Simultaneous-move games with mixed strategies

Until now, the strategy of each agent corresponded to a single action. Such strategies in simultaneous-move games are called **pure strategies**.

Ex: (prisoners' dilemma)

Cooperate

Both pure strategies

Defect

No equilibrium: no pair of actions best response to each other

		HINGIS	
		DL	CC
SELES	DL	50	80
	CC	90	20

pure strategy in a simultanous-move game = action

also = expectation on other's action

Take a step back in time:

Hingis' expectation on other's action is uncertain.

Thus

Hingis' best-response is uncertain

We want to model this uncertainty

To handle these issues, we will introduce:

1. **mixed strategies**

2. **mixed strategy Nash equilibrium**

What is a mixed strategy?

It specifies that a pure strategy be chosen randomly

that is,

it assigns probabilities to the agent's pure strategies.

Choose a mixed strategy =
choose a probability (of playing) for each pure strategy

A mixed strategy is a probability distribution on pure strategies

Intuition for a mixed strategy

A population of agents (say tennis players)

Interpretation 1:

Some percentage p of them plays DL all the time

The remaining percentage (1-p) of them plays CC all the time

Interpretation 2:

Each agent sometimes plays DL (p percent of the time)

and sometimes plays CC ((1-p) percent of the time)

Example: (chicken game between Mercedes and BMW drivers)

Some Mercedes drivers always Swerve, some always go straight

Example:(tennis game) sometimes DL, sometimes CC

Example: (the chicken game)

I throw a coin. If heads, I swerve; if tails, I go straight.

So a mixed strategy is two probability numbers:

1/2 for Swerve (**in general** p for Swerve)

1/2 for Straight (**in general** 1-p for Straight)

NOTE:

Every **pure strategy** is a degenerate mixed strategy

It simply says that this pure strategy be chosen 100% of the time.

That is, you assign the probability number 1 to that pure strategy and the probability number 0 to all other strategies.

A (strategic) game with mixed strategies is

1. A set of players N
2. For each player i in N, a set of his strategies: ?
3. For each player i in N, his payoff function: ?

Defining mixed strategies

S_i the set of pure strategies of player i

$\Pi(S_i)$ the set probability distributions on S_i

= the set of mixed strategies of player i

π_i in $\Pi(S_i)$ is a typical mixed strategy for i

$\pi_i(s_i)$ the probability of player i playing pure strategy s_i

$\pi_i : S_i \rightarrow [0,1]$ is a function such that the sum of $\pi_i(s_i)$ numbers is 1

$$\sum_{s_i \in S_i} \pi_i(s_i) = 1$$

Defining payoffs of mixed strategies

$u_i(\,.\,)$ player i's payoffs from pure strategies

Example: $u_i(\,s_i\,,\,s_{-i}\,)$

$U_i(\,.\,)$ player i's payoffs from mixed strategies

Example: $U_i(\,\pi_i\,,\,\pi_{-i})$

Important assumption:

$U_i(\,\pi_i\,,\,\pi_{-i})$ is the **expected payoff** of *i* from lottery $(\,\pi_i\,,\,\pi_{-i}\,)$

it is a weighted average of $u_i(\,s_i\,,\,s_{-i}\,)$ values where

the weight of $u_i(\,s_i\,,\,s_{-i}\,)$ is $\pi_1\,(s_1).\,\pi_2\,(s_2).\;\ldots\;\pi_n\,(s_n)$

The expected payoff of a lottery

Example:

Say you will get payoff X_1 with probability p_1

X_2 with probability p_2

……….

X_n with probability p_n

Then your expected payoff is the weighted average:

$$p_1 X_1 + p_2 X_2 + \ldots + p_n X_n$$

Agent i's expected payoff from the mixed strategy profile: $(\pi_1, \ldots, \pi_n)$

$$U_i(\pi_1, \pi_2, \ldots, \pi_n) = \sum_{\substack{(s_1,\ldots,s_n) \\ \in S_1 \times \ldots S_n}} u_i(s_1, \ldots, s_n)\ \pi_1(s_1)\ \ldots\ \pi_n(s_n)$$

The summation is over all pure strategy profiles.

Agent i's payoff from the pure strategy profile $(s_1, \ldots, s_n)$

The probability of the pure strategy profile $(s_1, \ldots, s_n)$ being played

$\pi_{Seles}(DL) = 0.4 \qquad \pi_{Seles}(CC) = 0.6$

$\pi_{Hingis}(DL) = 0.7 \qquad \pi_{Hingis}(CC) = 0.3$

		HINGIS	
		0.7 DL	0.3 CC
SELES	0.4 DL	0.28 50	0.12 80
	0.6 CC	0.42 90	0.18 20

$$U_i(\pi_1, \pi_2) = 0.28 \; 50 \; + 0.12 \; 80 \; + 0.42 \; 90 \; + 0.18 \; 20$$

$$= 65$$

Seles

$\pi_1(DL) = p \qquad \pi_1(CC) = (1-p)$

$\pi_2(DL) = q \qquad \pi_2(CC) = (1-q)$

		HINGIS	
		DL	CC
SELES	DL	50	80
	CC	90	20

$U_i(\pi_1, \pi_2) =$

A (strategic) game with mixed strategies is

1. A set of players N
2. For each player i in N,

   a set of his pure strategies: S_i

   and from that a set of his mixed strategies: $\Pi(S_i)$

3. For each player i in N,

   his payoff function on pure strategies: $u_i(\,.\,)$

and from that his payoff function on mixed strategies: $U_i(\,.\,)$

Payoffs are no more ordinal

With pure strategies, two games are equivalent if in them players' ranking of outcomes are identical

Ex: prisoner's dilemma and students doing a joint project

This is no more true when mixed strategies are allowed

Because, the players' ranking of lotteries might be different in the two games.

Example:

Consider the lottery which gives 0.25 probability to each cell

		COLUMN	
		L	C
ROW	T	2, 1	1, 0
	B	1, 0	-2, 1

		COLUMN	
		L	C
ROW	T	8, 1	1, 0
	B	1, 0	-2, 1

Nash equilibrium in mixed strategies:

Specify a mixed strategy for each agent

that is, choose a mixed strategy profile

with the property that

each agent's mixed strategy is a

best response

to her opponents' strategies.

Intuition for mixed strategy Nash equilibrium

It is a steady state of the society in which the frequency of each action is fixed

(with pure strategies it was a fixed action instead)

Seles vs. Hingis: a zero-sum game with no pure strategy Nash eq.

What would the expectations be in reality?

		HINGIS	
		DL	CC
SELES	DL	50	80
	CC	90	20

FIGURE 5.1 Seles's Success Percentages in the Tennis Point

Extending the game

Allow mixed strategies π_H for Hingis:

play DL with probability q and

play CC with probability (1-q)

If Seles plays DL, her expected payoff is

$$U_{Seles}(DL, \pi_H) = 50\,q + 80\,(1-q)$$

If Seles plays CC, her expected payoff is

$$U_{Seles}(CC, \pi_H) = 90\,q + 20\,(1-q)$$

Extending the game table for the column player

An infinite number of new columns

		HINGIS		
		DL	CC	q-Mix
SELES	DL	50	80	$50q + 80(1 - q)$
	CC	90	20	$90q + 20(1 - q)$

Given a q-choice for Hingis, what will Seles choose?

FIGURE 5.4 Payoff Table with Hingis's Mixed Strategy

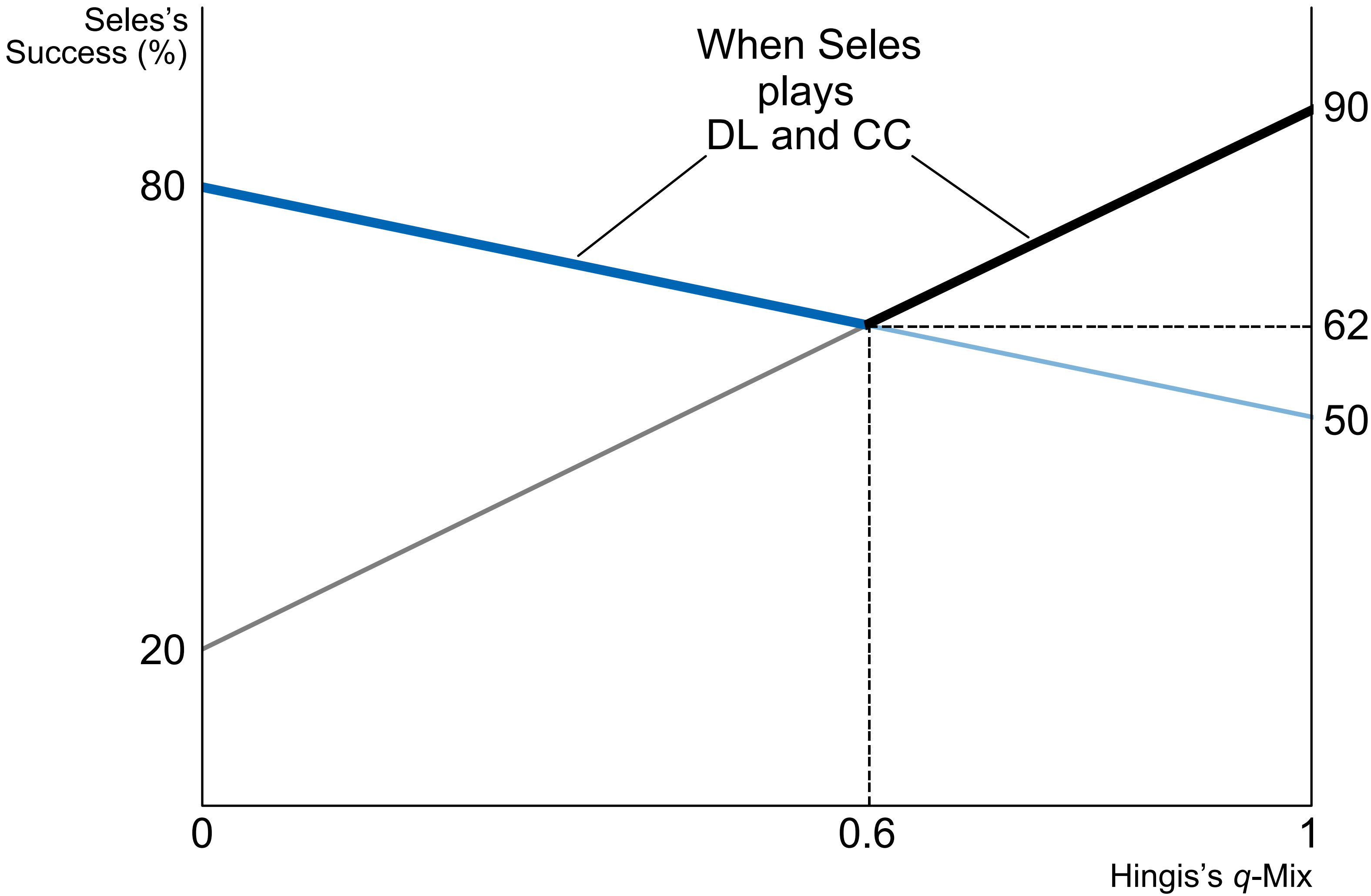


FIGURE 5.5 Diagrammatic Solution for Hingis's *q*-Mix

eles' best responses to different q choices of Hingis:

Against q s.t. $q < 0.6$ — Seles plays DL

Against q s.t. $0.6 < q$ — Seles plays CC

Against $q = 0.6$ — Both pure strategies best response

If two pure strategies are both best responses, then any mixture of them is also a best response

Fix $q = 0.6$

Then $U_{Seles}(DL, \pi_H) = U_{Seles}(CC, \pi_H) = 62$

Fix a mixed strategy π_S for Seles:
play DL with probability p and
play CC with probability (1-p)

Seles' expected payoff from (π_S, π_H) is

$$U_{Seles}(\pi_S, \pi_H) = pq\, U_{Seles}(DL, DL) + p(1-q)\, U_{Seles}(DL, CC) + (1-p)q\, U_{Seles}(CC, DL) + (1-p)(1-q)\, U_{Seles}(CC, CC)$$

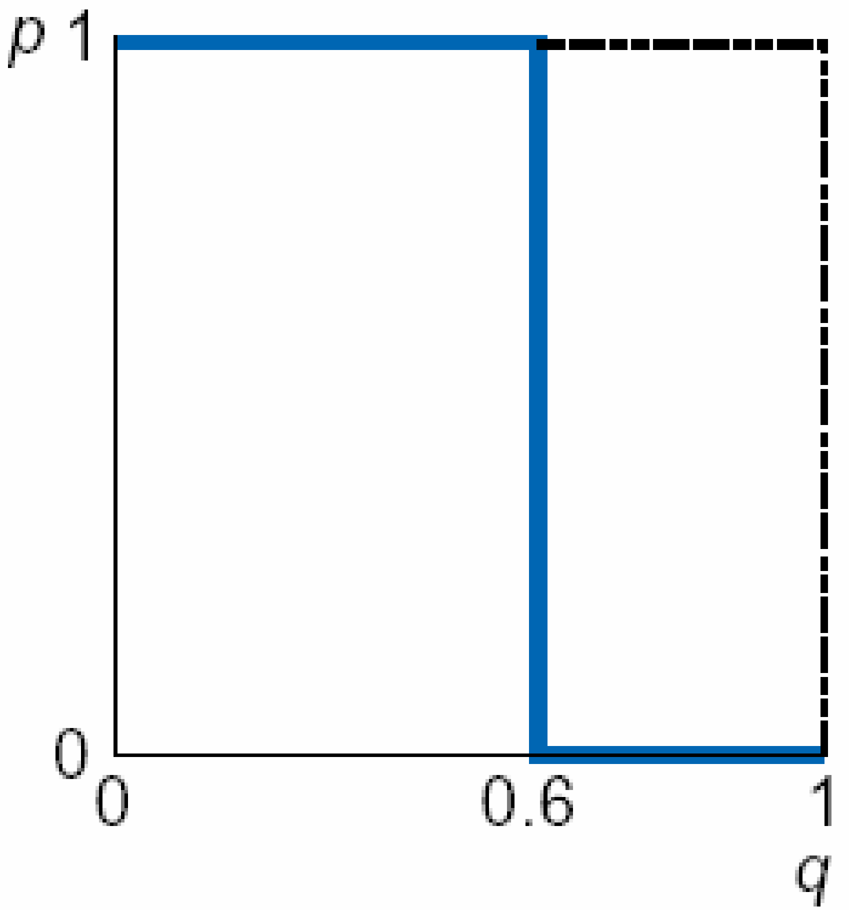
$$= p\, U_{Seles}(DL, \pi_H) + (1-p)\, U_{Seles}(CC, \pi_H)$$

$$= p\, 62 + (1-p)\, 62 = 62\,.$$

Seles' best responses to different q choices of Hingis:

Against q s.t. $q < 0.6$	Seles plays $p = 1$
Against q s.t. $0.6 < q$	Seles plays $p = 0$
Against $q = 0.6$	Seles plays any p in [0,1]

The best response curve of Seles:



What will Hingis do?

Fix a mixed strategy π_S for Seles:

play DL with probability p and

play CC with probability (1-p)

If Hingis plays DL, her expected payoff is:

$$U_{Hingis}(\pi_S, DL) = 100 - [50p + 90(1-p)] = 100 - U_{Seles}(\pi_S, DL)$$

If Hingis plays CC, her expected payoff is:

$$U_{Hingis}(\pi_S, CC) = 100 - [80p + 20(1-p)] = 100 - U_{Seles}(\pi_S, CC)$$

Extending the game table for the row player

An infinite number of new rows

		HINGIS	
		DL	CC
SELES	DL	50	80
	CC	90	20
	p-Mix	$50p + 90(1 - p)$	$80p + 20(1 - p)$

Given a p-choice for Seles, what will Hingis choose?

FIGURE 5.2 Payoff Table with Seles's Mixed Strategy

Seles' payoffs (Hingis' payoffs are 100 minus these)

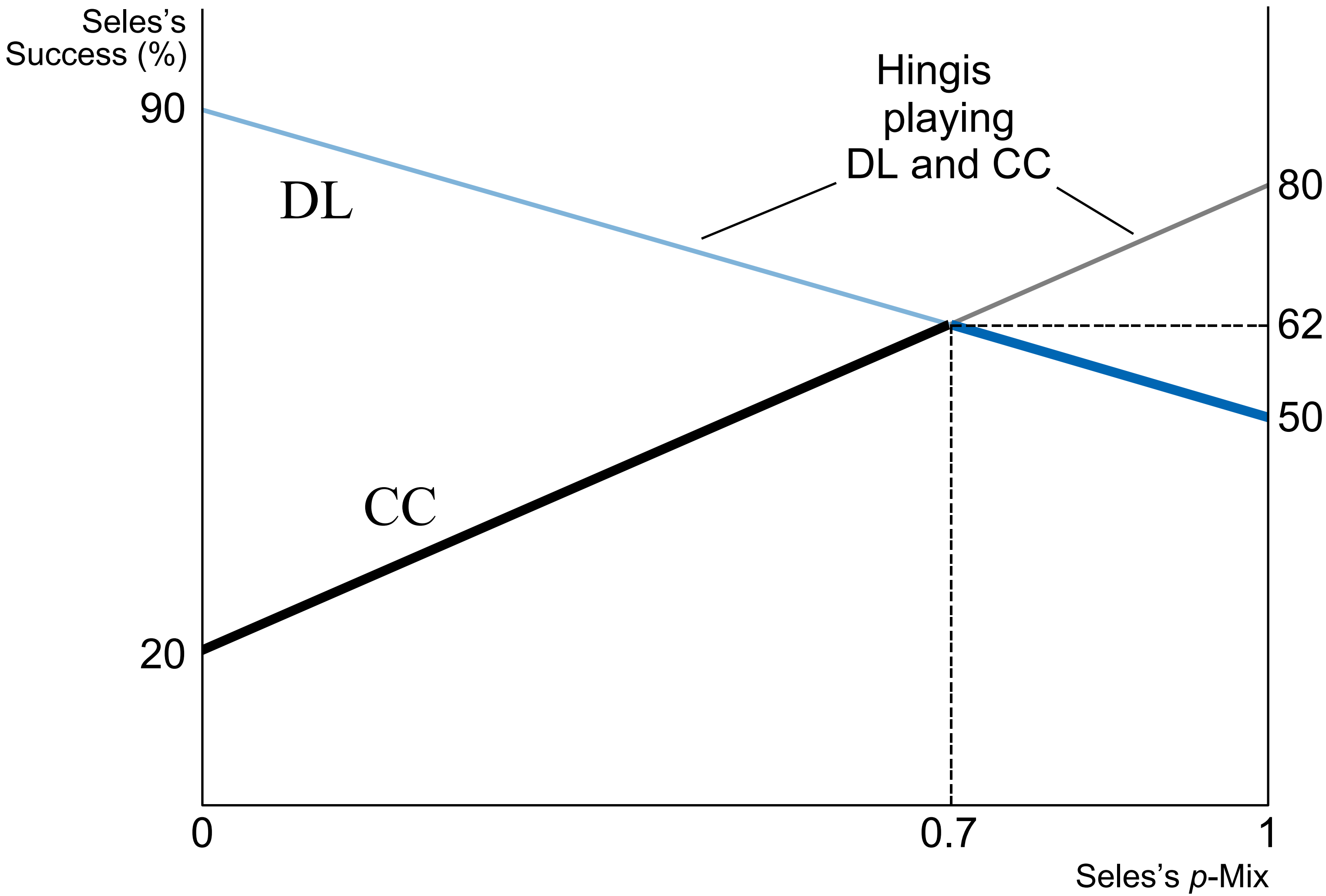


FIGURE 5.3 Diagrammatic Solution for Seles's *p*-Mix

Hingis' payoffs

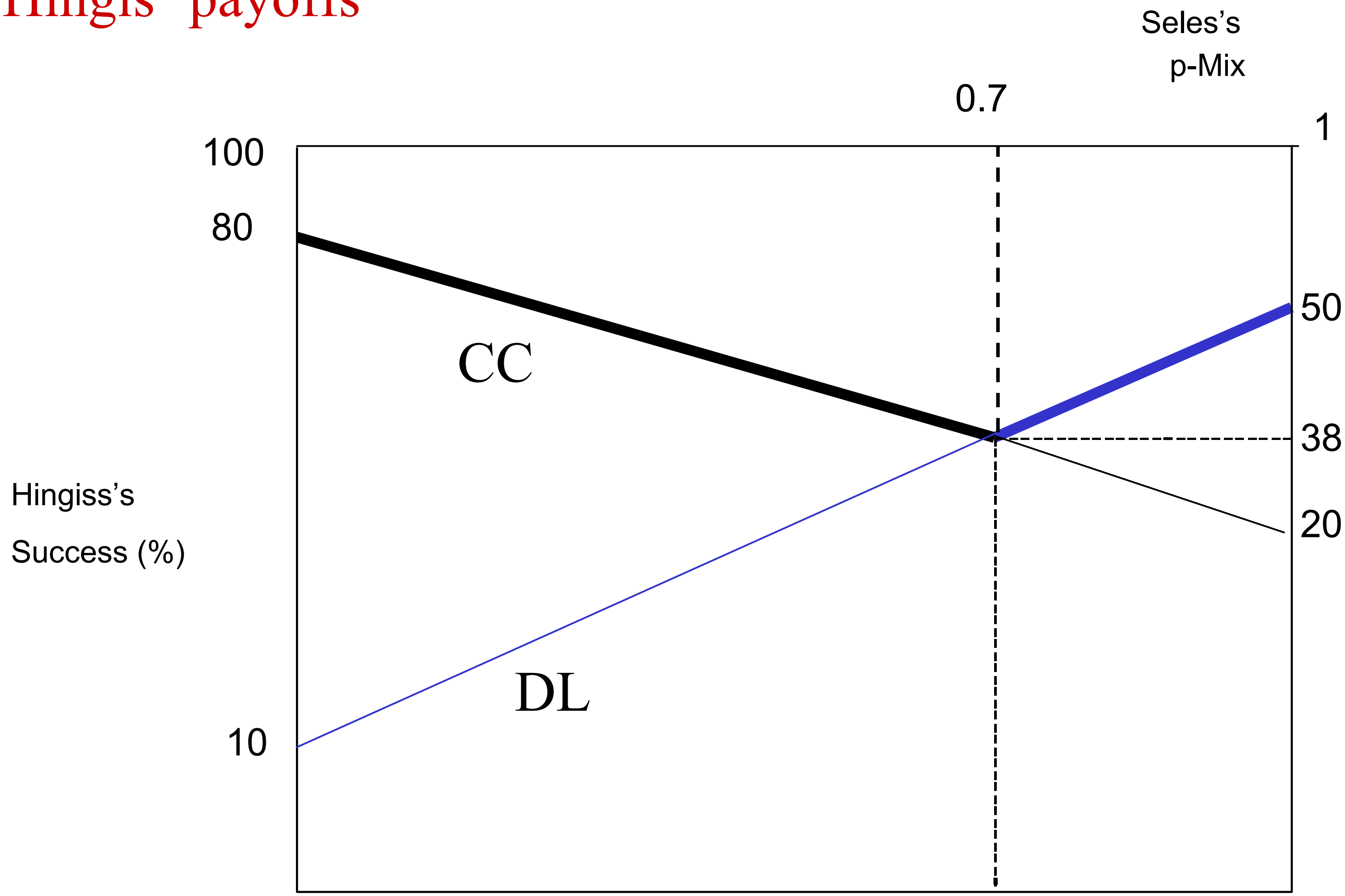


FIGURE 5.3 Diagrammatic Solution for Seles's *p*-Mix

Hingis' best responses to different p choices of Seles:

Against p s.t. $p < 0.7$ — Hingis plays CC

Against p s.t. $0.7 < p$ — Hingis plays DL

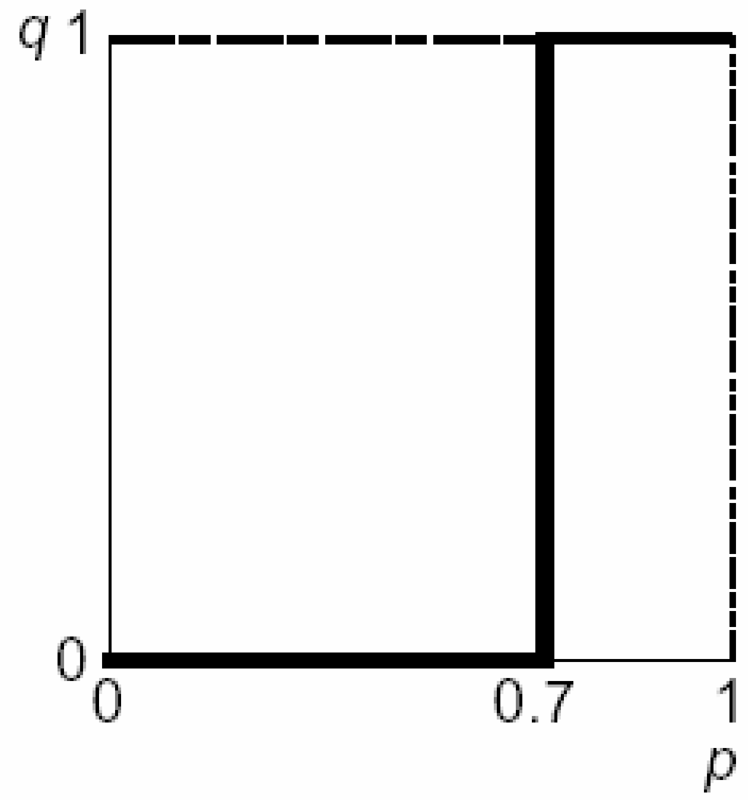
Against $p = 0.7$ — Both pure strategies best response

(=> all mixed strategies best response)

Hingis' best responses to different p choices of Seles:

Against p s.t. $p < 0.7$	Hingis plays $q = 0$
Against p s.t. $0.7 < p$	Hingis plays $q = 1$
Against $p = 0.7$	Hingis plays any q in [0,1]

The best response curve of Hingis:



Best response curve of Hingis

Best response curve of Seles

Best response curves combined

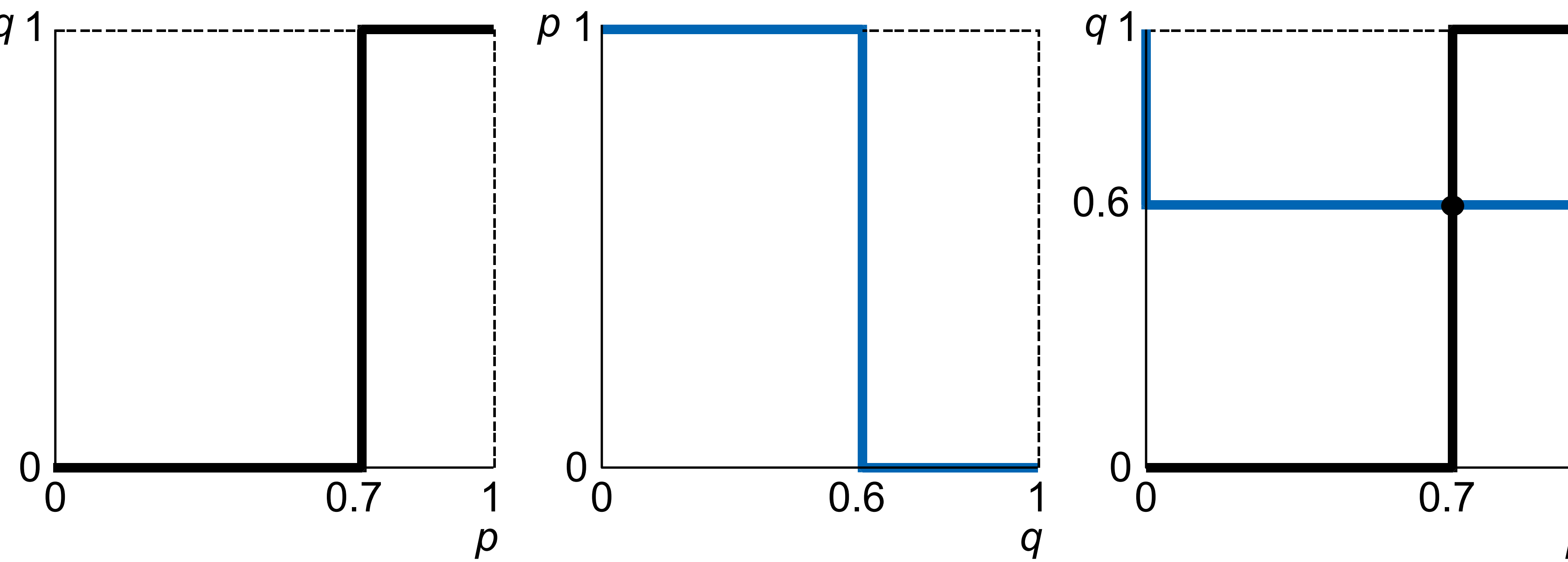


FIGURE 5.6 Best-Response Curves and Equilibrium

The mixed strategy Nash equilibrium is

$$((0.7 , 0.3) , (0.6 , 0.4))$$

and the payoff profile resulting from this equilibrium is

$$(62 , 100\text{-}62)$$

Note:

At the equilibrium, Seles gets the same expected payoff from DL and CC

$$U_{Seles} (DL , \pi_H) = U_{Seles} (CC , \pi_H) = 62$$

Similarly, Hingis gets the same payoff from DL and CC

$$U_{Hingis} (\pi_S , DL) = U_{Hingis} (\pi_S , CC) = 100 - 62$$

THIS IS SOMETHING GENERAL !

Very Useful Proposition:

If a player is playing a mixed strategy as a best response

and if he assigns positive probability to two his actions (say A and B

then his expected payoffs from these two actions are equal

Proof:

Suppose his expected payoff from A was larger than B

(note that his current payoff is a weighted average of these)

Then he can transfer some probability (i.e. weight) from B to A and increase his payoff

But this means, his original strategy was not a best response, a contradiction. So, the expected payoffs of A and B must be equal.

Use this proposition to develop a faster way of finding equilibrium

Dixit and Skeath call it: "leave the opponent indifferent" method

		HINGIS		
		DL	CC	q-Mix
SELES	DL	50	80	$50q + 80(1 - q)$
	CC	90	20	$90q + 20(1 - q)$
	p-Mix	$50p + 90(1 - p)$	$80p + 20(1 - p)$	

Conterintuitive change in mixture probabilities

Hingis' payoffs from DL has increased: will her q choice increase?

		HINGIS		
		DL	CC	q-Mix
SELES	DL	30	80	$30q + 80(1 - q)$
	CC	90	20	$90q + 20(1 - q)$
	p-Mix	$30p + 90(1 - p)$	$80p + 20(1 - p)$	

		DEAN		
		Swerve	Straight	q-Mix
JAMES	Swerve	0, 0	–1, 1	$-(1-q) = q-1$, $(1-q)$
	Straight	1, –1	–2, –2	$q-2(1-q) = 3q-2$, $-q-2(1-q) = q-2$
	p-Mix	$1-p$, $-(1-p) = p-1$	$-p-2(1-p) = p-2$, $p-2(1-p) = 3p-2$	

FIGURE 5.7 Mixing Strategies in Chicken

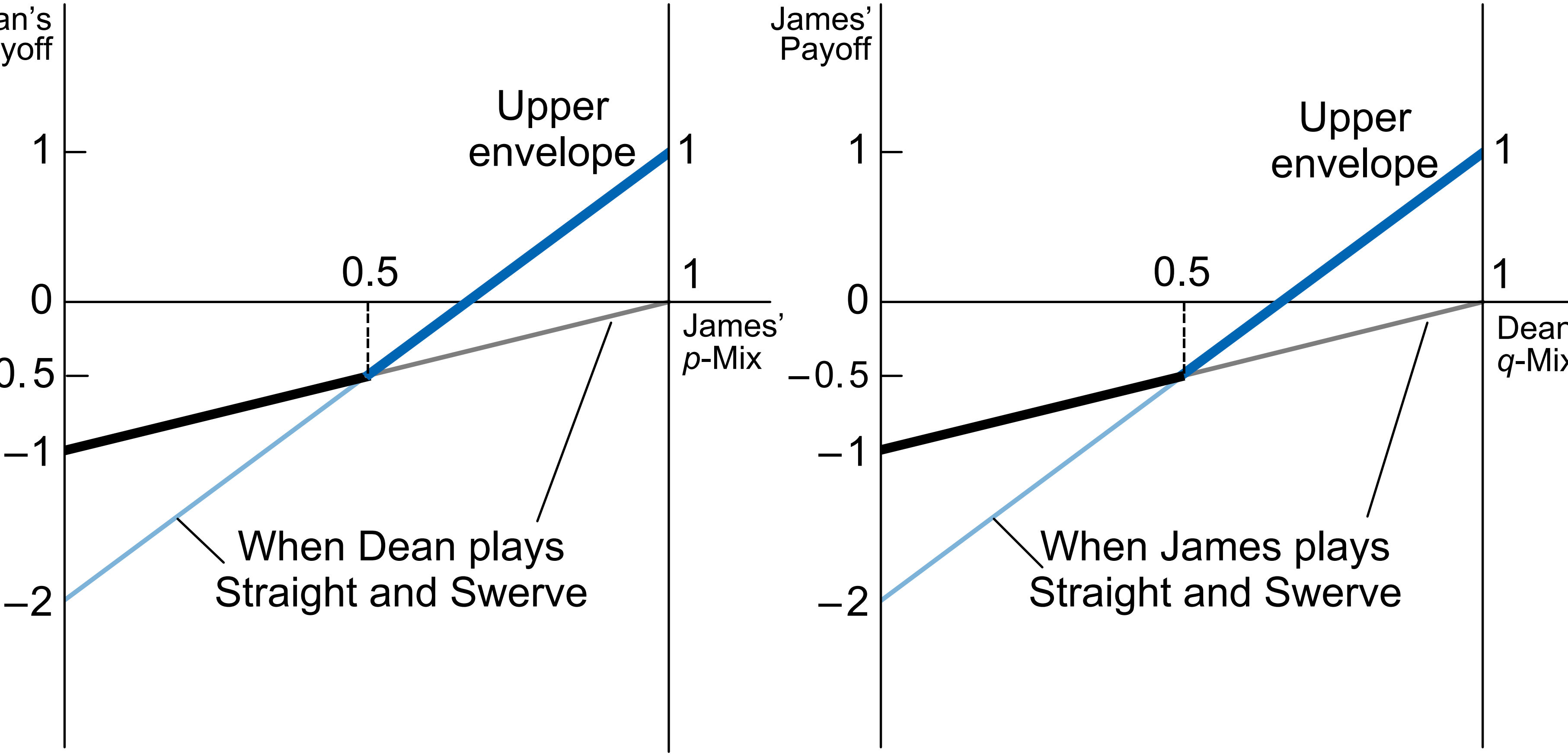


FIGURE 5.8 Optimal Responses with Mixed Strategies in Chicken

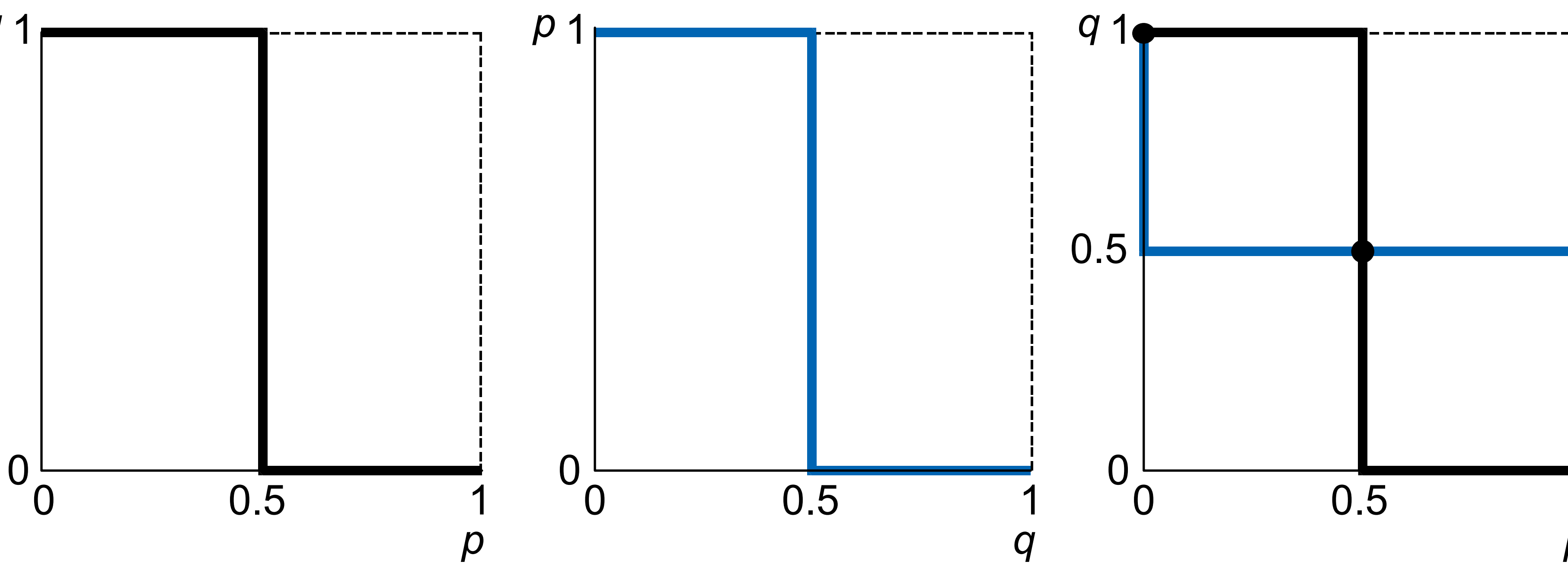


FIGURE 5.9 Best-Response Curves and Mixed Strategy Equilibria in Chicken

Observe 1:

A pure strategy Nash equilibrium of a game in pure strategies,

after mixed strategies are allowed,

continues to be an equilibrium

They are now mixed strategy equilibria where agents choose very simple (degenerate) mixed strategies (i.e., play this action with probability 1 and play everything else with probability 0)

Observe 2:

A Nash equilibrium in mixed strategies where each agent plays a pure strategy with probability one,

after mixed strategies are dropped,

continues to be an equilibrium in pure strategies.

		HUMANITIES FACULTY		
		Lab	Theater	q-Mix
SCIENCE FACULTY	Lab	2, 1	0, 0	$2q$, q
	Theater	0, 0	1, 2	$1 - q$, $2(1 - q)$
	p-Mix	$2p$, p	$1 - p$, $2(1 - p)$	

FIGURE 5.10 Mixing in the Battle-of-the-Two-Cultures Game

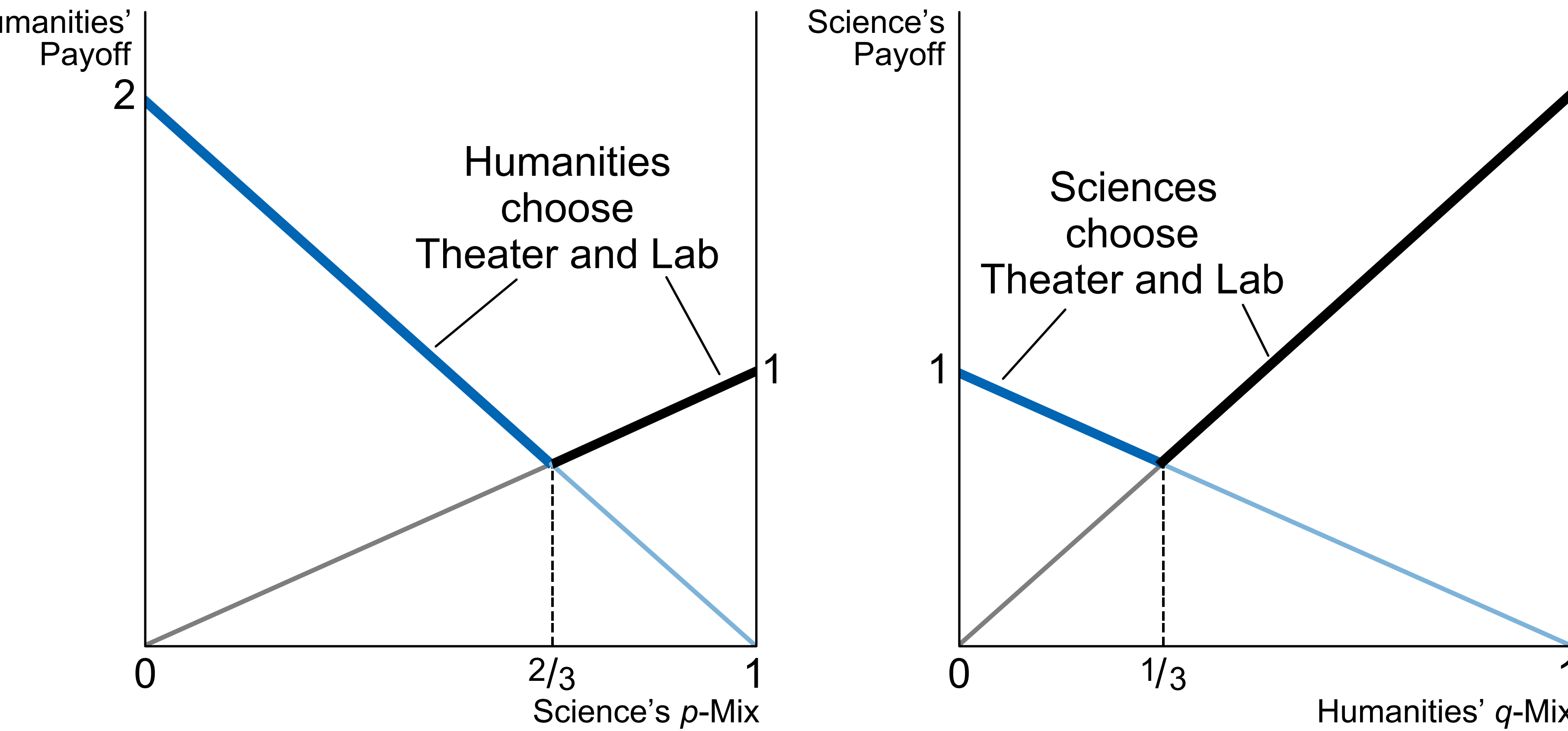


FIGURE 5.11 Best Responses with Mixed Strategies in the Battle of the Two Cultures

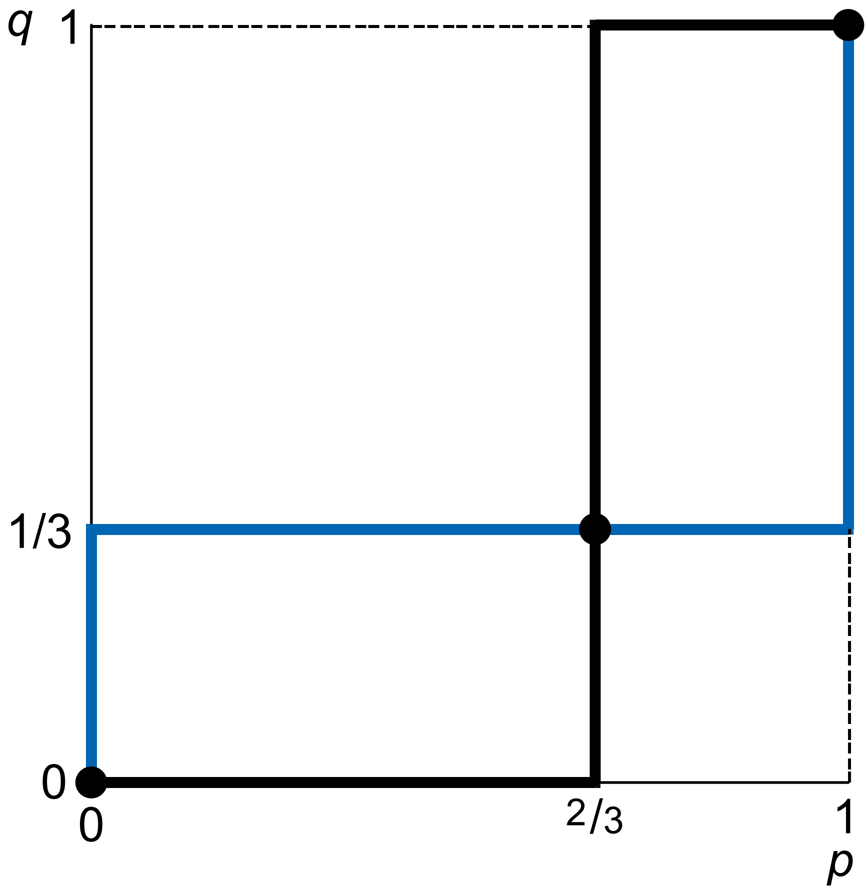


FIGURE 5.12 Mixed-Strategy Equilibria in the Battle of the Two Cultures