MIXED STRATEGIES

when there are

MORE THAN TWO ACTIONS

The best response analysis gets complicated

Get help from the concept of domination

Given a strategic form game with mixed strategies

$$G = (N , (\Pi(S_1), \ldots , \Pi(S_n)) , (U_1, \ldots ,U_n))$$

for an Agent i, π_i **strictly dominates** π'_i if

1. for *every* strategy profile

   $$\pi_{-i} = (\pi_1 , \ldots , \pi_{i-1} , \pi_{i+1} , \ldots , \pi_n)$$

   of the agents other than i,

$$U_i (\pi_i , \pi_{-i}) > U_i (\pi'_i , \pi_{-i})$$

Given a strategic form game with mixed strategies

$$G = (N, (\Pi(S_1), \ldots, \Pi(S_n)), (U_1, \ldots, U_n))$$

Agent *i's* pure strategy s_i **is strictly dominated** if

he has a mixed strategy that strictly dominates it, that is,

- There is a mixed strategy π_i of agent *i* such that for *every* strategy profile

$$\pi_{-i} = (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n)$$

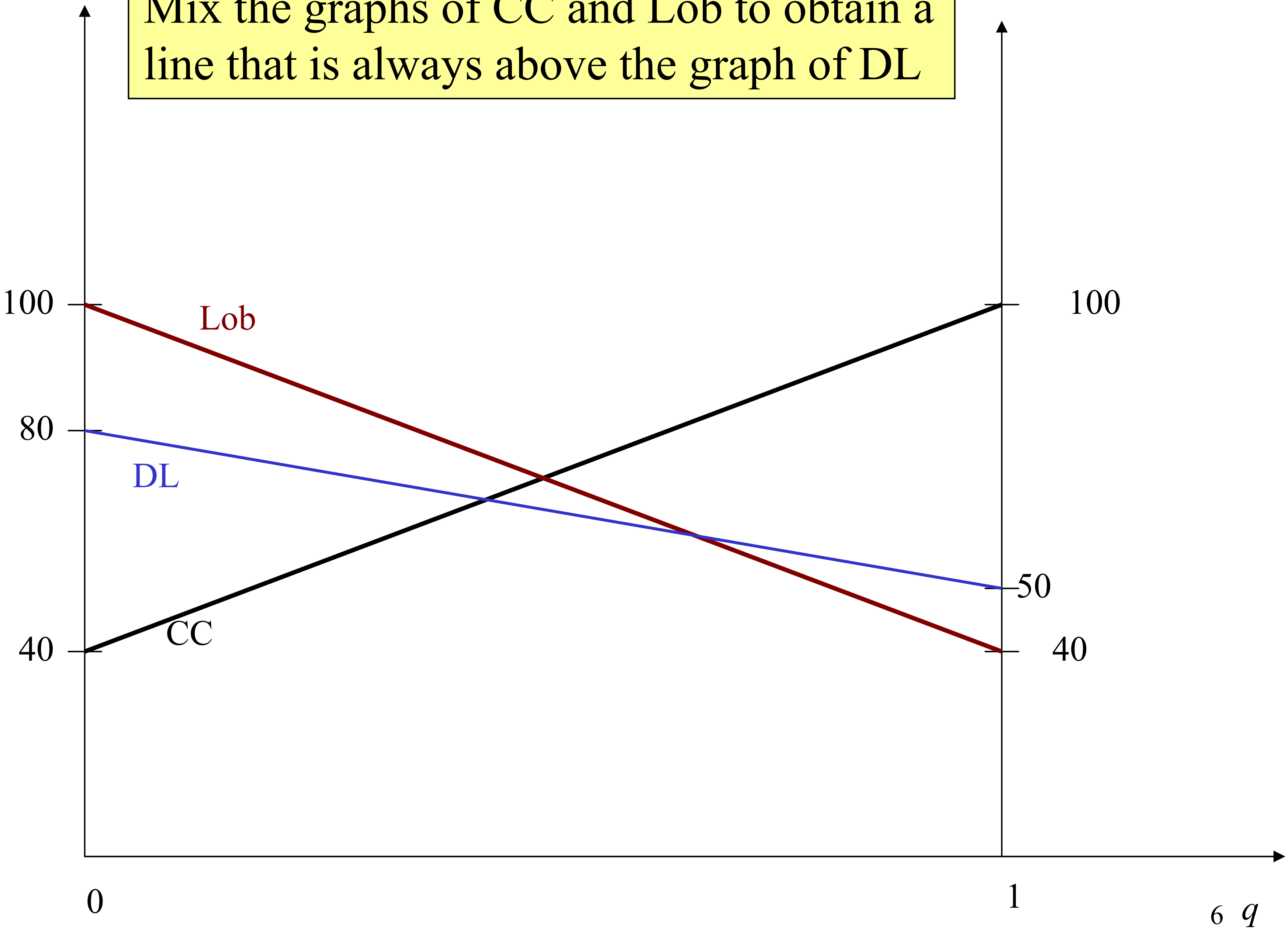
of the agents other than *i*,

$$u_i(\pi_i, \pi_{-i}) > u_i(s_i, \pi_{-i})$$

A pure strategy that is not dominated by pure strategies can be strictly dominated by a mixed strategy:

		HINGIS		
		DL	CC	q-Mix
SELES	DL	50	80	$50q + 80(1 - q)$
	CC	100	40	$100q + 40(1 - q)$
	Lob	40	100	$40q + 100(1 - q)$
	p-Mix	$50p_1 + 100p_2 + 40(1 - p_1 - p_2)$	$80p_1 + 40p_2 + 100(1 - p_1 - p_2)$	

Mix the graphs of CC and Lob to obtain a line that is always above the graph of DL



Strictly dominated strategies are never played.

Using this knowledge, you can eliminate some pure strategies.

The simplified game:

		HINGIS		
		DL	CC	q-Mix
SELES	CC	100	40	$100q + 40(1 - q)$
	Lob	40	100	$40q + 100(1 - q)$
	p-Mix	$100p_2 + 40(1 - p_2)$	$40p_2 + 100(1 - p_2)$	

Weak Domination

Does not really help.
Players can assign positive probability to weakly dominated actions and still play a best response.

What if there is no domination ?

Very Useful Proposition:

If a player is playing a mixed strategy as a best response

and if he assigns positive probability to two his actions (say A and B

then his expected payoffs from these two actions are equal

		HINGIS		
		DL	CC	q-Mix
SELES	DL	50	80	$50q + 80(1 - q)$
	CC	90	20	$90q + 20(1 - q)$
	Lob	70	60	$70q + 60(1 - q)$
	p-Mix	$50p_1 + 90p_2 + 70(1 - p_1 - p_2)$	$80p_1 + 20p_2 + 60(1 - p_1 - p_2)$	

FIGURE 5.17 Payoff Table for Tennis Point with Lob

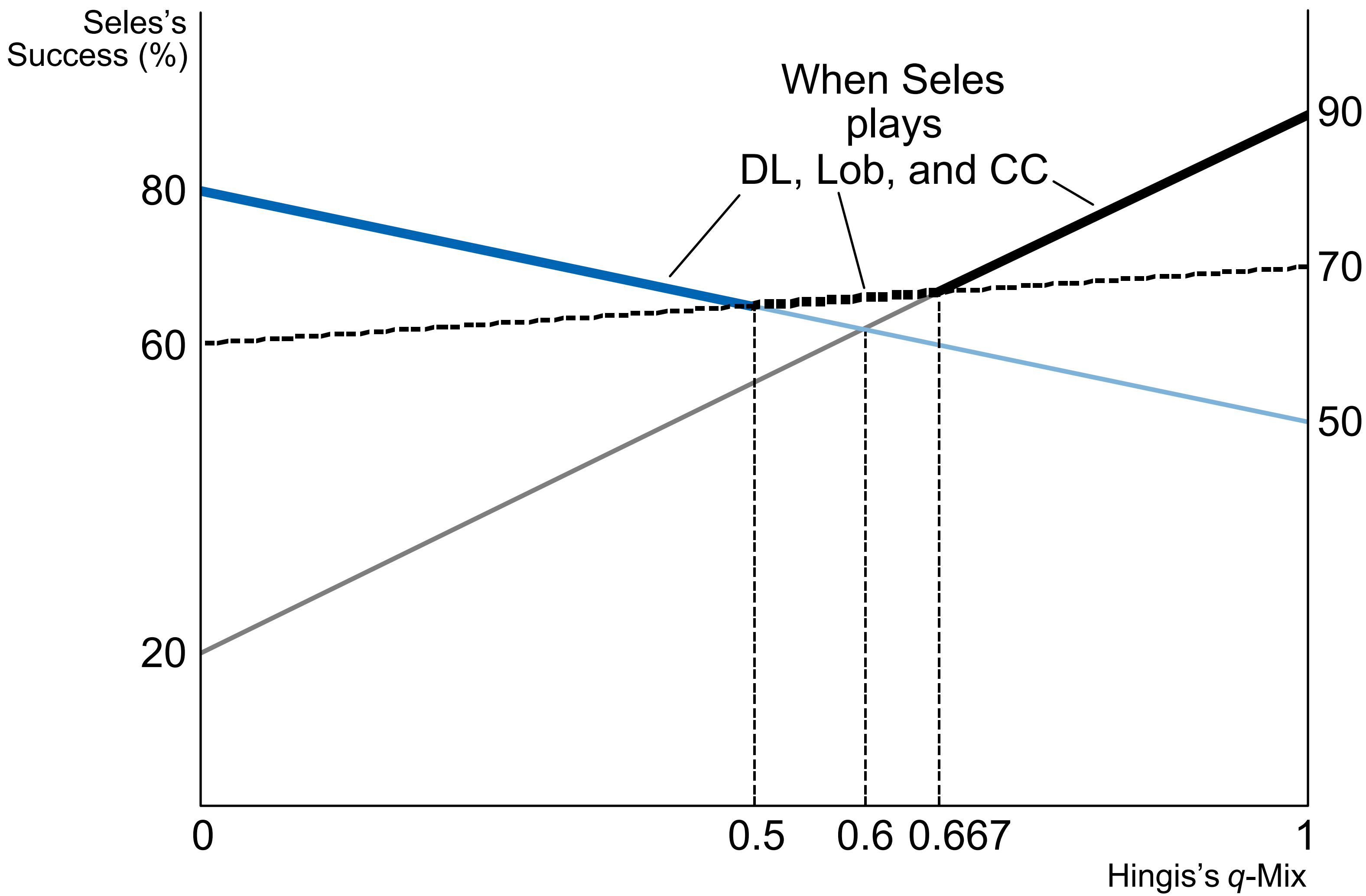


FIGURE 5.18 Diagrammatic Solution for Hingis's q-Mix

Seles' best responses to different q choices of Hingis:

Against $q < 0.5$ Seles plays DL $p_1 = 1$ $p_2 = 0$

Against $q = 0.5$ Seles mixes DL and Lob any p_1 in [0,1] $p_2 = 0$

Against $0.5 < q < 0.667$ Seles plays Lob $p_1 = 0$ $p_2 = 0$

Against $q = 0.667$ Seles mixes CC and Lob $p_1 = 0$ any p_2 in [0,1]

Against $q > 0.667$ Seles plays CC $p_1 = 0$ $p_2 = 1$

Claims:

In a Nash equilibrium, **Hingis does NOT play**

1. $q < 0.5$

   then Seles plays DL, then Hingis plays DL, contradiction

2. $0.5 < q < 0.667$

   then Seles plays Lob, then Hingis plays CC, contradiction

3. $q = 0.667$

   then Seles mixes Lob and CC, then Hingis plays CC, contradiction

4. $q > 0.667$

   then Seles plays CC, then Hingis plays CC, contradiction

When $q = 0.5$:

		HINGIS		
		DL	CC	q-Mix
SELES	DL	50	80	$50q + 80(1 - q)$
	Lob	70	60	$70q + 60(1 - q)$
	p-Mix	$50p_1 + 70(1 - p_1)$	$80p_1 + 60(1 - p_1)$	

Nash equilibrium: $q = 0.5, \quad p_2 = 0, \quad p_1 = ?$

FIGURE 5.19 Payoff Table after Eliminating Seles's CC Strategy

100 minus Hingis' payoffs from DL and CC (as a function of p_1)

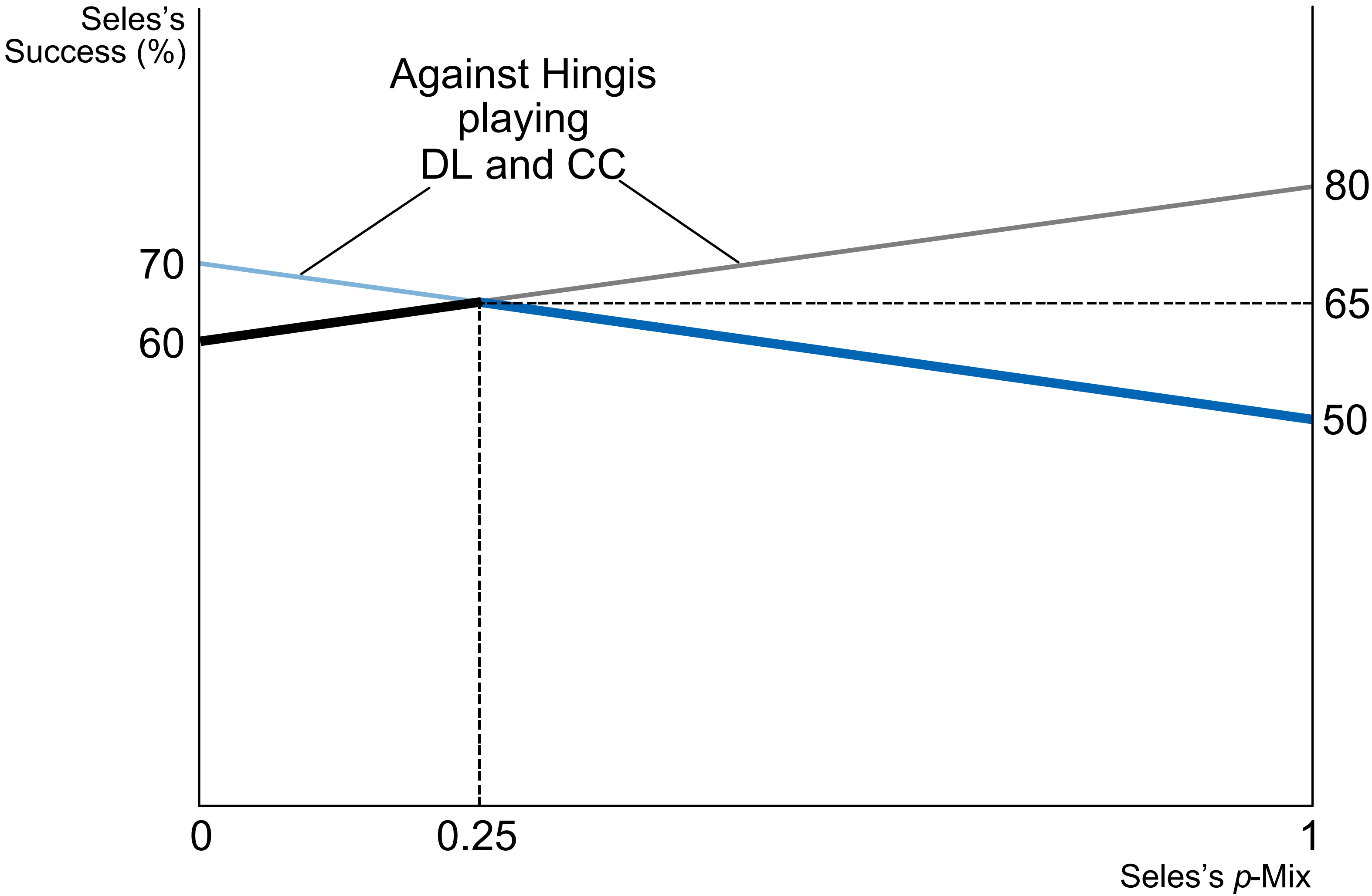
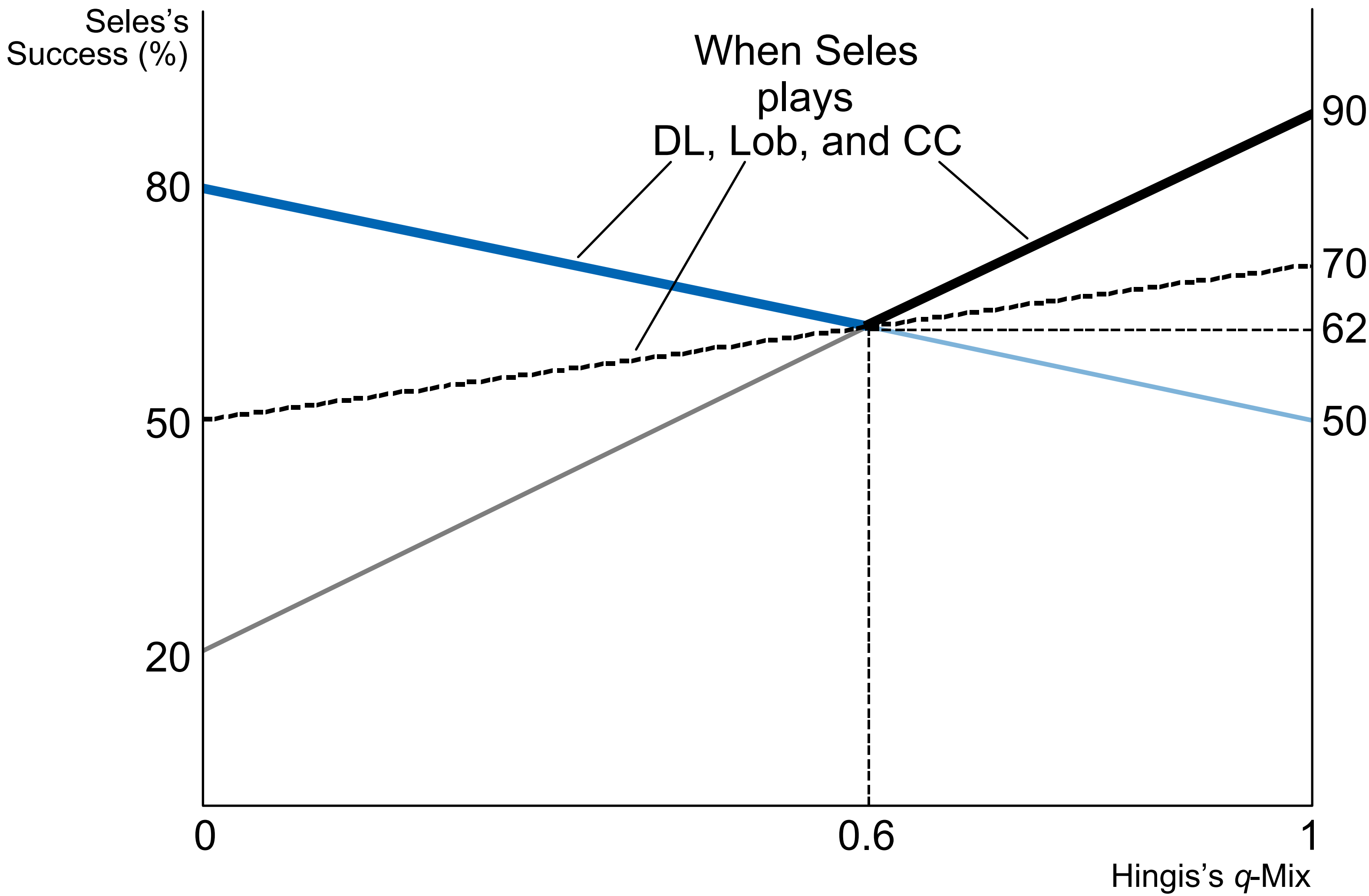


FIGURE 5.20 Diagrammatic Solution for Seles's *p*-Mix

		HINGIS		
		DL	CC	q-Mix
SELES	DL	50	80	$50q + 80(1 - q)$
	CC	90	20	$90q + 20(1 - q)$
	Lob	70	50	$70q + 50(1 - q)$
	p-Mix	$50p_1 + 90p_2 + 70(1 - p_1 - p_2)$	$80p_1 + 20p_2 + 50(1 - p_1 - p_2)$	

FIGURE 5.21 Payoff Table for Tennis Point with Lob: The Coincidence Case



Question: Is Lob weakly dominated ?

FIGURE 5.22 Diagrammatic Solution for Hingis's *q*-Mix: The Coincidence Case

eles' best responses to different q choices of Hingis:

gainst $q < 0.6$ Seles plays DL $p_1 = 1 \quad p_2 = 0$

gainst $q = 0.6$ Seles mixes DL, CC and Lob any p_1 p_2 in [0,1]

gainst $q > 0.6$ Seles plays CC $p_1 = 0 \quad p_2 = 1$

Claims:

In a Nash equilibrium, **Hingis does NOT play**

1. $q < 0.6$

then Seles plays DL, then Hingis plays DL, contradiction

4. $q > 0.6$

then Seles plays CC, then Hingis plays CC, contradiction

How to make $q = 0.6$ part of an equilibrium?

Seles must choose a mixed strategy such that Hingis will receive the same payoff from DL and CC

How to make $q = 0.6$ part of an equilibrium?

Seles must choose a mixed strategy such that Hingis will receive the same payoff from DL and CC

For this, solve

$$50\ p_1 + 90\ p_2 + 70\ (1 - p_1 - p_2) = 80\ p_1 + 20\ p_2 + 50\ (1 - p_1 - p_2)$$

=>

$$50\ p_2 + 20 = 50\ p_1 \quad \Rightarrow \quad p_2 = p_1 - 0.4$$

Now remember that $p_2 + p_1 <= 1$

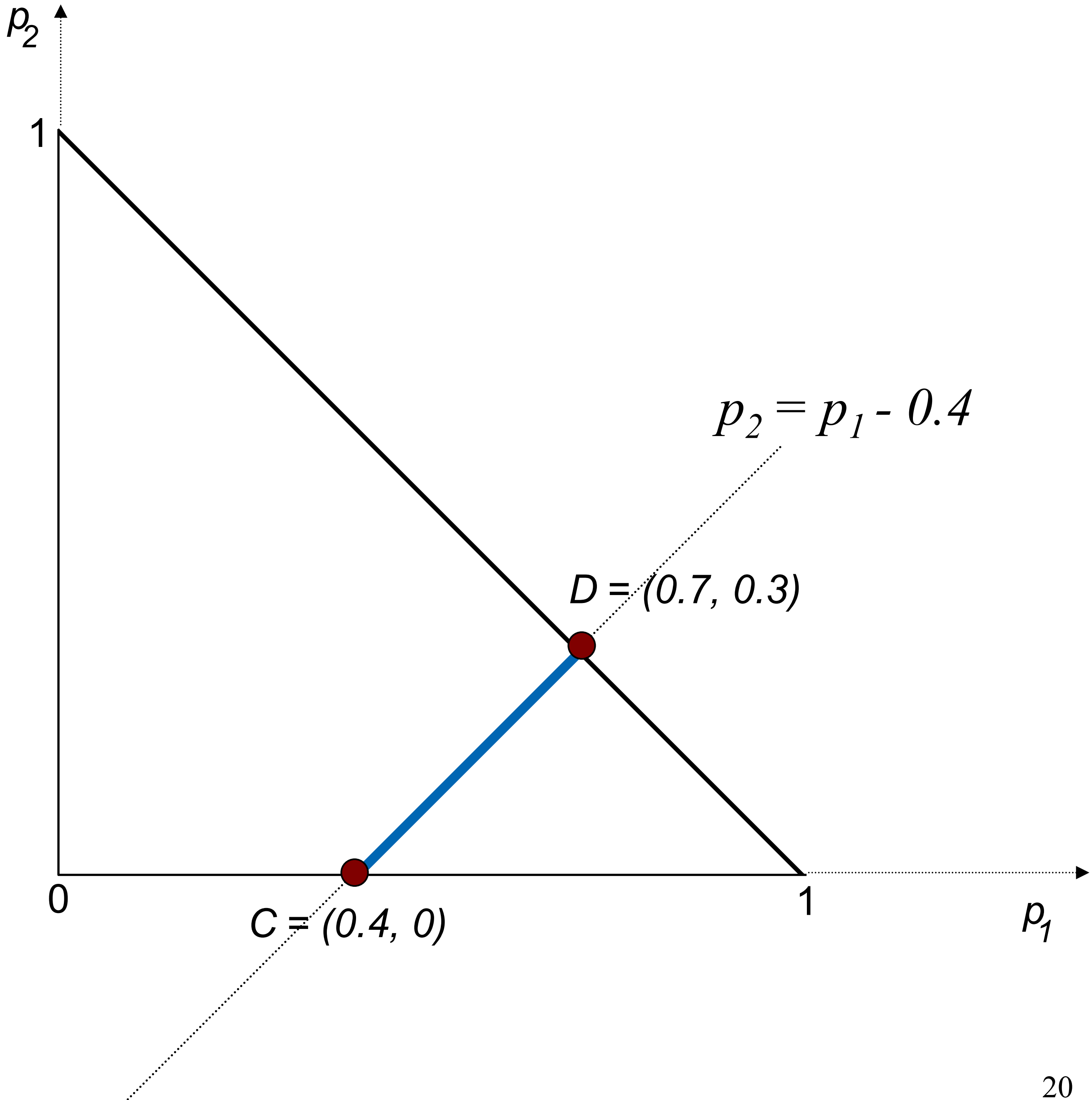


FIGURE 5.23 Seles's Indeterminate *p*-Mix in the Coincidence Case

The mixed strategy Nash equilibria are

$$\big(\,(\,p_1\,,\,p_2\,,\,1-p_1-p_2\,)\,,\,(\,0.6\,,\,0.4\,)\,\big)$$

where p_1 is between 0.4 and 0.7 and

$$p_2 = p_1 - 0.4$$

NOTE: Even though there is an infinite number of equilibria, the resulting payoff profile is unique: (62, 38)

A 3x3 game:

	X	Y	Z	Mixed
U	5 , 9	3 , 7	4 , 14	$4+q_1-q_2$ $14-5q_1-7q_2$
M	4 , 3	6 , 4	4 , 5	$4+2q_2$ $5-2q_1-q_2$
D	3 , 4	4 , 6	5 , 1	$5-2q_1-q_2$ $1+3q_1+5q_2$
Mixed	$3+2p_1+p_2$ $4+5p_1-p_2$	$4-p_1+2p_2$ $6+p_1-2p_2$	$5-p_1-p_2$ $1+13p_1+4p_2$	

1. Is there a pure strategy equilibrium? No

2. Is there a mixed equilibrium where all pure strategies are assigned positive probabilities? That is, where

$$p_1 > 0 \qquad p_2 > 0 \qquad 1 - p_1 - p_2 > 0 \qquad \text{and}$$

$$q_1 > 0 \qquad q_2 > 0 \qquad 1 - q_1 - q_2 > 0$$

$p_1 > 0 \quad p_2 > 0 \quad 1 - p_1 - p_2 > 0$ implies (by our proposition)

$$4 + q_1 - q_2 = 4 + 2\, q_2$$ and

$$4 + q_1 - q_2 = 5 - 2\, q_1 - q_2$$

Solving this, one gets $q_1 = 1/3$ and $q_2 = 1/9$

The proposition tells us that if Column's q-mix is not this one, Row can't play a mixed strategy of the above kind as a best response.

That is, if there is an equilibrium where Row plays a mixed strategy that satisfies $p_1 > 0 \quad p_2 > 0 \quad 1 - p_1 - p_2 > 0$, then in that equilibrium Col must play $q_1 = 1/3$ and $q_2 = 1/9$.

$q_1 > 0$ $q_2 > 0$ $1 - q_1 - q_2 > 0$ implies (by our proposition)

$$4 + 5\,p_1 - p_2 = 6 + p_1 - 2\,p_2 \qquad \text{and}$$

$$4 + 5\,p_1 - p_2 = 1 + 13\,p_1 + 4p_2$$

Solving this, one gets $p_1 = 7/12$ and $\boldsymbol{p_2 = -1/3}$

This is a contradiction. So there is no equilibrium in which Col plays a mixed strategy where $q_1 > 0$ $q_2 > 0$ $1 - q_1 - q_2 > 0$

What about a mixed strategy where

$$1 - q_1 - q_2 = 0 \text{ but } q_1 > 0 \text{ and } q_2 > 0 \text{ ?}$$

Then $\quad 4 + 5\,p_1 - p_2 = 6 + p_1 - 2\,p_2 \quad$ and

$$4 + 5\,p_1 - p_2 > 1 + 13\,p_1 + 4\,p_2$$

This can be rewritten as

$p_2 = 2 - 4\,p_1 \quad$ and

$$p_2 < 3/5 - 8/5\,p_1$$

No p_1 and p_2 value simultaneously satisfies these conditions.

So there is no equilibrium where $1 - q_1 - q_2 = 0$ but $q_1 > 0$ and $q_2 > 0$

What about a mixed strategy where

$q_2 = 0$ but $q_1 > 0$ and $1 - q_1 - q_2 > 0$?

Then $4 + 5\,p_1 - p_2 = 1 + 13\,p_1 + 4\,p_2$ and

$4 + 5\,p_1 - p_2 > 6 + p_1 - 2\,p_2$

This can be rewritten as

$p_2 > 2 - 4\,p_1$ and

$p_2 = 3/5 - 8/5\,p_1$

No p_1 and p_2 value simultaneously satisfies these conditions.

So there is no equilibrium where $q_2 = 0$ but $q_1 > 0$ and $1 - q_1 - q_2 > 0$

What about a mixed strategy where

$q_1 = 0$ but $q_2 > 0$ and $1 - q_1 - q_2 > 0$?

Then $4 + 5\,p_1 - p_2 < 6 + p_1 - 2\,p_2$ and

$6 + p_1 - 2\,p_2 = 1 + 13\,p_1 + 4\,p_2$

This can be rewritten as

$p_2 < 2 - 4\,p_1$ and

$p_2 = 5/6 - 2\,p_1$

Any p_1 and p_2 value such that

$0 \leq p_1 \leq 5/12$ and $p_2 = 5/6 - 2\,p_1$

satisfies this condition.

If Col chooses $q_1 = 0$ but $q_2 > 0$ and $1 - q_1 - q_2 > 0$

Then for Row, U is a dominated strategy.

Thus, in equilibrium, $p_1 = 0$.

Since $p_2 = 5/6 - 2\, p_1$, this implies $p_2 = 5/6$.

This means, Row is assigning a positive probability to both M and D.

For this to be part of an equilibrium, our proposition says

$$4 + 2q_2 = 5 - 2q_1 - q_2 = 5 - q_2$$

Solving, we get $q_2 = 1/3$.

So the equilibrium is as follows:

$$\left((0, 5/6, 1/6) \quad , \quad (0, 1/3, 2/3) \right)$$

Collective Action Games

+

Games with a very large number of players

COLLECTIVE ACTION GAMES

- **Social problems concerning collective action**
  - multiple-person games with too many players
  - unsatisfactory outcomes
  - social interest vs. private incentives

Societies usually have problems in implementing outcomes that are considered to be good for everybody.

Helping the poor

Planting trees and not burning them later

Keeping the environment clean

Obeying the traffic laws

A Simple Example

- **Two farmers**: need an irrigation project
    - it is a **pure public good** (nonexcludable and nonrival)

      like national defense

      compare it to a private good (like a sandwich)
    - who is going to build it?

    - **Strategies**: **participate** or **shirk**
- b_1 and c_1 **:** benefit and cost of project when **1** person builds
- b_2 and c_2 **:** benefit and cost of project when **2** persons build

What is an individual's payoff?

The benefit minus the cost (if she participated)

or

The benefit (if she shirked)

What is the best for the society?

The outcome that maximizes

the sum of individual payoffs

Why sum? (utilitarianism)

Why utilitarianism? Because the Dixit-Skeath book uses it.

What is the best for the society?

- Pareto optimality (Vilfredo Pareto)

  An outcome is Pareto-optimal if there is no alternative outcome which gives all agents an at least as high payoff and some agents a higher payoff

- Egalitarianism (John Rawls)

  The Egalitarian-optimal outcome maximizes the smallest payoff in the society

- Utilitarianism (John Stuart Mill)

  The Utilitarian-optimal outcome maximizes the total payoff in the society

b_1 c_1

b_2 c_2

General Case

		YOUR NEIGHBOR	
		Build	Not
YOU	Build	$b_2 - c_2$, $b_2 - c_2$	$b_1 - c_1$, b_1
	Not	b_1, $b_1 - c_1$	0, 0

Utilitarian payoff: your payoff + your neighbor's payoff

Utilitarian optimum: maximizes the utilitarian payoff

Egalitarian payoff: minimum {your payoff, your neighbor's payoff}

Egalitarian optimum: maximizes the egalitarian payoff

$b_1 = 6 \qquad c_1 = 7$

$b_2 = 8 \qquad c_2 = 4$

Prisoners' Dilemma Game

		YOUR NEI GHBOR	
		Build	Not
YOU	Build	4, 4	-1,6
	Not	6, - 1	0, 0

Utilitarian optimum: (Build,Build)

Egalitarian optimum: (Build,Build)

Pareto optima: (Build,Build), (Not, Build), (Build, Not)

Nash equilibrium: (Not, Not)

FIGURE 11.1 Collective Action as a Prisoners' Dilemma: Version I

$b_1 = 6$ $c_1 = 7$

$b_2 = 6.3$ $c_2 = 4$

Prisoners' Dilemma Game

		YOUR NEIGHBOR	
		Build	Not
YOU	Build	2.3, 2.3	–1, 6
	Not	6, –1	0, 0

Utilitarian optimum: (Build,Not) or (Not,Build)

Egalitarian optimum: (Build, Build)

Pareto optima: (Build,Build), (Not, Build), (Build, Not)

Nash equilibrium: (Not,Not)

FIGURE 11.2 Collective Action as a Prisoners' Dilemma: Version II

$b_1 = 6$ $c_1 = 4$

$b_2 = 8$ $c_2 = 3$

Chicken Game

		YOUR NEIGHBOR	
		Build	Not
YOU	Build	5, 5	2, 6
	Not	6, 2	0, 0

Utilitarian optimum: (build,build)

Egalitarian optimum: (build,build)

Pareto optima: (Build,Build), (Not, Build), (Build, Not)

Nash equilibrium: (build,not) and (not,build)

FIGURE 11.3 Collective Action as Chicken: Version I

$b_1 = 3$ $c_1 = 7$

$b_2 = 8$ $c_2 = 4$

Assurance Game

		YOUR NEIGHBOR: Build	YOUR NEIGHBOR: Not
YOU	Build	4, 4	–4, 3
	Not	3, –4	0, 0

Utilitarian optimum: (build,build)

Egalitarian optimum: (build,build)

Pareto optima: (Build,Build)

Nash equilibrium: (build,build) and (not,not)

FIGURE 11.4 Collective Action as an Assurance Game

What about in a large group?

N agents for a public project

An agent's benefit if n people participates: $b(n)$

Cost of participating if n people participates: $c(n)$

Two strategies: Shirk or Participate

An agent's payoff depends on what the others are doing

If n people are participating:

payoff of a shirking agent: $s(n)=b(n)$

payoff of a participating agent: $p(n)=b(n)-c(n)$

An agent compares $s(n)$ and $p(n+1)$

Social payoff from *n* participants

$$T(n) = n\,p(n) + (N - n)\,s(n)$$

The marginal social gain from a one person increase in participants

$$T(n+1) - T(n) = p(n+1) - s(n)$$

Marginal private gain (the part that derives individual choice)

$$+\, n\,(p(n+1) - p(n))$$

Externality on participants

$$+\,(N - n - 1)\,(s(n+1) - s(n))$$

Externality on shirkers

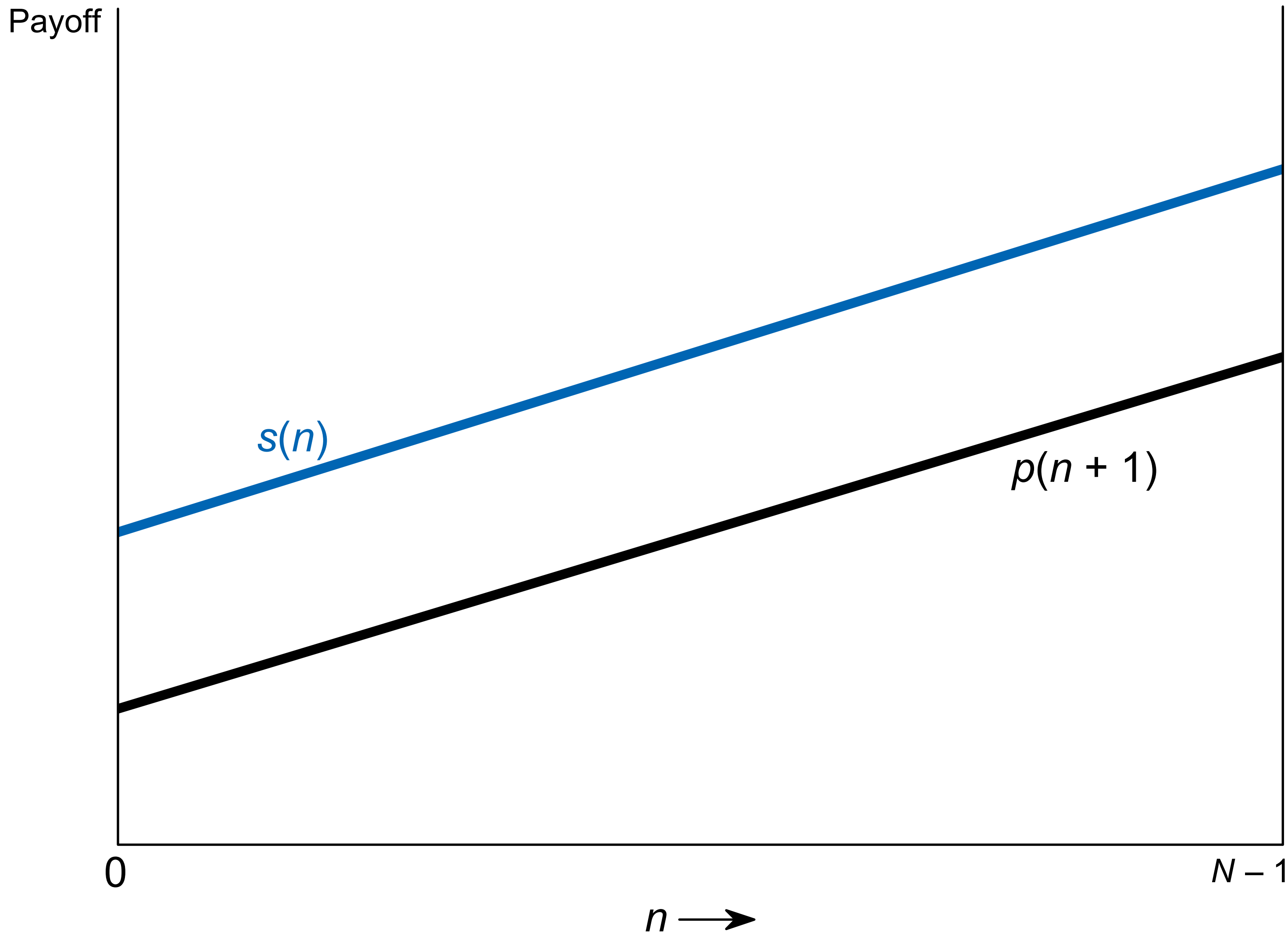


FIGURE 11.6 Multiperson Prisoners' Dilemma

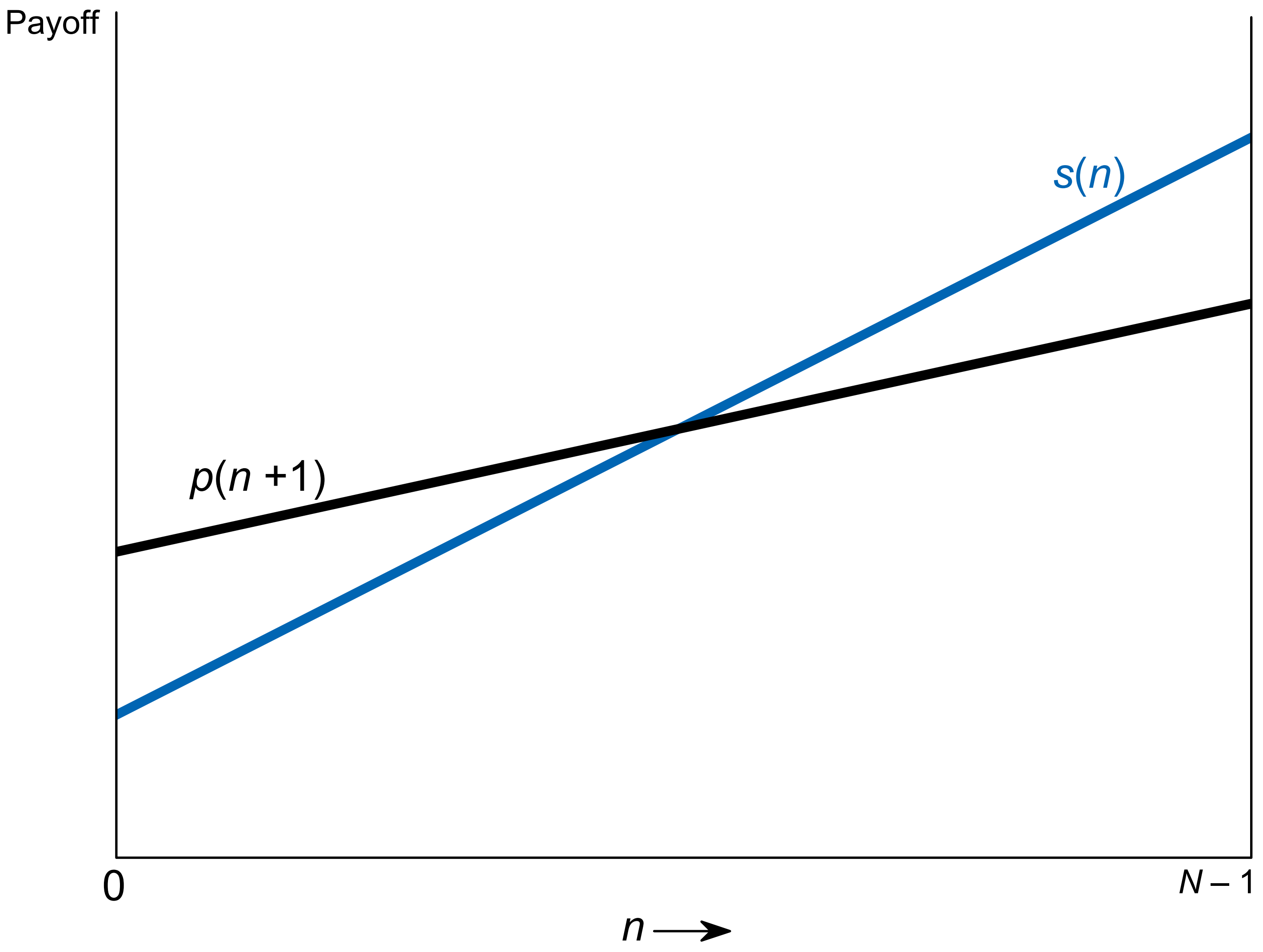


FIGURE 11.7 Multiperson Chicken

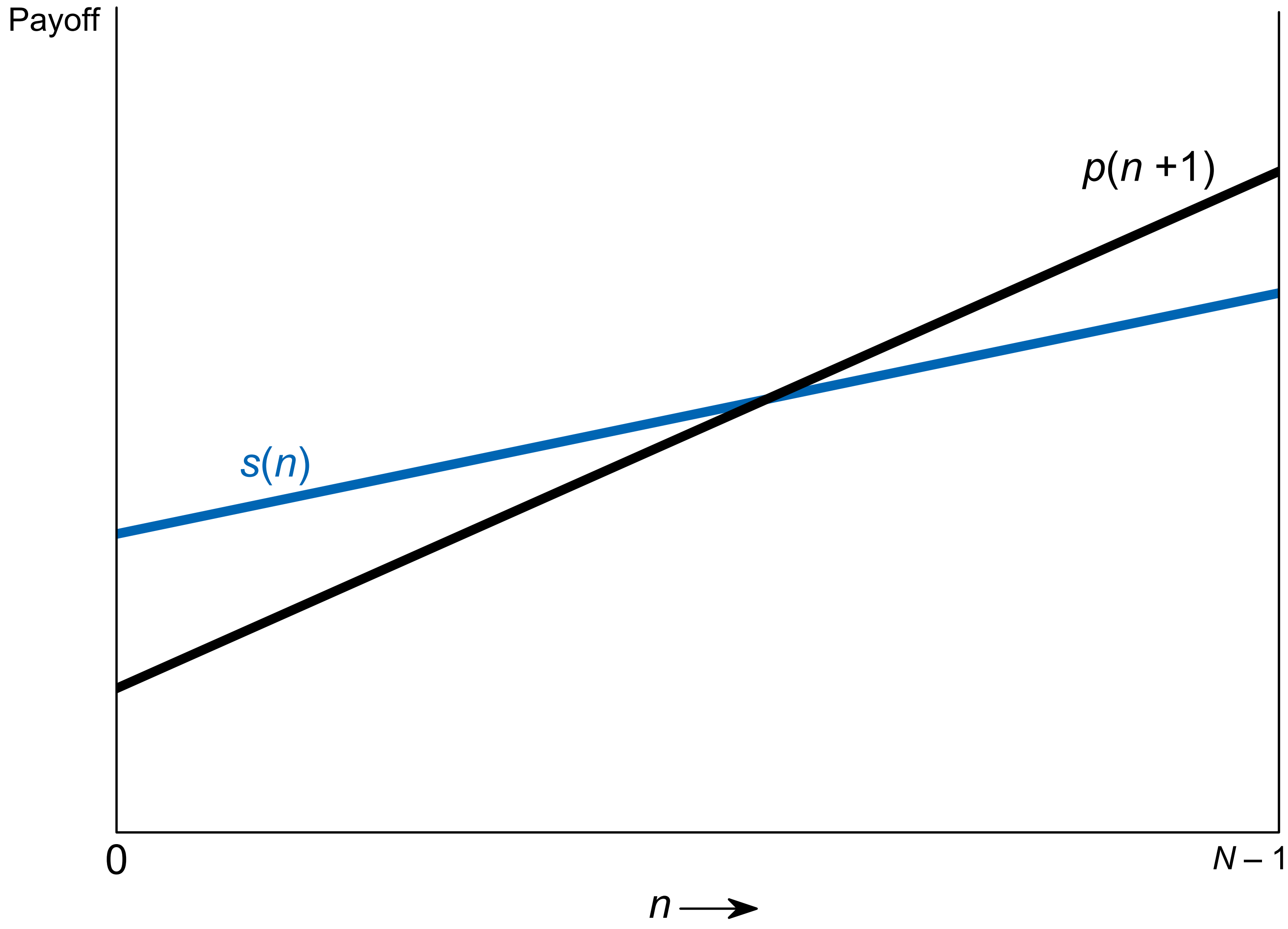


FIGURE 11.8 Multiperson Assurance Game

The route choice from home to work:

6000 drivers

Two routes from the suburbs to the city

Local route: always takes 45 minutes

Expressway: takes 15 minutes if there are not more than 2000 drivers

After that, increases 0.01 minutes with every additional driver

Want to

model it as a collective-action game

find Nash equilibria

find the social optimum

The route choice game:

Payoff: gain from traffic out of an hour

Shirkers' payoffs (from the local route)

$$s(n) = 15$$

Participants' payoffs (from the expressway)

$$p(n) = \begin{cases} 45 & \text{if } n <= 2000 \\ 45 - 0.01\,(n - 2000) & \text{if } n > 2000 \end{cases}$$

Nash equilibria: $n = 4999$, 5000

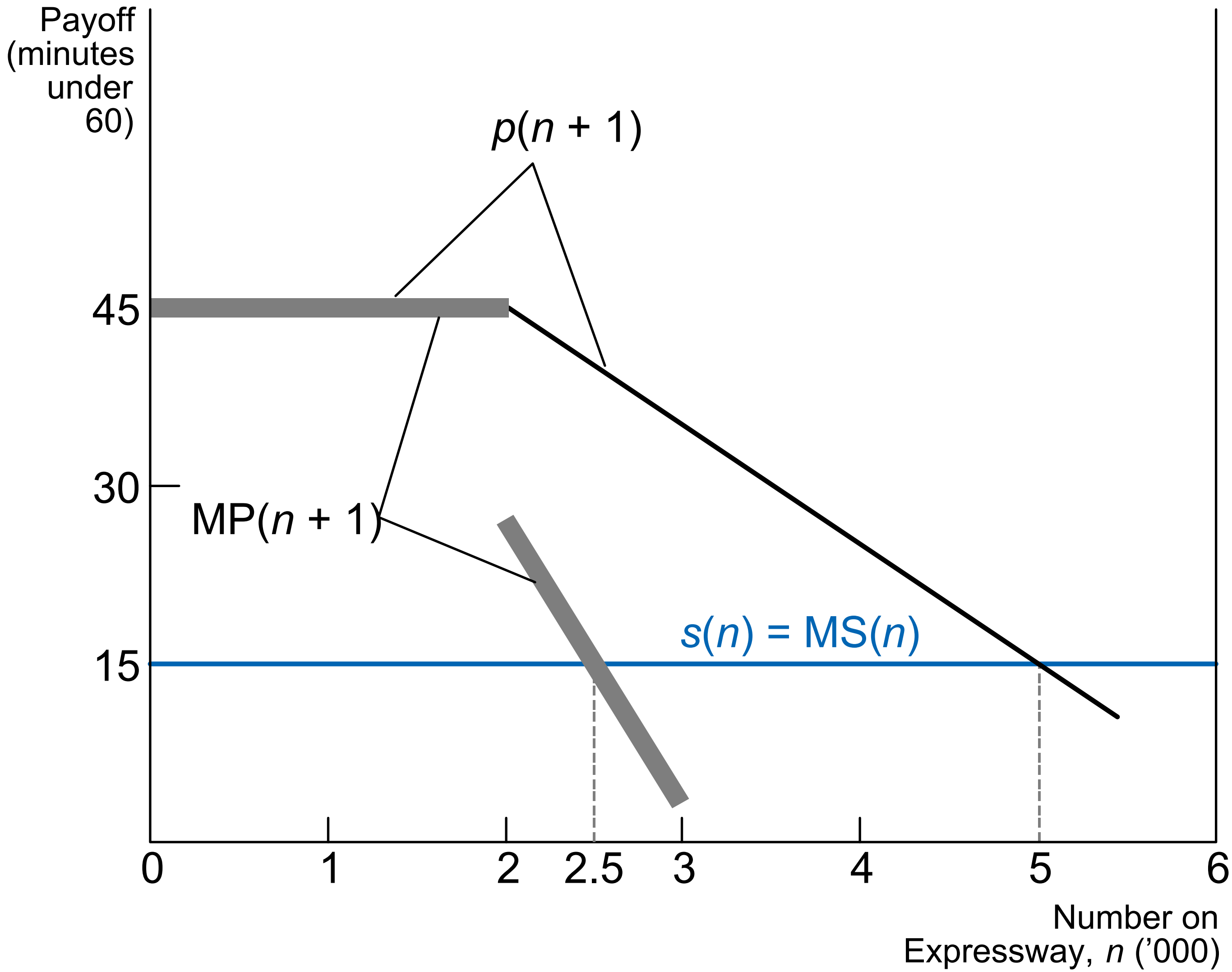


FIGURE 11.9 Equilibrium and Optimum in Route-Choice Game

Finding the social optimum

$$T(n) = n\,p(n) + (6000 - n)\,s(n)$$

$$T(n) = \begin{cases} n\,45 + (6000 - n)\,15 & \text{if } n <= 2000 \\ n\,(45 - 0.01\,(n - 2000)) + (6000 - n)\,15 & \text{if } n > 2000 \end{cases}$$

$$T(n) = \begin{cases} 90000 + 30\,n & \text{if } n <= 2000 \\ 90000 + 50\,n - 0.01\,n^2 & \text{if } n > 2000 \end{cases}$$

$$T'(n) = \begin{cases} 30 & \text{if } n <= 2000 \\ 50 - 0.02\,n & \text{if } n > 2000 \end{cases}$$

$$T''(n) = \begin{cases} 0 & \text{if } n <= 2000 \\ -\,0.02 & \text{if } n > 2000 \end{cases}$$

$$\boxed{50 - 0.02\,n = 0 \qquad => \qquad n = 2500}$$

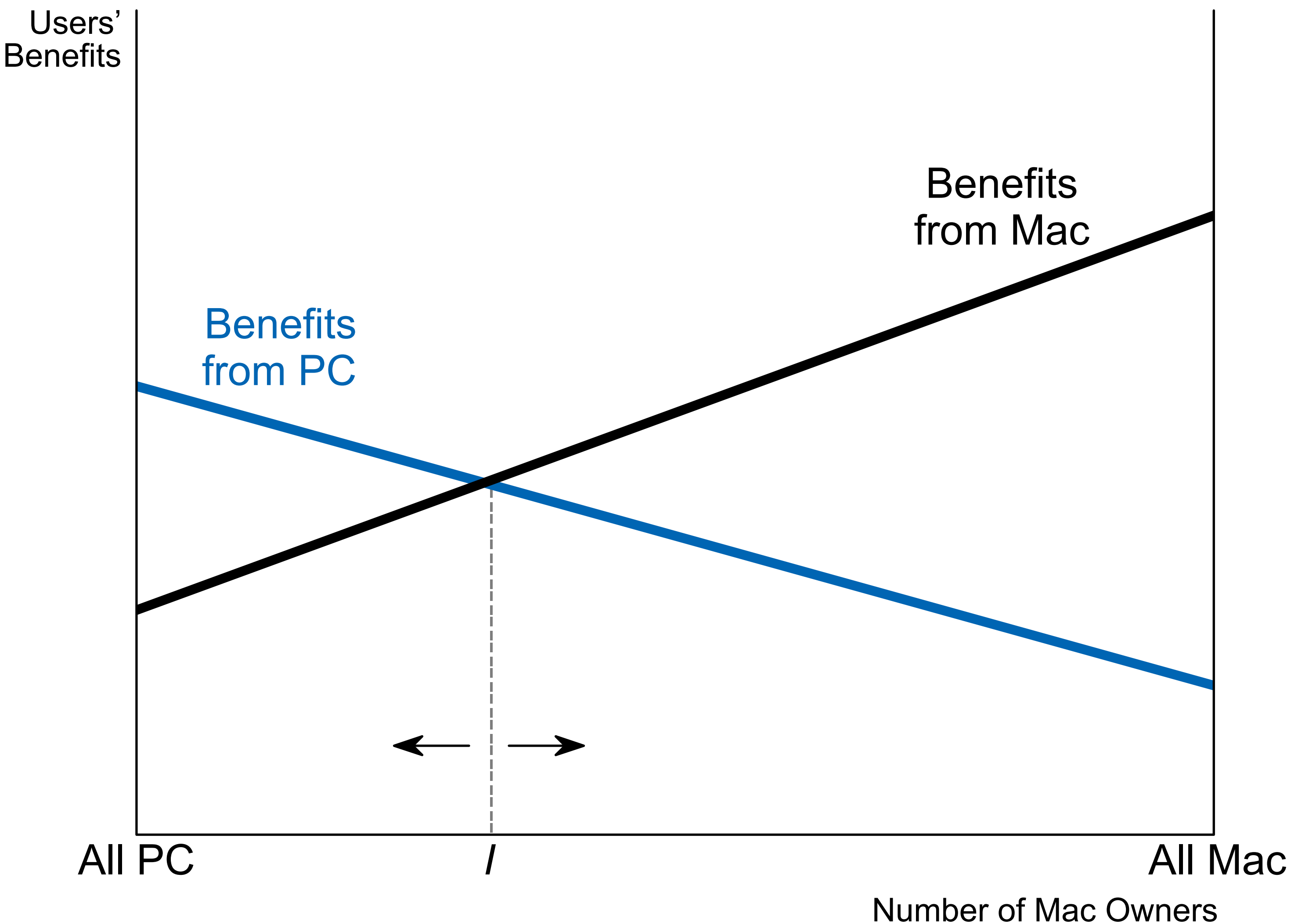


FIGURE 11.10 Payoffs in Computer Choice Game

The differential version of marginal social gain:

$$T'(n) = p(n) - s(n)$$ → Marginal private gain (the part that derives individual choice)

$$+\, n\, p'(n)$$ → Externality on participants

$$+\, (N - n)\, s'(n)$$ → Externality on shirkers