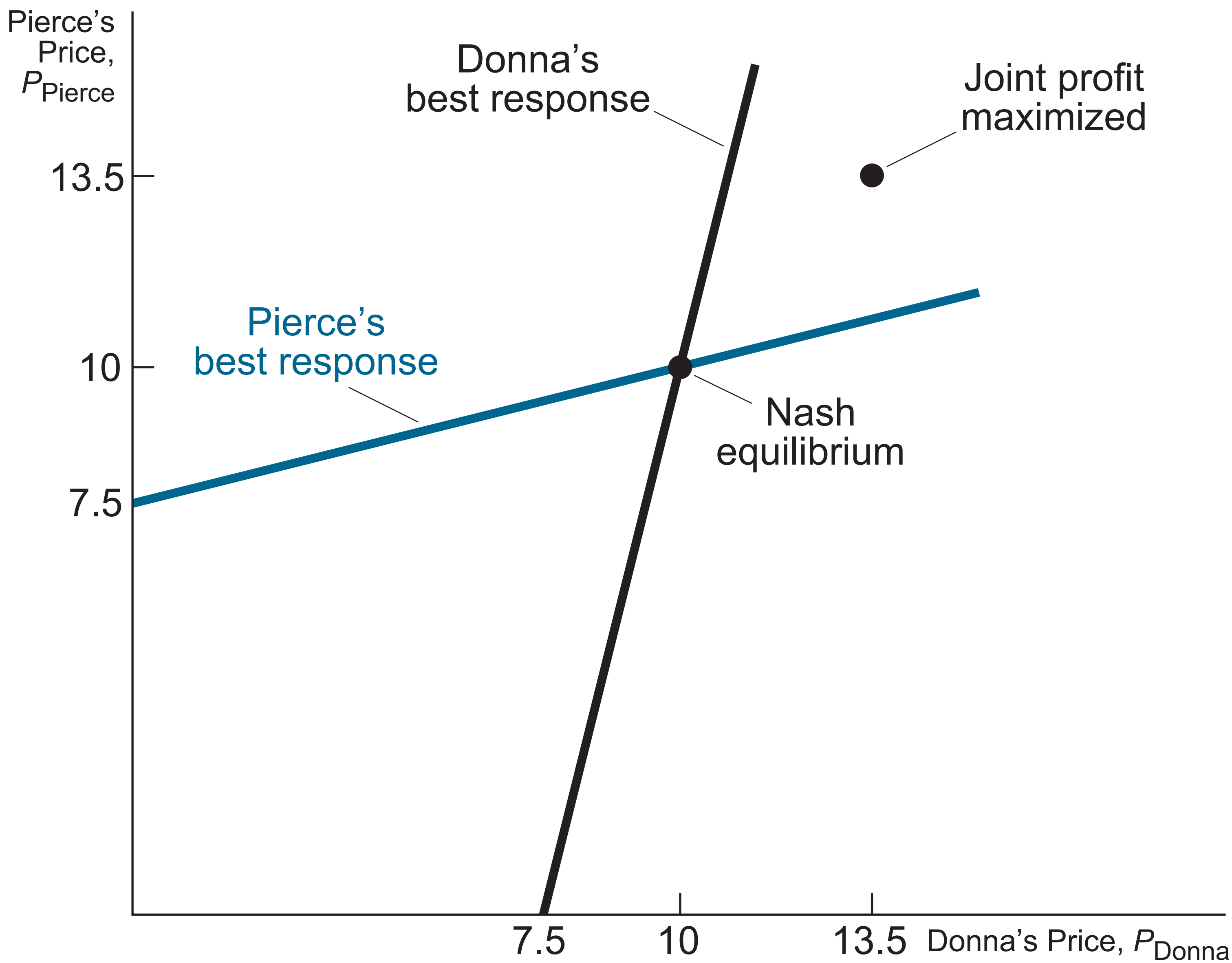


FIGURE 4.7 Best-Response Curves and Equilibrium in the Pizza Pricing Game