GAMES WITH SEQUENTIAL MOVES

- Chess, card games, bidding in some auctions, voting in some committees, entry-exit models in industrial organization,…

Players take actions sequentially

When I am choosing my action, I know what people who acted before me has done

The Senate Race

- Two candidates running to represent Tuzla in the parliament.
  - Gray: the incumbent senator
    the stronger one
  - Green: the newcomer (from the Greens party)
    the weaker one

They play the following game:

First: Gray chooses whether to give **ads** or **not**

Second: Green chooses whether to stay **in** or **out** of the race

NOTE:

Green chooses between the actions in and out after she observes Gray's choice

Possible **outcomes** of the game are:

(No Ads, Out) (Ads,Out) (No Ads, In) (Ads, In)

To describe a game, we need to specify:

1. Players (agents)

Gray and Green

2. Strategies for each player

??????????????

3. Payoffs

Gray's ranking of the outcomes:

(No Ads, Out) (Ads, Out) (No Ads, In) (Ads, In)

Gray's payoffs (choose numbers that represent her ranking)

4 3 2 1

To describe a game, we need to specify:

1. Players (agents)

Gray and Green

2. Strategies for each player

??????????????

3. Payoffs

Green's ranking of the outcomes:

(No Ads, In) (Ads, Out) (No Ads, Out) (Ads, In)

Green's payoffs (choose numbers that represent her ranking)

4 3 2 1

Is there a nice way to summarize this story in a figure ?????

YES!!!!!

Use a game tree (a.k.a, the extensive form)

Game Tree:

Made up of two things

1. Nodes there are two types of nodes

Decision nodes

nodes at which a player has to take an action

Terminal nodes

nodes at which there is no action to take

they represent possible outcomes of the game

2. Branches

they represent actions

they come out of decision nodes and connect nodes

The Game Tree in the Senate Race game:

Made up of two things

1. Nodes

Decision nodes

Gray has a node in which he has to act between Ads and No

What about Green?

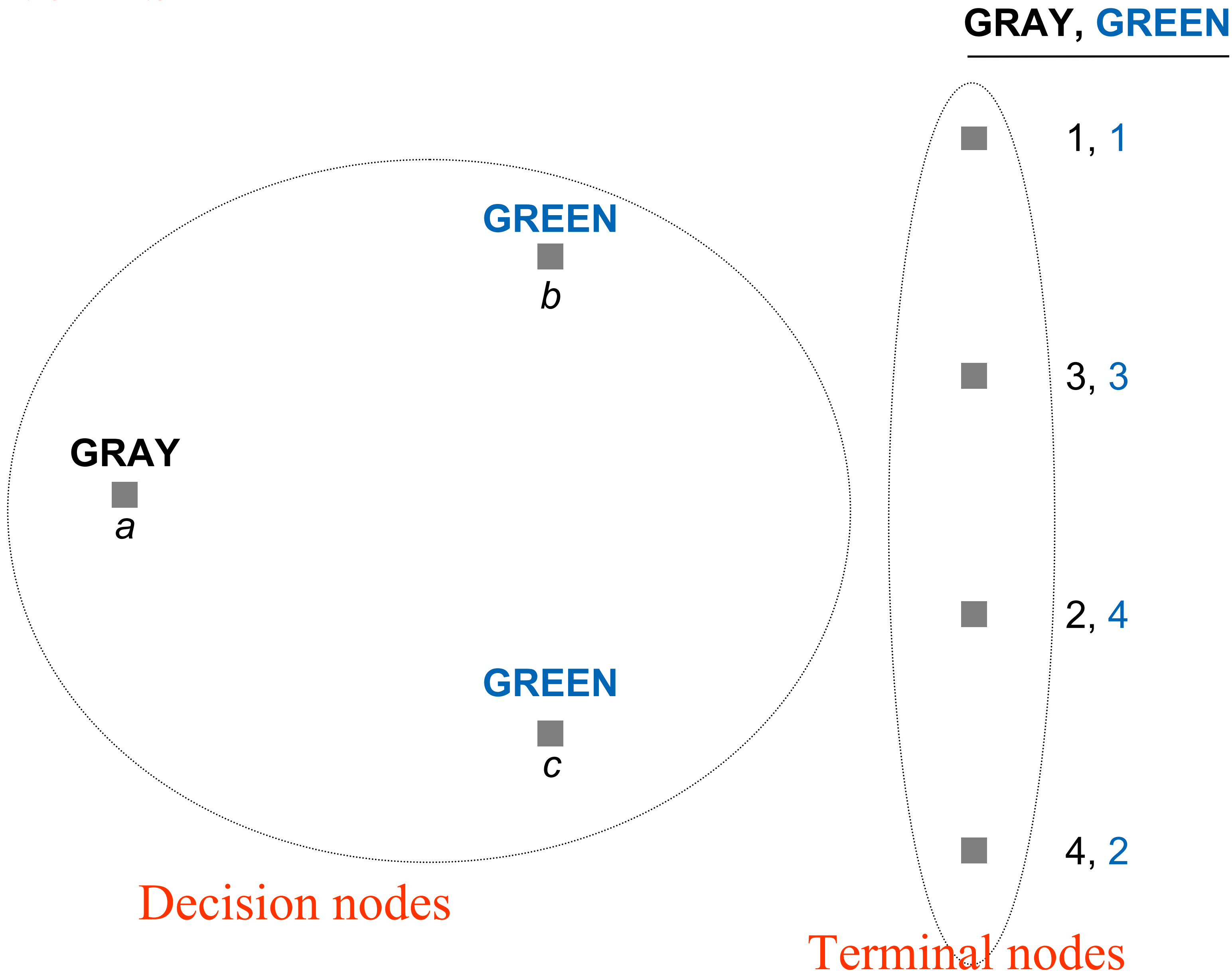
Terminal nodes

Each of (No Ads, Out) (Ads, Out) (No Ads, In) (Ads, In) is represented by a terminal node

2. Branches

"Ads" is a branch

NODES:


GRAY, GREEN
GREEN
b
GRAY
a
GREEN
c
1, 1
3, 3
2, 4
4, 2
Decision nodes
Terminal nodes

NODES AND BRANCHES:

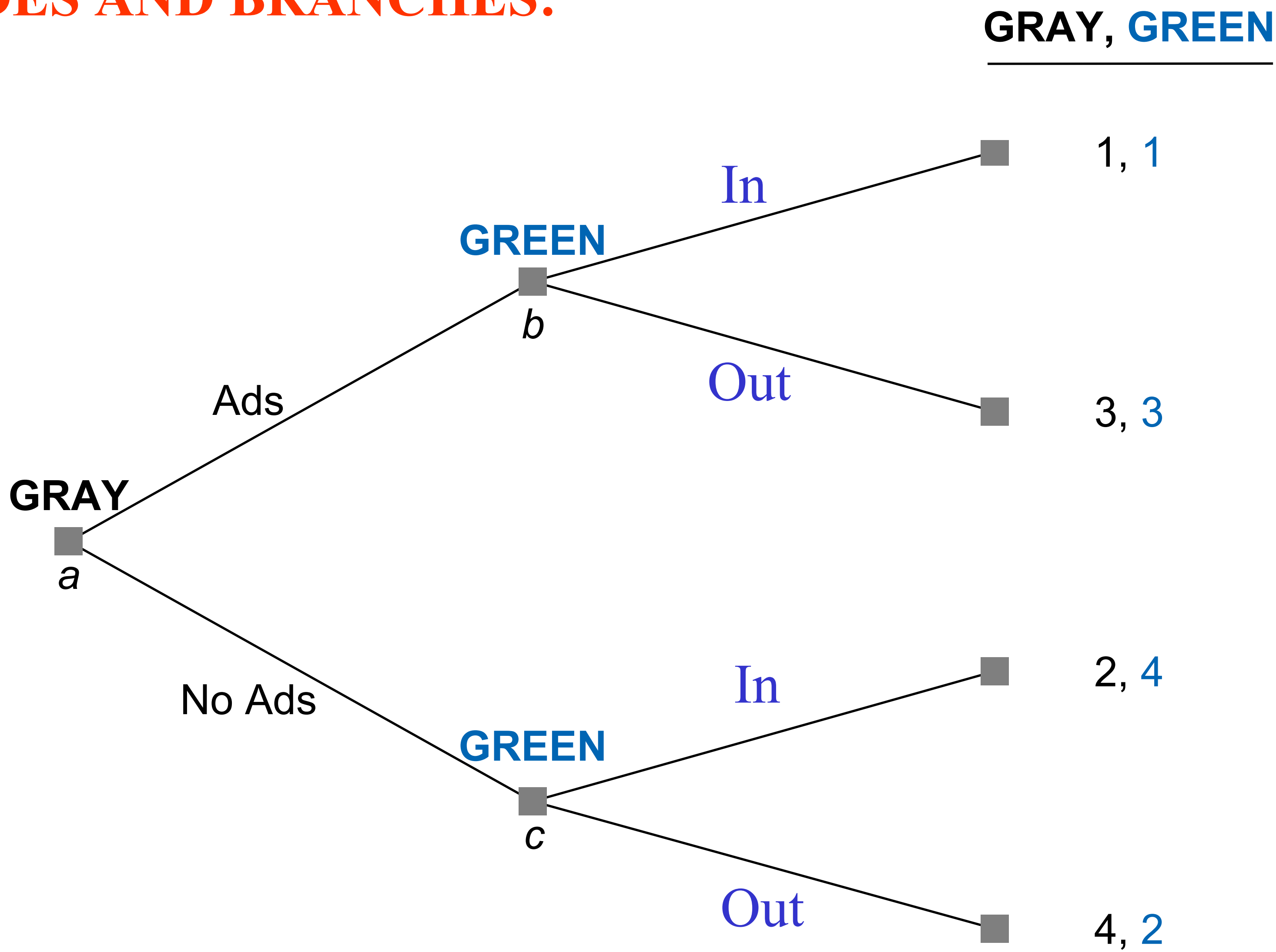


FIGURE 3.1 Tree for Senate Race Game

(a) Pruning at terminal nodes

Backward induction = Rollback technique

GRAY, GREEN

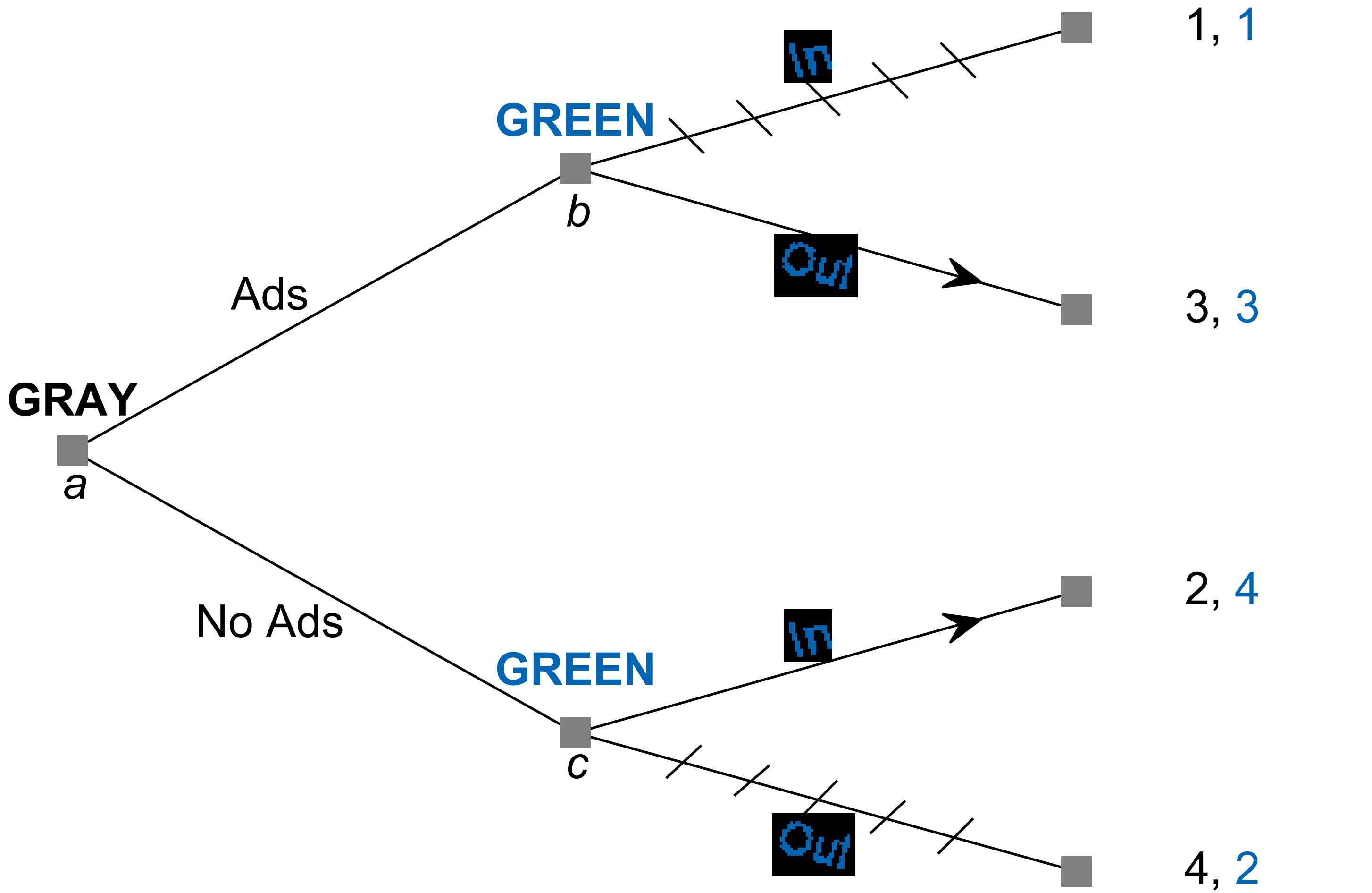


FIGURE 3.2 A Using Rollback

(b) Fully pruned tree

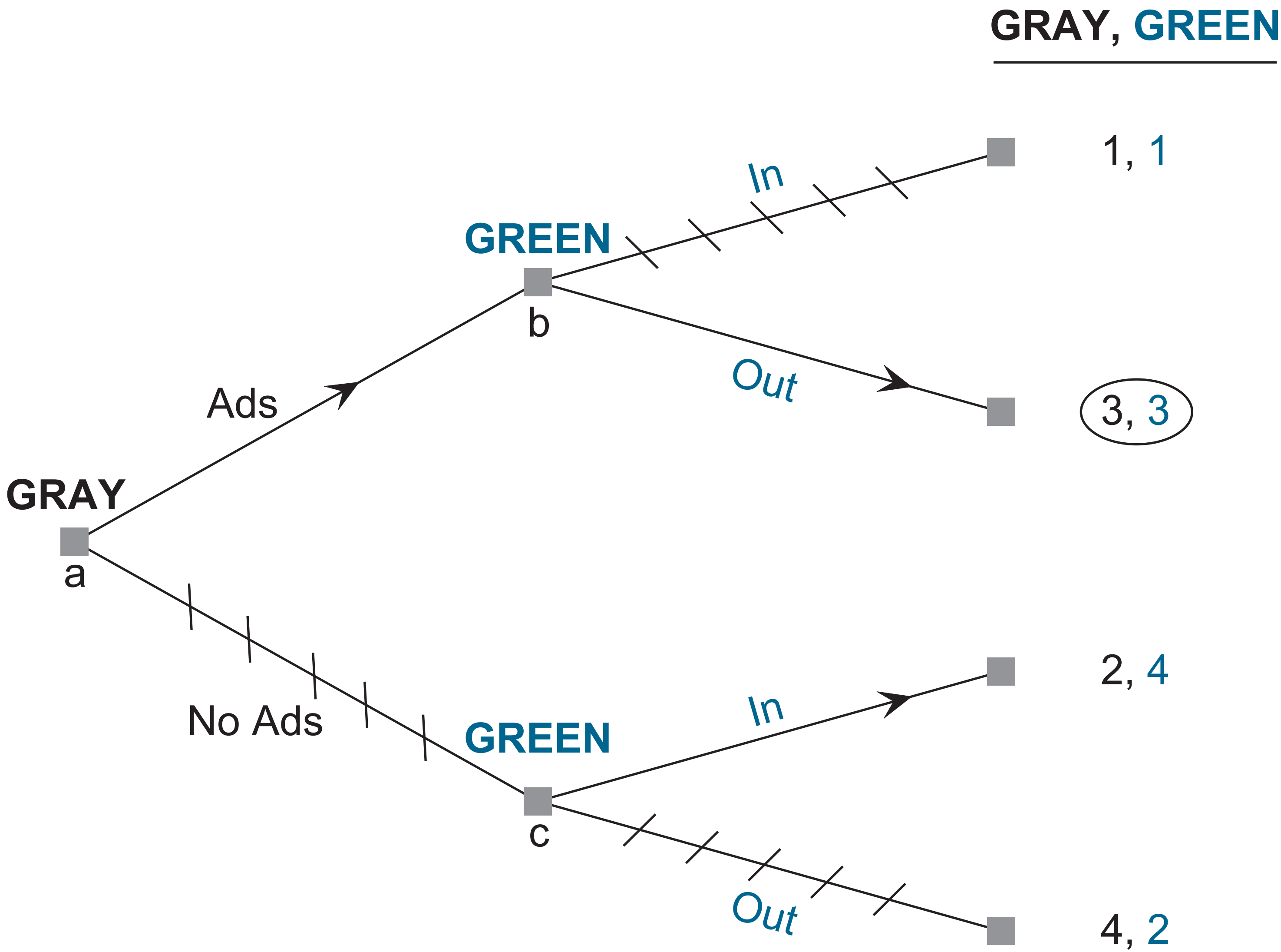


FIGURE 3.2 B Using Rollback

The senate game: formal representation

$$G = (N , S_{gray} , S_{green} , u_{gray} , u_{green})$$

where

$N = \{ Gray , Green \}$

$S_{gray} = \{ Ads , NoAds \}$

$S_{green} = \{$If Ads then In and if NoAds then In,

If Ads then In and if NoAds then Out,

If Ads then Out and if NoAds then In,

If Ads then Out and if NoAds then Out $\}$

and

u_{gray}(Ads , If Ads then In and if NoAds then In) = 1

u_{gray}(Ads , If Ads then In and if NoAds then Out) = 1

u_{gray}(Ads , If Ads then Out and if NoAds then In) = 3

u_{gray}(Ads , If Ads then Out and if NoAds then Out) = 3

u_{gray}(NoAds , If Ads then In and if NoAds then In) = 2

u_{gray}(NoAds , If Ads then In and if NoAds then Out) = 4

u_{gray}(NoAds , If Ads then Out and if NoAds then In) = 2

u_{gray}(NoAds , If Ads then Out and if NoAds then Out) = 4

u_{green}(Ads , If Ads then In and if NoAds then In) = 1

u_{green}(Ads , If Ads then In and if NoAds then Out) = 1

u_{green}(Ads , If Ads then Out and if NoAds then In) = 3

u_{green}(Ads , If Ads then Out and if NoAds then Out) = 3

u_{green}(NoAds , If Ads then In and if NoAds then In) = 4

u_{green}(NoAds , If Ads then In and if NoAds then Out) = 2

u_{green}(NoAds , If Ads then Out and if NoAds then In) = 4

u_{green}(NoAds , If Ads then Out and if NoAds then Out) = 2

A strategy profile is a list of strategies, **one for each agent**

That is,

one from S_{gray} = { Ads , NoAds } and

one from S_{green} = {If Ads then In and if NoAds then In,

If Ads then In and if NoAds then Out,

If Ads then Out and if NoAds then In,

If Ads then Out and if NoAds then Out }

Example:

(NoAds ; If Ads then Out and if NoAds then In)

A Nash equilibrium is a strategy profile with the property that

no agent can increase her payoff by changing her strategy in the profile

More on this to come later

The rollback (a.k.a. backward induction) technique gives you an equilibrium:

(Ads ; If Ads then Out and if NoAds then In)

To verify, we ask two questions:

1. Can Gray increase her payoff by changing from Ads to NoAds ?

2. Can Green increase her payoff by changing from

If Ads then Out and if NoAds then In

to any one of her other strategies ?

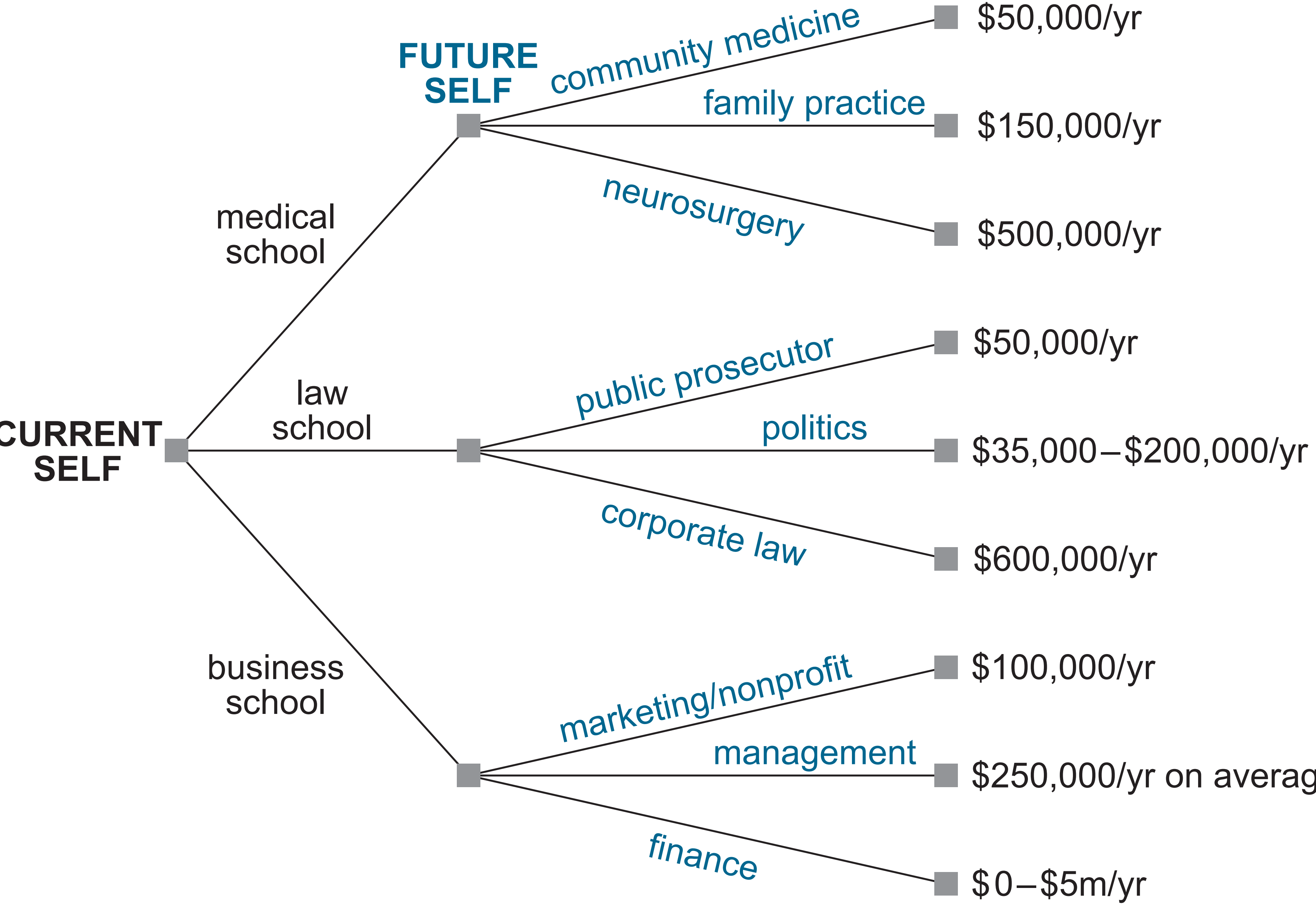


FIGURE 3.3 “One”-Player Game

First-mover advantage

Second-mover advantage

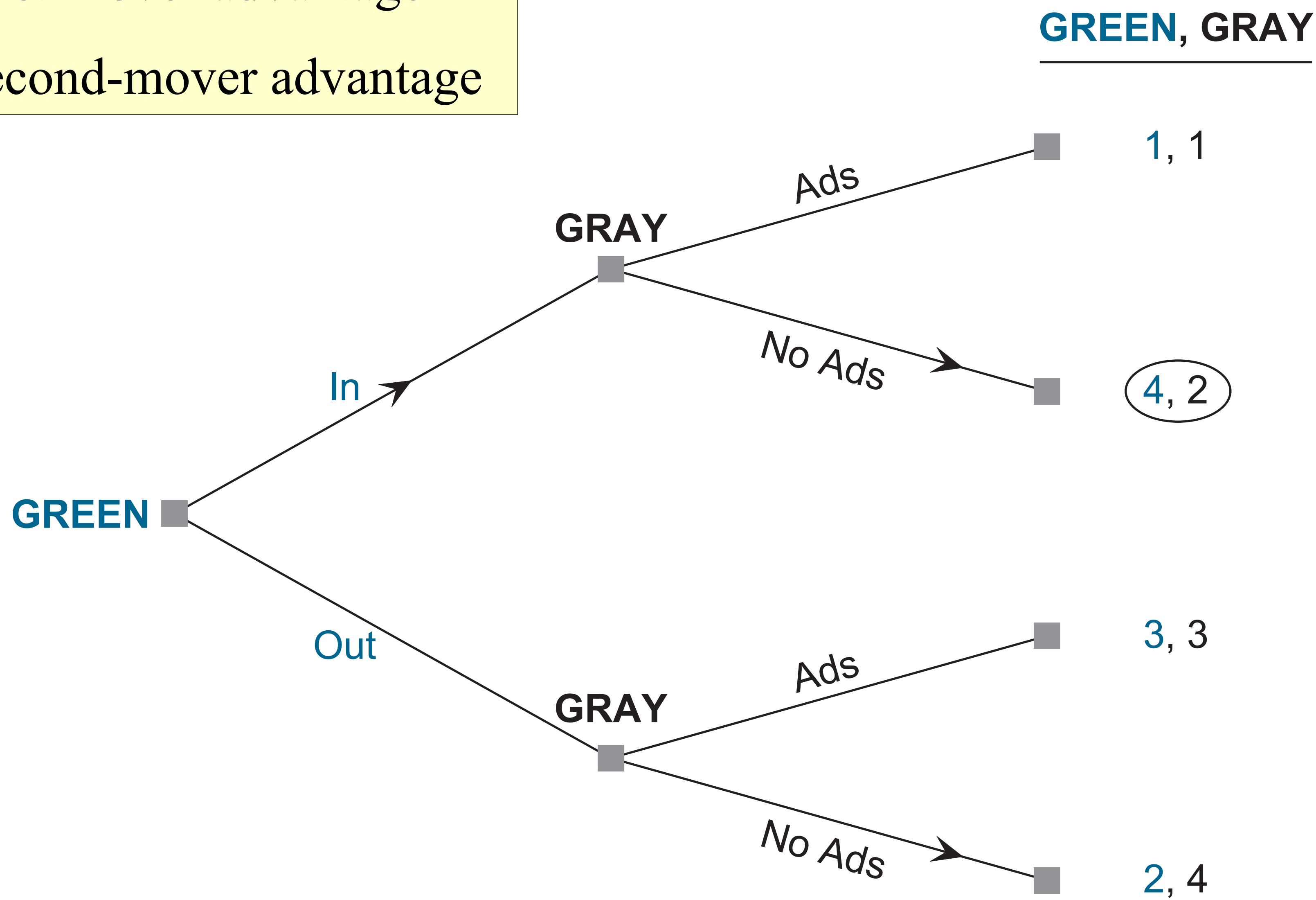


FIGURE 3.4 Change of Move Order in the Senate Race Game

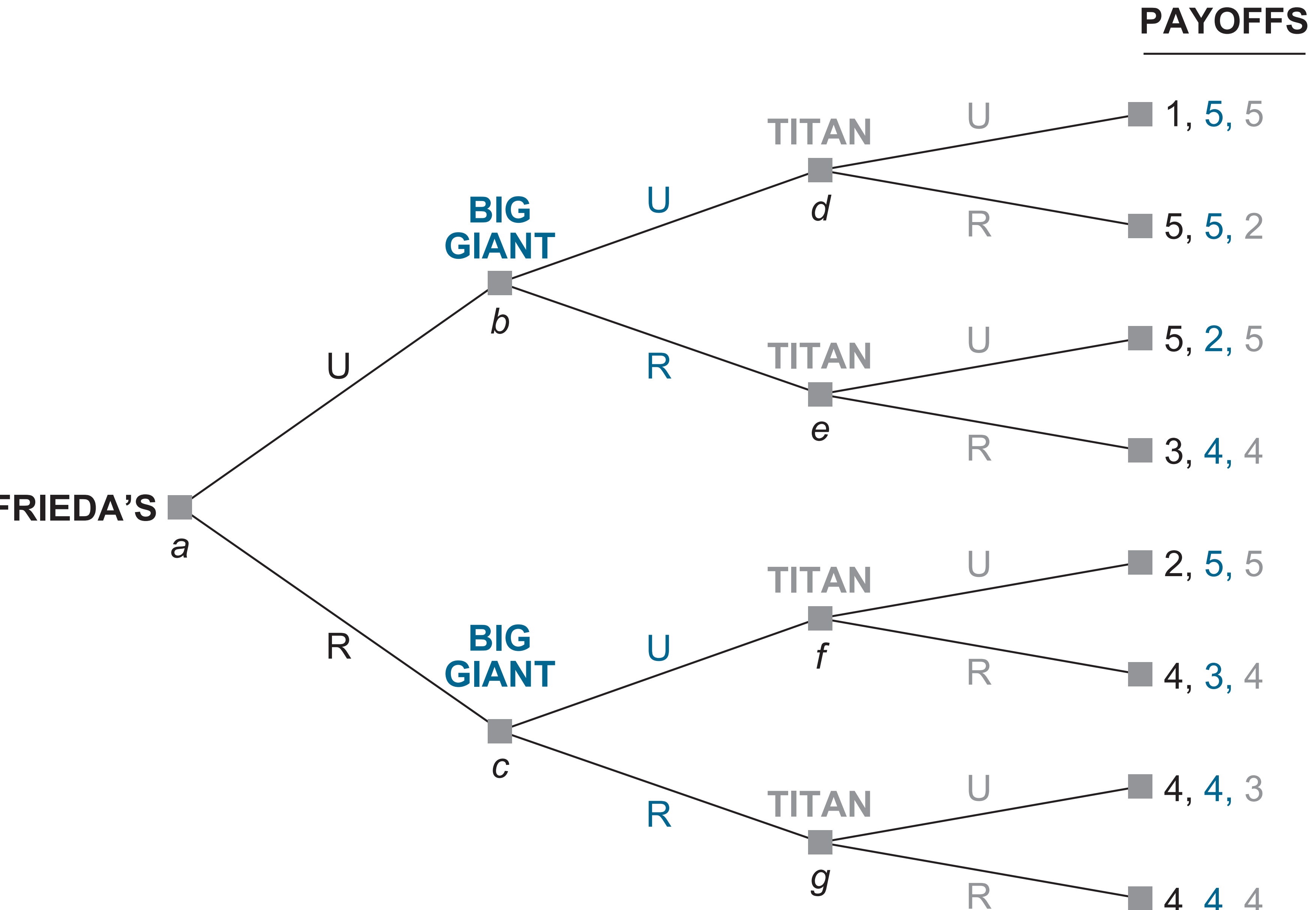


FIGURE 3.5 Three-Player Game Tree

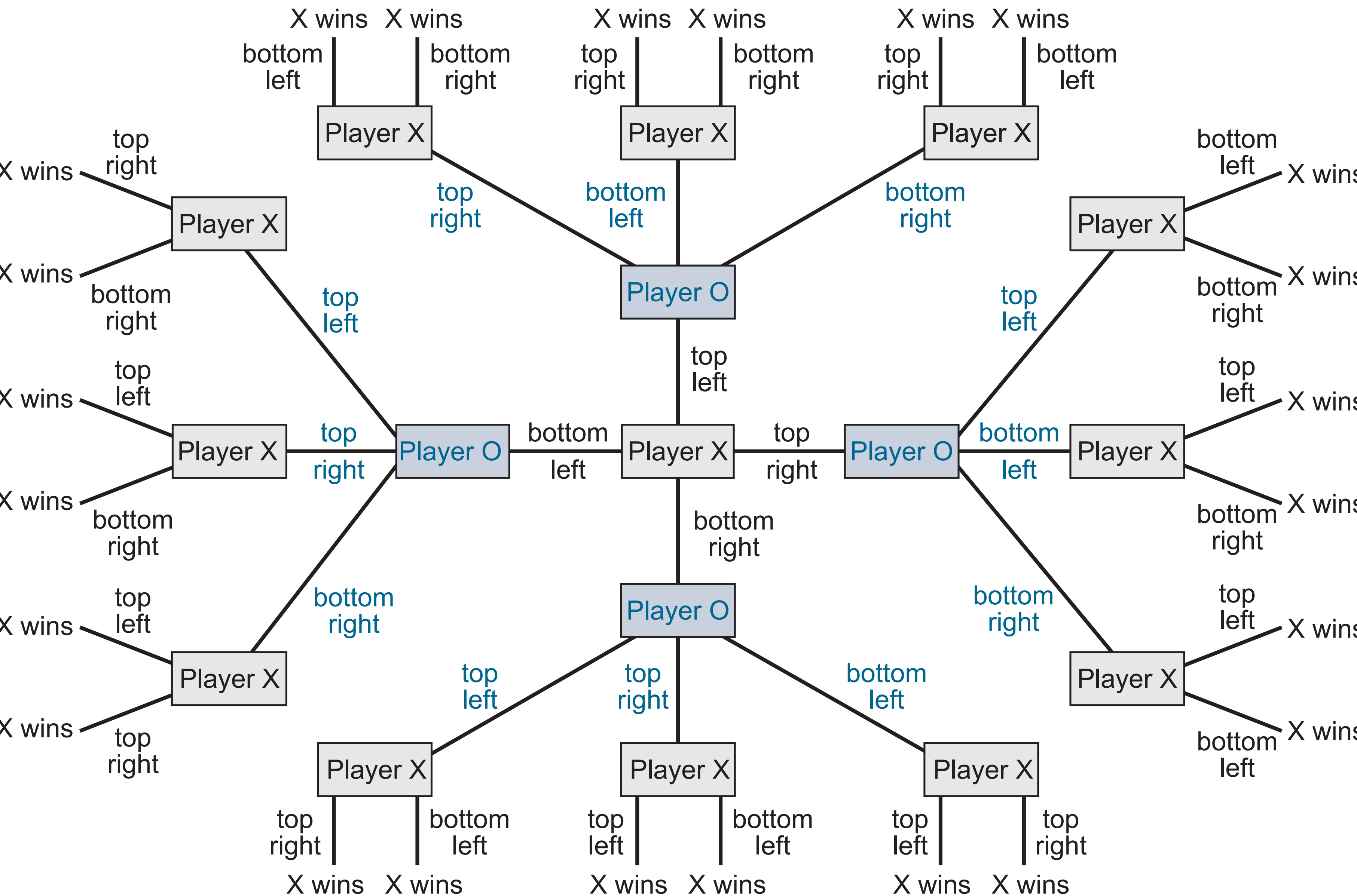


FIGURE 3.6 A More Complex Tree

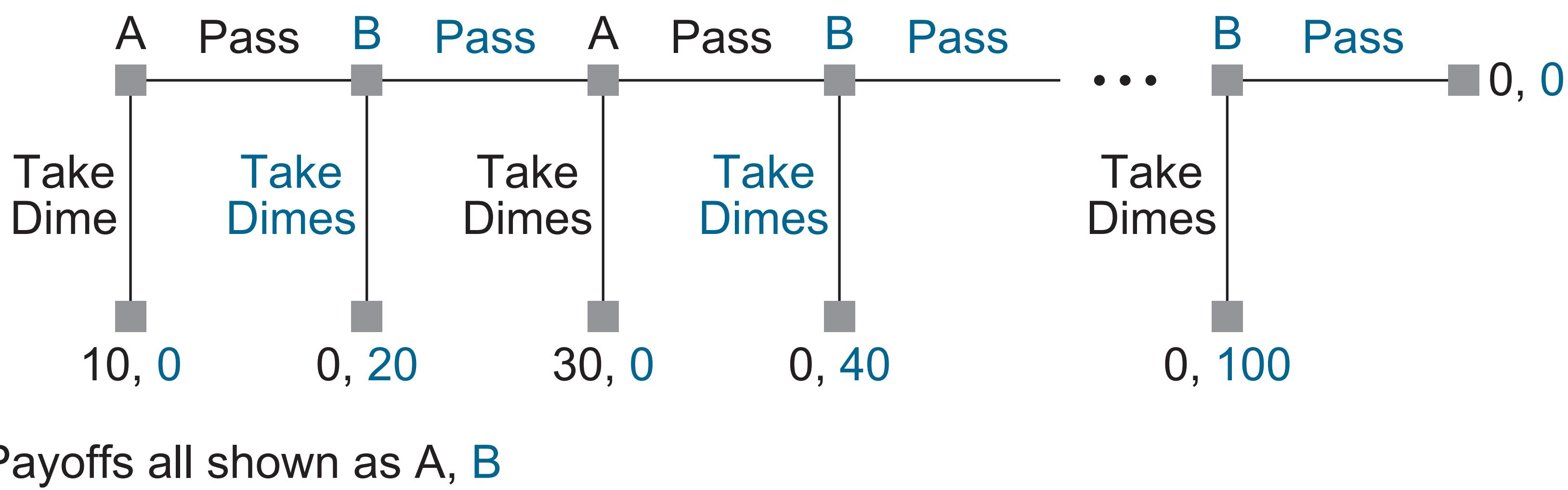
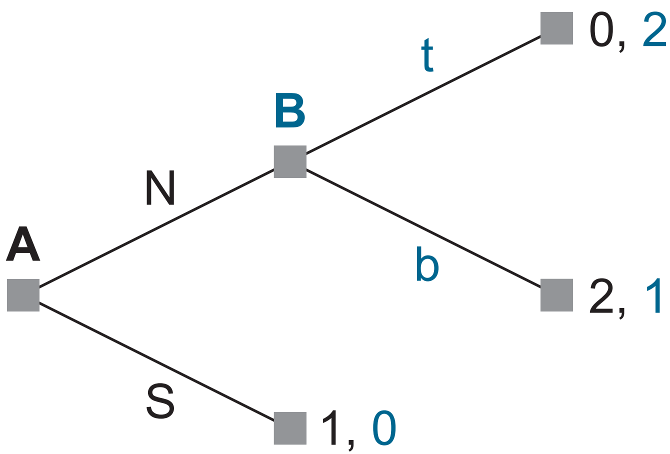
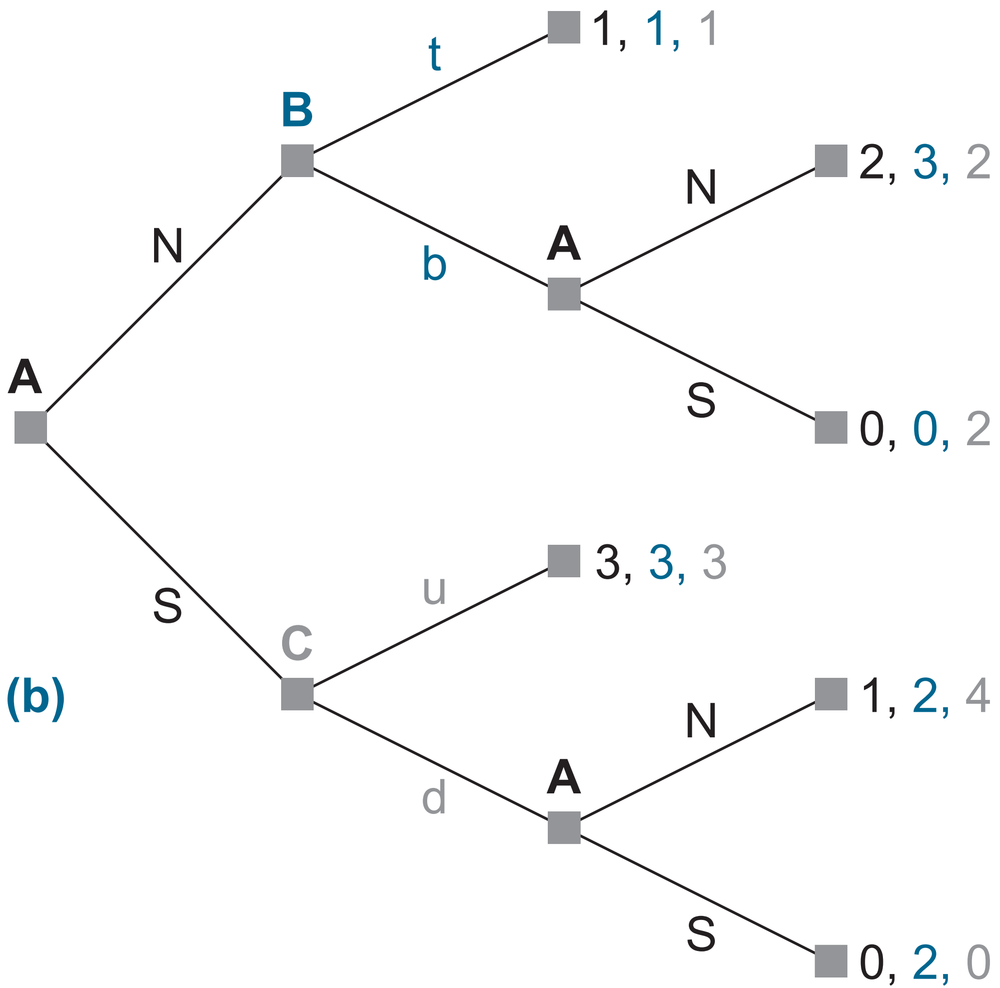


FIGURE 3.7 The Centipede Game

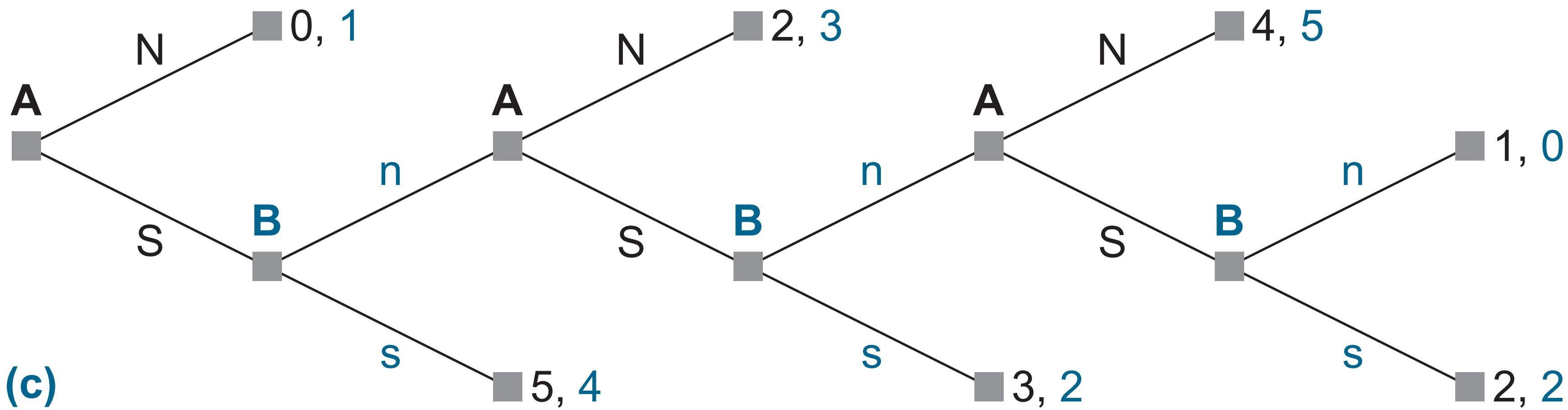


(a)



(b)

EXERCISE 3.2 ab



EXERCISE 3.2 c

Sequential-move games

extensive form (i.e. the game tree).

rollback technique

Simultaneous-move games

strategic form (i.e. the game table).

dominant or dominated strategies, cell-by-cell, and minimax techniques

Next: analyse relationship between the two

WHY ???

To be able to

1. relate the solution techniques used for the two types of games.
2. solve games that are mixtures of sequential and simultaneous moves.

(a) **Extensive form** of the Senate Race Game

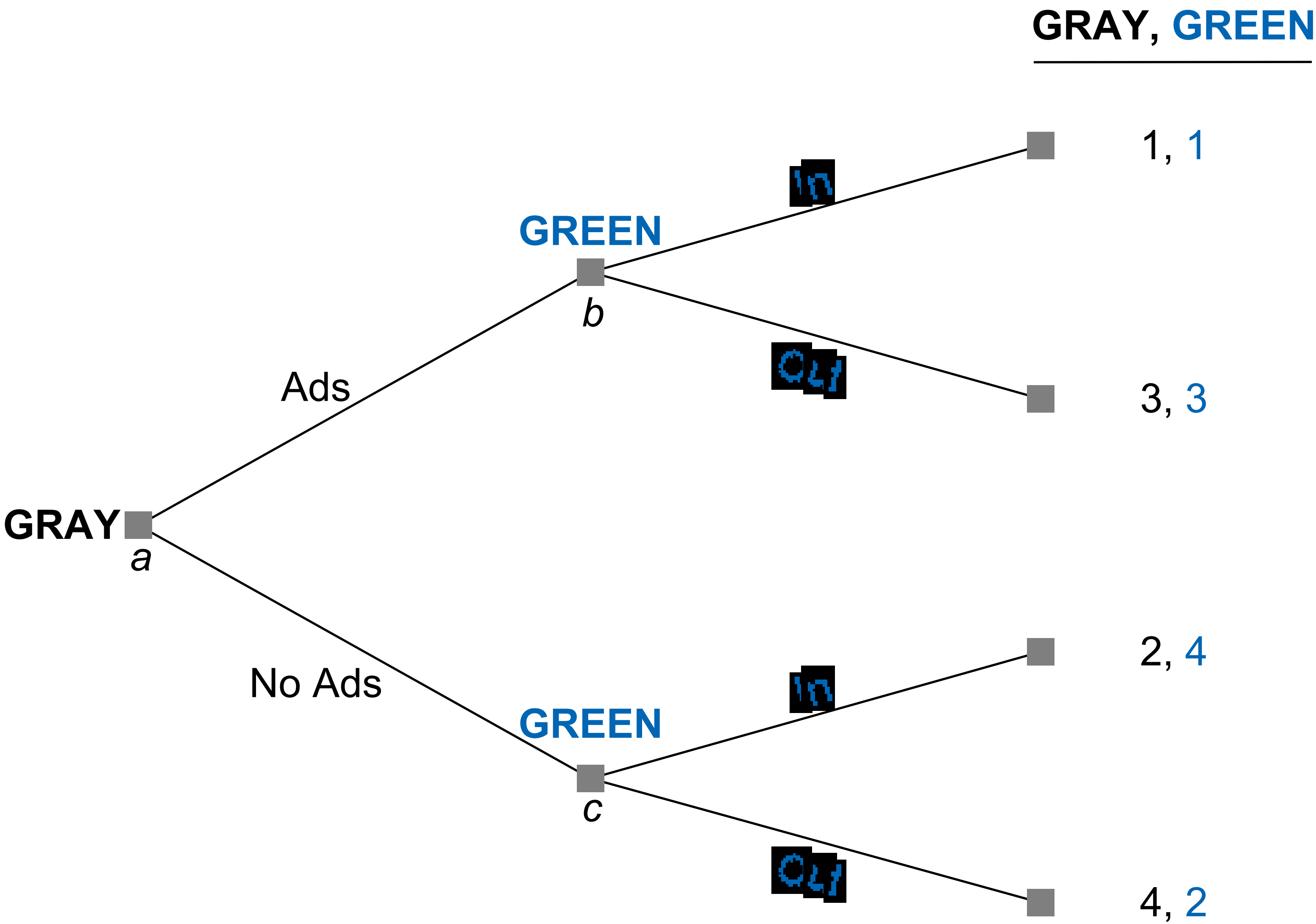


FIGURE 6.1 A Senate Race Game

This table represents an interaction different than the Senate Race game !

		GREEN	
		In	Out
GRAY	Ads	1, 1	3, 3
	No Ads	2, 4	4, 2

IGURE 6.8 Simultaneous-Play Version of the Senate Race Game

How to translate an extensive-form game into strategic-form?

1. Determine the agents' strategies

    a. What are the pure strategies of GRAY?

    b. What are the pure strategies of GREEN?

3. Use these strategies to form the game table.

4. Determine the payoffs of each outcome from the original tree.

(b) **Strategic form** of the Senate Race Game

		GREEN			
		In, In	In, Out	Out, In	Out, Out
GRAY	Ads	1, 1	1, 1	3, 3	3, 3
	No Ads	2, 4	4, 2	2, 4	4, 2

FIGURE 6.1 B Senate Race Game

Applying rollback to the game tree gives us the strategy profile

(Ads ; (Ads=>Out, NoAds=>In))

This profile is a Nash equilibrium of the Senate Race game

(verify this claim by using the strategic form representation)

this is not a coincidence

Theorem: For every game, every strategy profile that is obtained by applying the rollback (a.k.a. backward induction) technique to the game tree is a Nash equilibrium of the game.

Checking the strategic form of the Senate Race game however gives us another Nash equilibrium

(NoAds ; (Ads=>In, NoAds=>In))

Thus

Theorem: The rollback technique does not give you all Nash equilibria of a game.

Note that this new equilibrium is based on a noncredible threat

Claim: every Nash equilibrium which is not obtained by the rollback technique is based on a noncredible threat.

Let us formalize this idea:

Definition: Given an extensive form game and a decision node that is not part of an information set, the part of the game tree that follows from that decision node is called a **subgame**.

Note that every subgame of a game are themselves games and they have their own Nash equilibria.

Definition: A Nash equilibrium $(s_1, \ldots, s_n)$ for the game G is called a **subgame perfect Nash equilibrium of G** if for every subgame G' of the game G, the restriction of $(s_1, \ldots, s_n)$ to G' is a Nash equilibrium of G'.

The relation between

subgame perfect Nash equilibria

and the

rollback technique:

Theorem: Given an extensive form game G, the strategy profile $(s_1,\dots,s_n)$ is a subgame perfect Nash equilibrium of G if and only if $(s_1,\dots,s_n)$ is obtained by the rollback technique.

Subgame perfect Nash equilibria are those Nash equilibria that are not based on noncredible threats.

The Theorem follows since the rollback technique does not allow noncredible threats.

(a) Strategic form

		JAPANESE NAVY	
		North	South
USAF	North	2	2
	South	1	3

FIGURE 6.2 A The Battle of the Bismarck Sea

How to translate a strategic-form game into extensive form?

1. Choose a player to be the first-mover, say USAF.

   (He **won't** actually move first, doesn't matter who you choose)

2. Draw USAF's decision node and branches denoting its actions.

3. At the end of each branch, there will be a decision node for Japan

4. Draw branches denoting Japan's actions.

5. Write down the payoffs corresponding to each outcome.

In the story, Japan **did not know** which action USAF took.

How to denote this in the game tree?

Use an information set:

it is a set of decision nodes (of a single player), the player **can not observe** which of these decision nodes the game has reached.

Information sets indicate incomplete information.

The player **has to choose the same action** in every decision node in the same information set.

NOTE: You can not apply the rollback technique when there are (non-singleton) information sets.

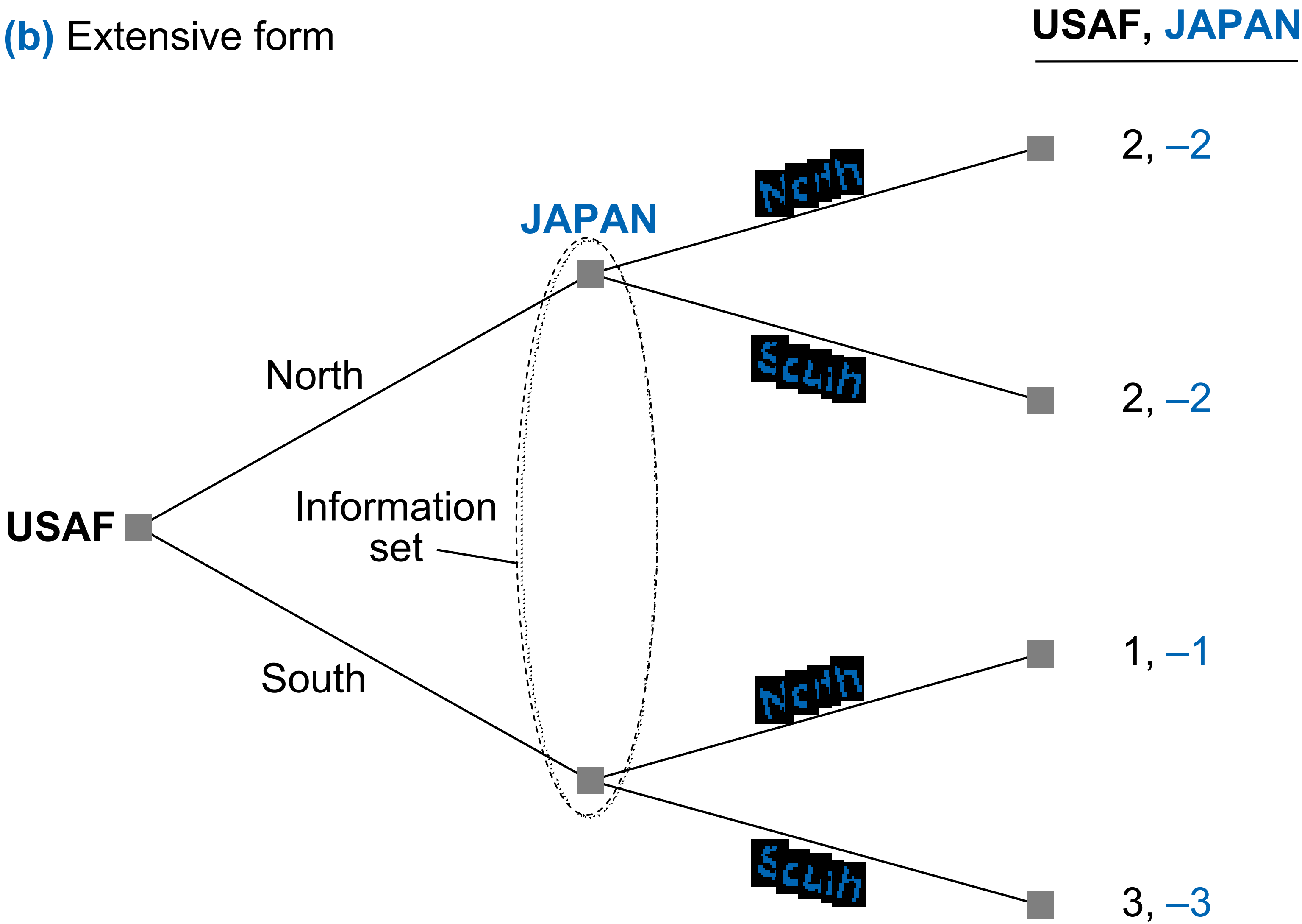


FIGURE 6.2 B The Battle of the Bismarck Sea

A game with both sequential and simultaneous moves

Two electronics firms: KUMQUAT and KIWIFRUIT

1. period: firms simultaneously choose their R&D budgets

   this determines the quality of their products

2. period: firms simultaneously choose their prices

   this determines their sales and profits

How to write down the strategies available to these firms?

How to find the equilibria of this game?

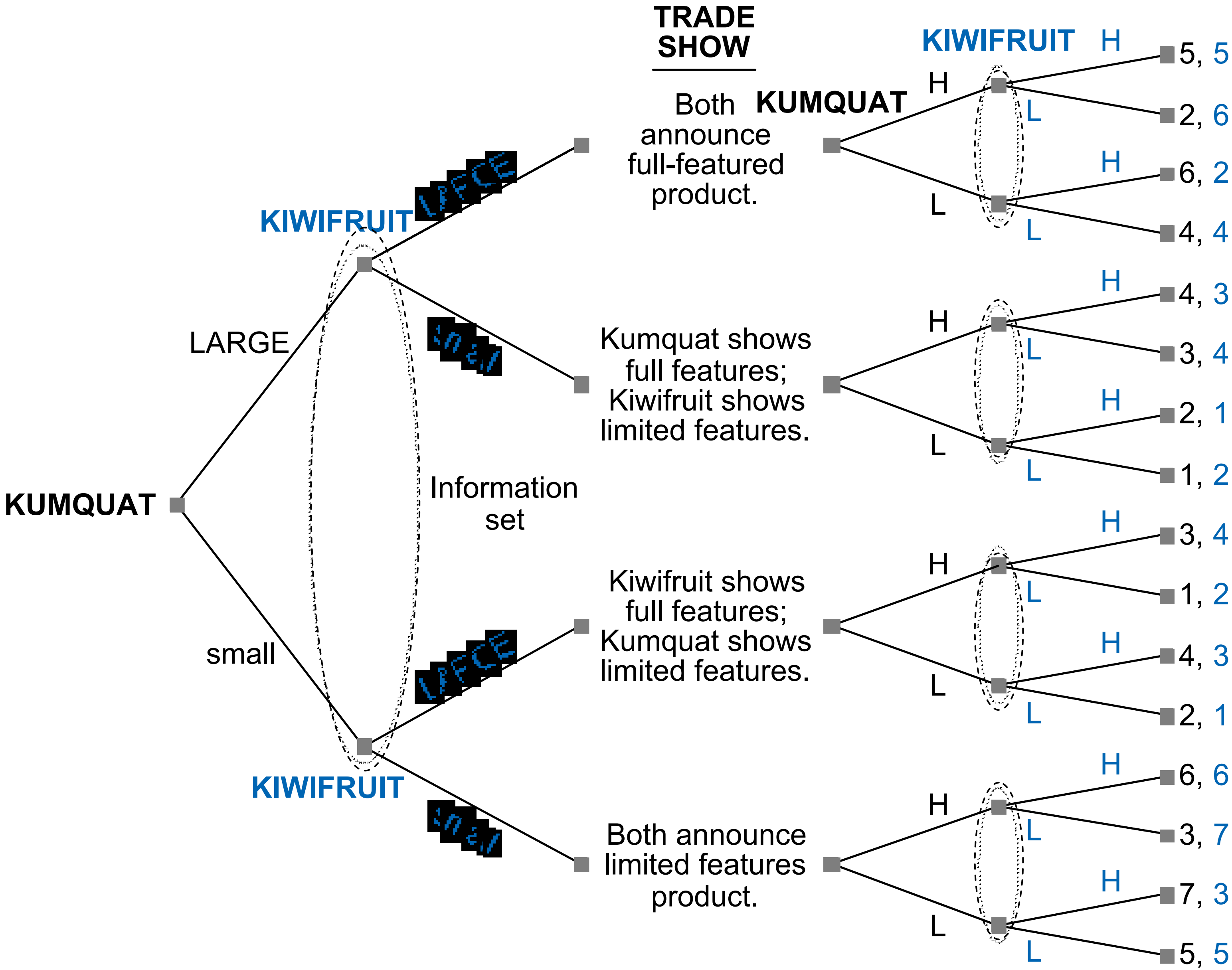


FIGURE 6.3 A Combination Game in Extensive Form

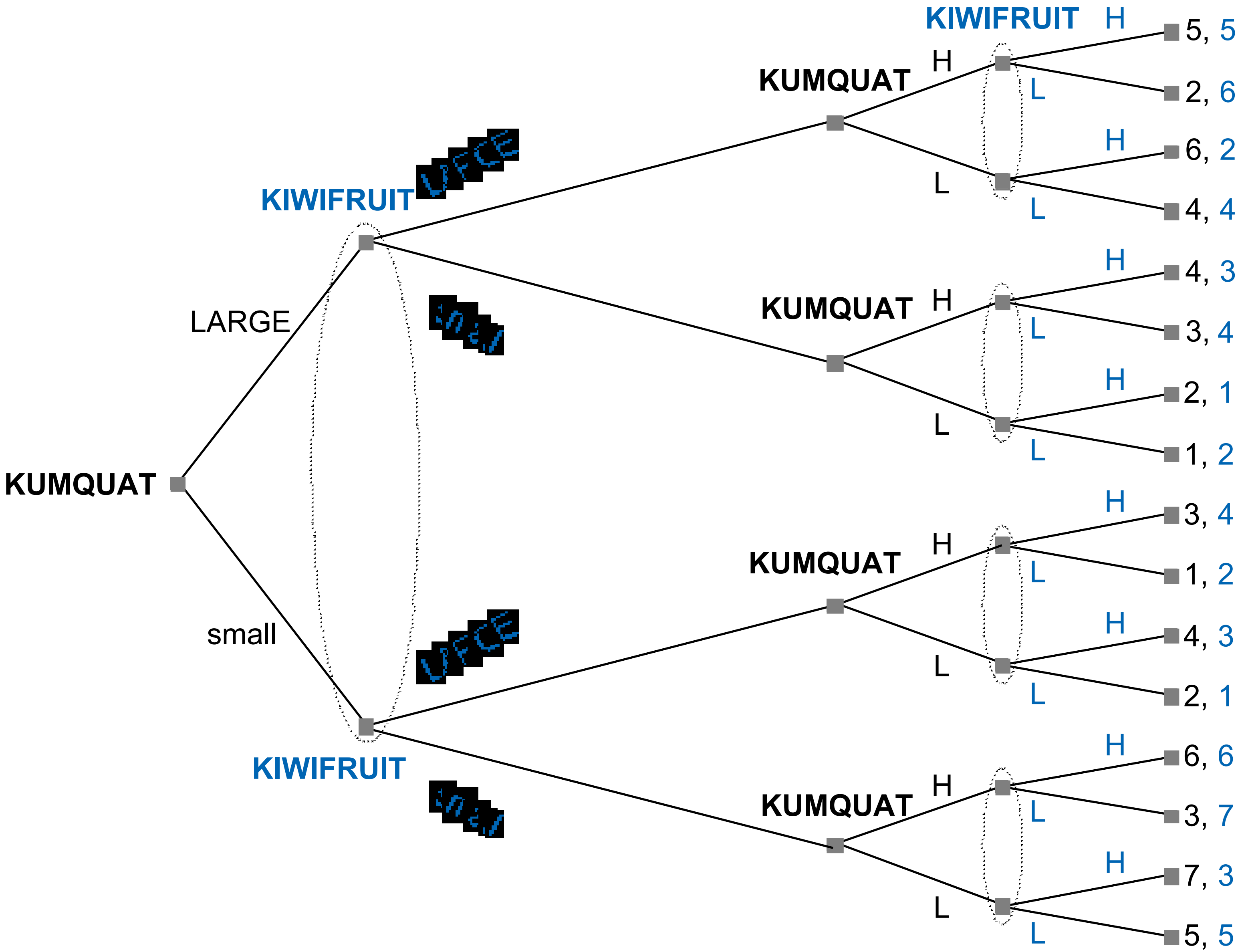
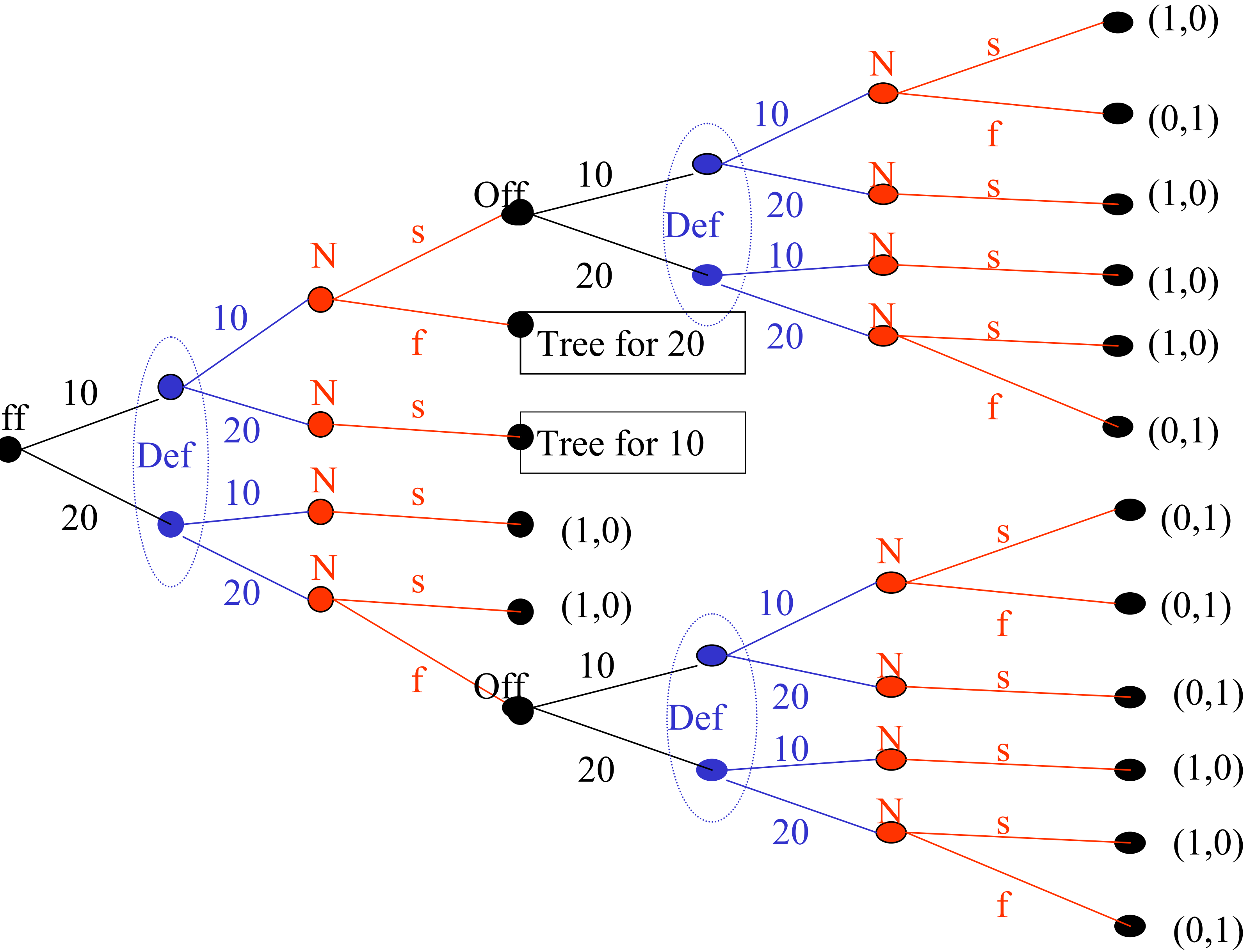


FIGURE 6.3 A Combination Game in Extensive Form


ff
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N
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s
s
s
f
Off
Off
Tree for 20
Tree for 10
(1,0)
(1,0)
10
20
10
20
Def
Def
10
20
10
20
10
20
10
20
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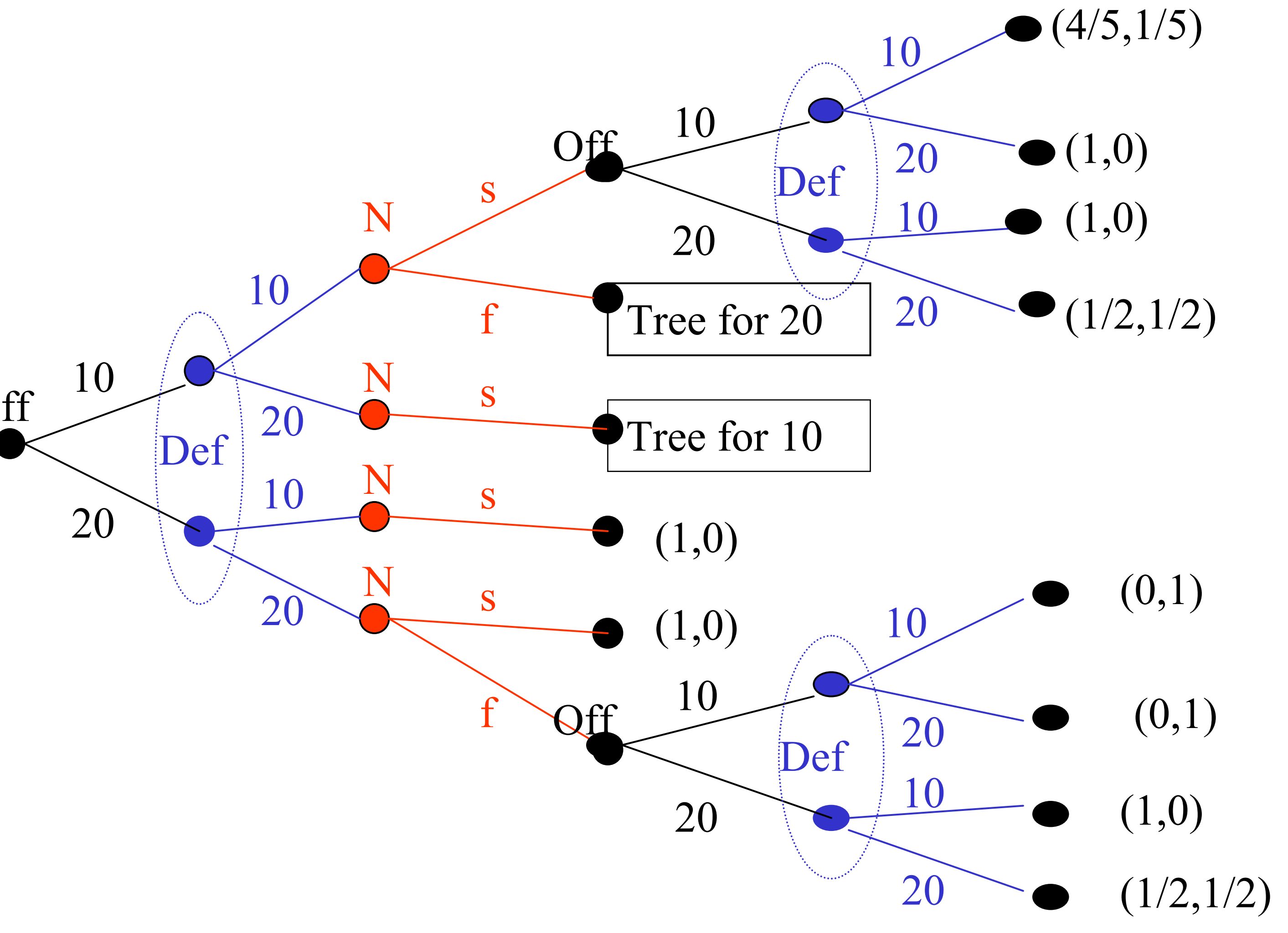
s
f
(1,0)
(0,1)
(1,0)
(1,0)
(1,0)
(0,1)
(0,1)
(0,1)
(0,1)
(1,0)
(1,0)
(0,1)

		DEFENSE	
		10	20
OFFENSE	10	4/5	1
	20	1	1/2

FIGURE 6.4 Success Probability Table in Two-Play Football Example


ff
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Def
10
20
10
20
N
N
N
N
s
f
s
s
s
f
Off
10
20
Def
10
20
10
20
(4/5,1/5)
(1,0)
(1,0)
(1/2,1/2)
Tree for 20
Tree for 10
(1,0)
(1,0)
Off
10
20
Def
10
20
10
20
(0,1)
(0,1)
(1,0)
(1/2,1/2)

		DEFENSE	
		10	20
OFFENSE	10	0	0
	20	1	$^1/_2$

FIGURE 6.5 Payoff Table for Fourth Down, 20 Yards to Go

		DEFENSE	
		10	20
OFFENSE	10	4/5	1
	20	1	1/2

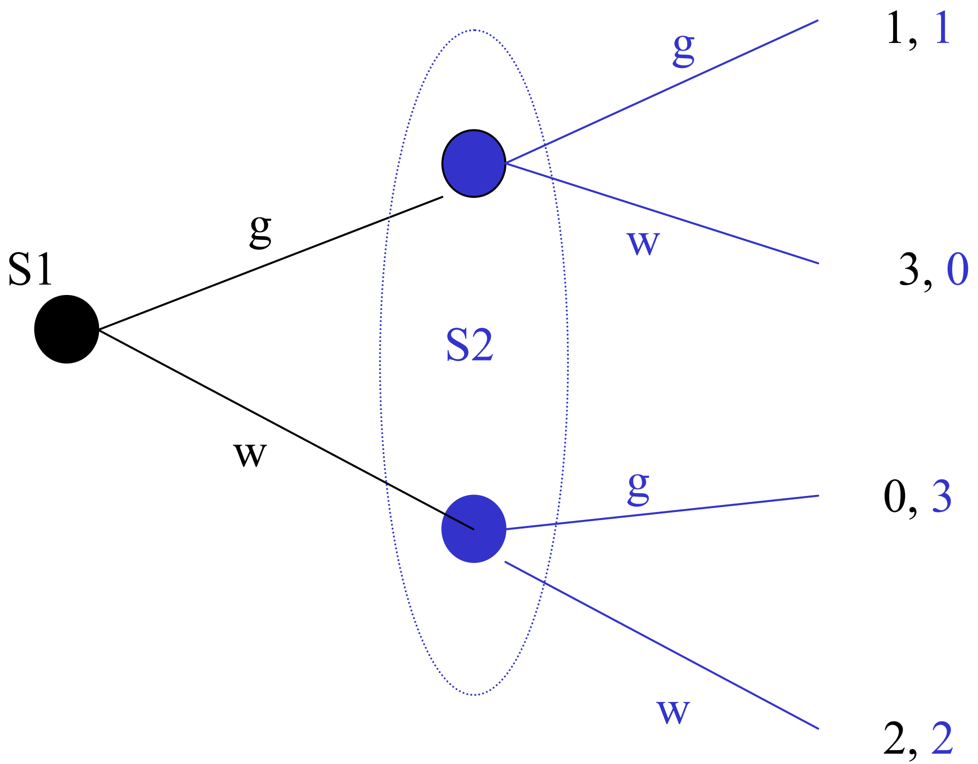
FIGURE 6.6 Payoff Table for Fourth Down, 10 Yards to Go

		DEFENSE	
		10	20
OFFENSE	10	11/14	6/7
	20	1	3/4

FIGURE 6.7 Payoff Table for Third Down

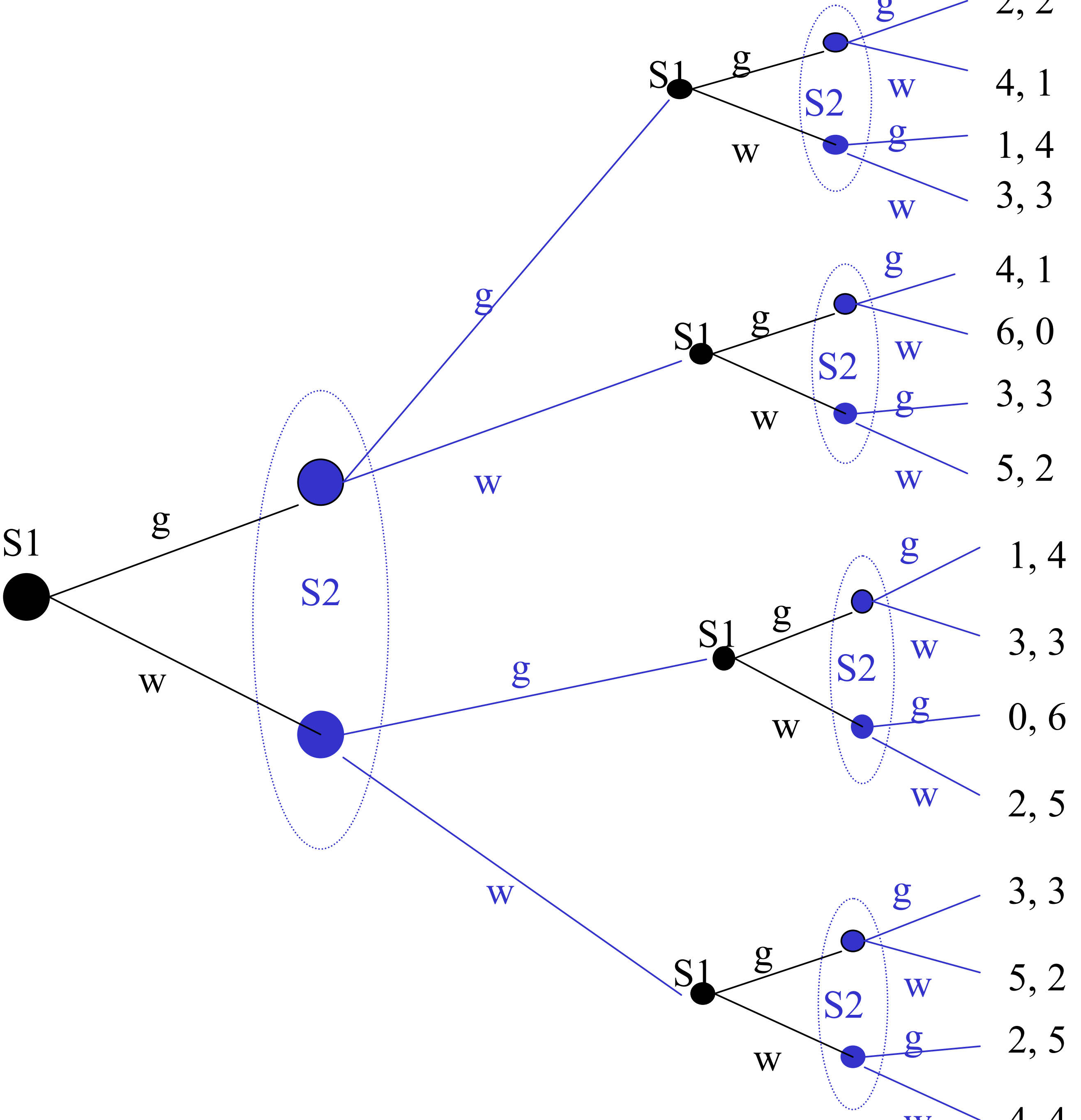
Prisoners' Dilemma

		Student 2	
		Goof off (Defect)	Work hard (Cooperate)
Student 1	Goof off (Defect)	1, 1	3, 0
	Work hard (Cooperate)	0, 3	2, 2


S1
g
w
S2
g
w
g
w
1, 1
3, 0
0, 3
2, 2

A twice-repeated prisoners’ dilemma game

1. Draw the game tree.
2. How many subgames does this game have?
3. What are the players’ strategies?
4. Apply rollback to find all subgame perfect Nash equilibria.


S1
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w
S2
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S1
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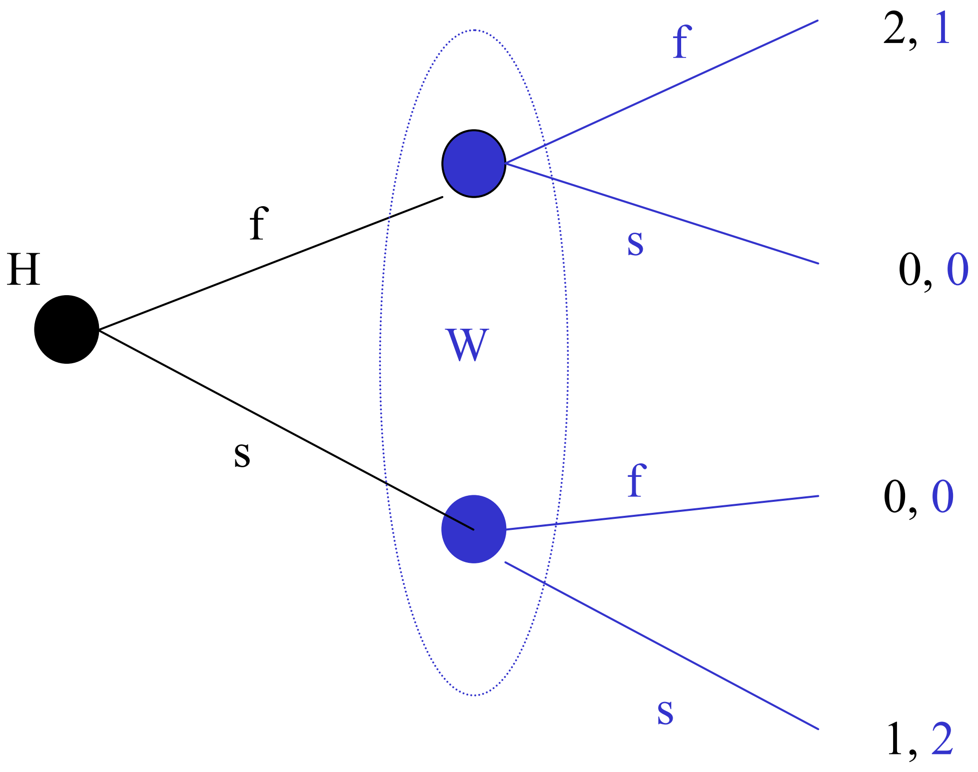
S1
S2
S2
S2
S2
2, 2
4, 1
1, 4
3, 3
4, 1
6, 0
3, 3
5, 2
1, 4
3, 3
0, 6
2, 5
3, 3
5, 2
2, 5
4, 4

Battle of the Sexes

		Wife	
		Football	Soap opera
Husband	Football	2, 1	0, 0
	Soap opera	0, 0	1, 2

Other examples: two politicians determining position on an issue

two merging firms choosing between PC and MAC


H
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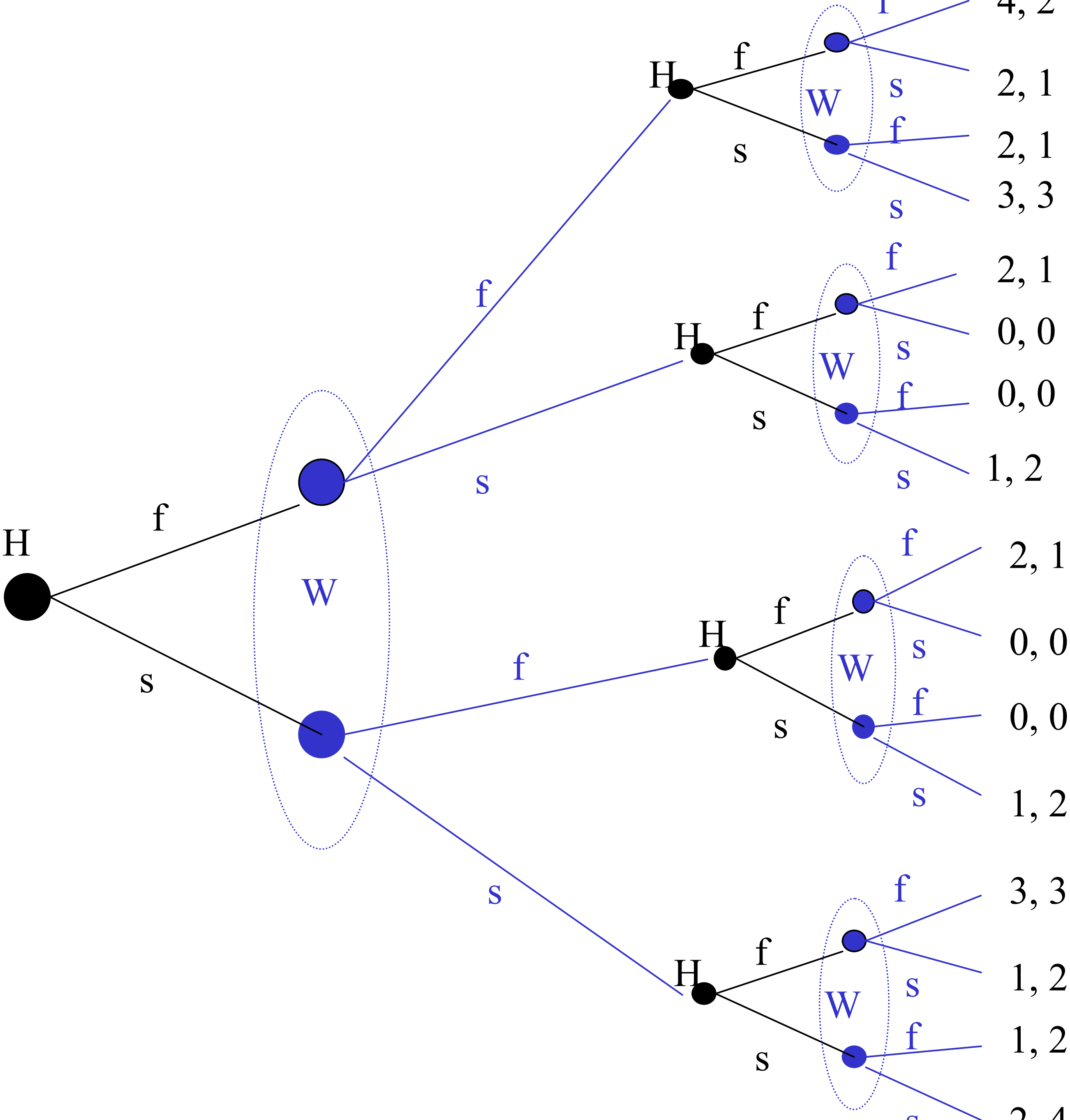
s
W
f
s
f
s
2, 1
0, 0
0, 0
1, 2

A twice-repeated battle-of-the-sexes game

1. Draw the game tree.
2. How many subgames does this game have?
3. What are the players' strategies?
4. Apply rollback to find all subgame perfect Nash equilibria.


H
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H
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2, 1
2, 1
3, 3
H
f
s
W
f
s
f
s
2, 1
0, 0
0, 0
1, 2
H
f
s
W
f
s
f
s
2, 1
0, 0
0, 0
1, 2
H
f
s
W
f
s
f
3, 3
1, 2
1, 2

Sequential moves and infinitely many actions

- Analyze the sequential versions of the following games.
- Relate your reasoning to the best-response analysis

The Sequential Price-setting Game Between Donna and Pierce

Donna's Deep Dish: moves first and chooses P_{Donna}
Pierce's Pizza Pies: moves second and chooses P_{Pierce}

Market surveys show that given the prices each sells (in 1000 pizzas per week):

$$Q_{Donna} = 12 - P_{Donna} + 0.5P_{Pierce}$$

$$Q_{Pierce} = 12 - P_{Pierce} + 0.5P_{Donna}$$

Note: If Pierce increases his price, his sales go down and Donna's sales go up

- Cost of each pizza: 3 USD

- Pierce's profit (i.e. his **payoff**) (in 1000 USD) is then

$$
\begin{aligned}
Y_{Pierce} &= P_{Pierce}Q_{Pierce} - 3Q_{Pierce} \\
&= (P_{Pierce} - 3)Q_{Pierce} \\
&= (P_{Pierce} - 3)(12 - P_{Pierce} + 0.5P_{Donna}) \\
&= (15 + 0.5P_{Donna})P_{Pierce} - P_{Pierce}^2 - 36 - 1.5P_{Donna}
\end{aligned}
$$

- Given P_{Donna}, Pierce will choose his price to maximize his payoff

  Taking the derivative of Y_{Pierce} with respect to P_{Pierce}

$$\frac{dY_{Pierce}}{dP_{Pierce}} = 15 + 0.5P_{Donna} - 2P_{Pierce}$$

  When Y_{Pierce} is maximized, this derivative is equal to 0

$$\frac{dY_{Pierce}}{dP_{Pierce}} = 15 + 0.5P_{Donna} - 2P_{Pierce} = 0$$

Solving for P_{Pierce} we have

$$P^*_{Pierce} = \frac{15 + 0.5P_{Donna}}{2} = 7.5 + 0.25P_{Donna}$$

This is the **best-response function** of Pierce.

NOTE: We have to verify that what we found by equating the derivative to 0 is a maximum (it can also be a minimum or a saddle-point). For this, we must check if the second derivative at P^*_{Pierce} is negative:

$$\frac{d^2Y_{Pierce}}{dP^2_{Pierce}} = -2$$

So it's O.K.. We have maximized Y_{Pierce} at P^*_{Pierce} and therefore, we have a best-ersponse function.

What about Donna?

- Donna knows that whatever price she chooses, Pierce will observe it and play a best response.
- So Donna's problem is to maximize her payoff when P_{Pierce} is given by the previous formula.

Donna's payoff function is

$$
\begin{aligned}
Y_D &= \left(15 + \frac{1}{2}P_P\right)P_D - P_D^2 - 36 - \frac{3}{2}P_P \\
&= \left(15 + \frac{1}{2}\left(\frac{15}{2} + \frac{1}{4}P_D\right)\right)P_D - P_D^2 - 36 - \frac{3}{2}\left(\frac{15}{2} + \frac{1}{4}P_D\right) \\
&= -\frac{7}{8}P_D^2 + \frac{147}{8}P_D - \frac{189}{4}
\end{aligned}
$$

The first and the second derivatives are

$$
\begin{aligned}
\frac{\partial Y_D}{\partial P_D} &= \frac{147}{8} - \frac{7}{4}P_D \\
\frac{\partial^2 Y_D}{\partial P_D^2} &= -\frac{7}{4}
\end{aligned}
$$

Thus, Donna's optimal price choice is

$$P_D = \frac{147}{8}\frac{4}{7} = 10.5$$

Seeing this price, Pierce responds with

$$P_P = \frac{15}{2} + \frac{1}{4}P_D = 10.125$$

In the simultaneous-move version of this game, we had calculated the Nash equilibrium

$$(10, 10)\,.$$

That is, being the first mover gives Donna the upper hand. She can now charge a higher price.

Comparing the outcomes of sequential and simultaneous move interactions

FRIEDA'S

Urban

		BIG GIANT	
		Urban	Rural
TITAN	Urban	5, 5, ~~1~~	5, ~~2~~, 5
	Rural	~~2~~, 5, 5	~~4~~, ~~4~~, ~~3~~

Rural

		BIG GIANT	
		Urban	Rural
TITAN	Urban	5, 5, 2	~~3~~, ~~4~~, ~~4~~
	Rural	~~4~~, ~~3~~, ~~4~~	4, 4, 4

FIGURE 6.9 Simultaneous-Move Mall Location Game

FRIEDA'S (Page player)

	Urban (U)				Rural (R)			
	GIANT (Column player)				**GIANT (Column player)**			
TITAN (Row player)	UU	UR	RU	RR	UU	UR	RU	RR
1: UUUU	5,5,1	5,5,1	5,2,5	5,2,5	5,5,2	3,4,4	5,5,2	3,4,4
2: UUUR	5,5,1	5,5,1	5,2,5	5,2,5	5,5,2	4,4,4	5,5,2	4,4,4
3: UURU	5,5,1	5,5,1	5,2,5	5,2,5	4,3,4	3,4,4	4,3,4	3,4,4
4: URUU	5,5,1	5,5,1	4,4,3	4,4,3	5,5,2	3,4,4	5,5,2	3,4,4
5: RUUU	2,5,5	2,5,5	5,2,5	5,2,5	5,5,2	3,4,4	5,5,2	3,4,4
6: UURR	5,5,1	5,5,1	5,2,5	5,2,5	4,3,4	4,4,4	4,3,4	4,4,4
7: URRU	5,5,1	5,5,1	4,4,3	4,4,3	4,3,4	3,4,4	4,3,4	3,4,4
8: RRUU	2,5,5	2,5,5	4,4,3	4,4,3	5,5,2	3,4,4	5,5,2	3,4,4
9: URUR	5,5,1	5,5,1	4,4,3	4,4,3	5,5,2	4,4,4	5,5,2	4,4,4
10: RURU	2,5,5	2,5,5	5,2,5	5,2,5	4,3,4	3,4,4	4,3,4	3,4,4
11: RUUR	2,5,5	2,5,5	5,2,5	5,2,5	5,5,2	4,4,4	5,5,2	4,4,4
12: URRR	5,5,1	5,5,1	4,4,3	4,4,3	4,3,4	4,4,4	4,3,4	4,4,4
13: RURR	2,5,5	2,5,5	5,2,5	5,2,5	4,3,4	4,4,4	4,3,4	4,4,4
14: RRUR	2,5,5	2,5,5	4,4,3	4,4,3	5,5,2	4,4,4	5,5,2	4,4,4
15: RRRU	2,5,5	2,5,5	4,4,3	4,4,3	4,3,4	3,4,4	4,3,4	3,4,4
16: RRRR	2,5,5	2,5,5	4,4,3	4,4,3	4,3,4	4,4,4	4,3,4	4,4,4

FIGURE 6.11 Mall Location Game in Strategic Form

(a) Simultaneous play

		WIFE	
		Confess	Deny
HUSBAND	Confess	10 yr, 10 yr	1 yr, 25 yr
	Deny	25 yr, 1 yr	3 yr, 3 yr

FIGURE 6.12 A Three Versions of the Prisoners' Dilemma Game

(b) Sequential play—Husband moves first

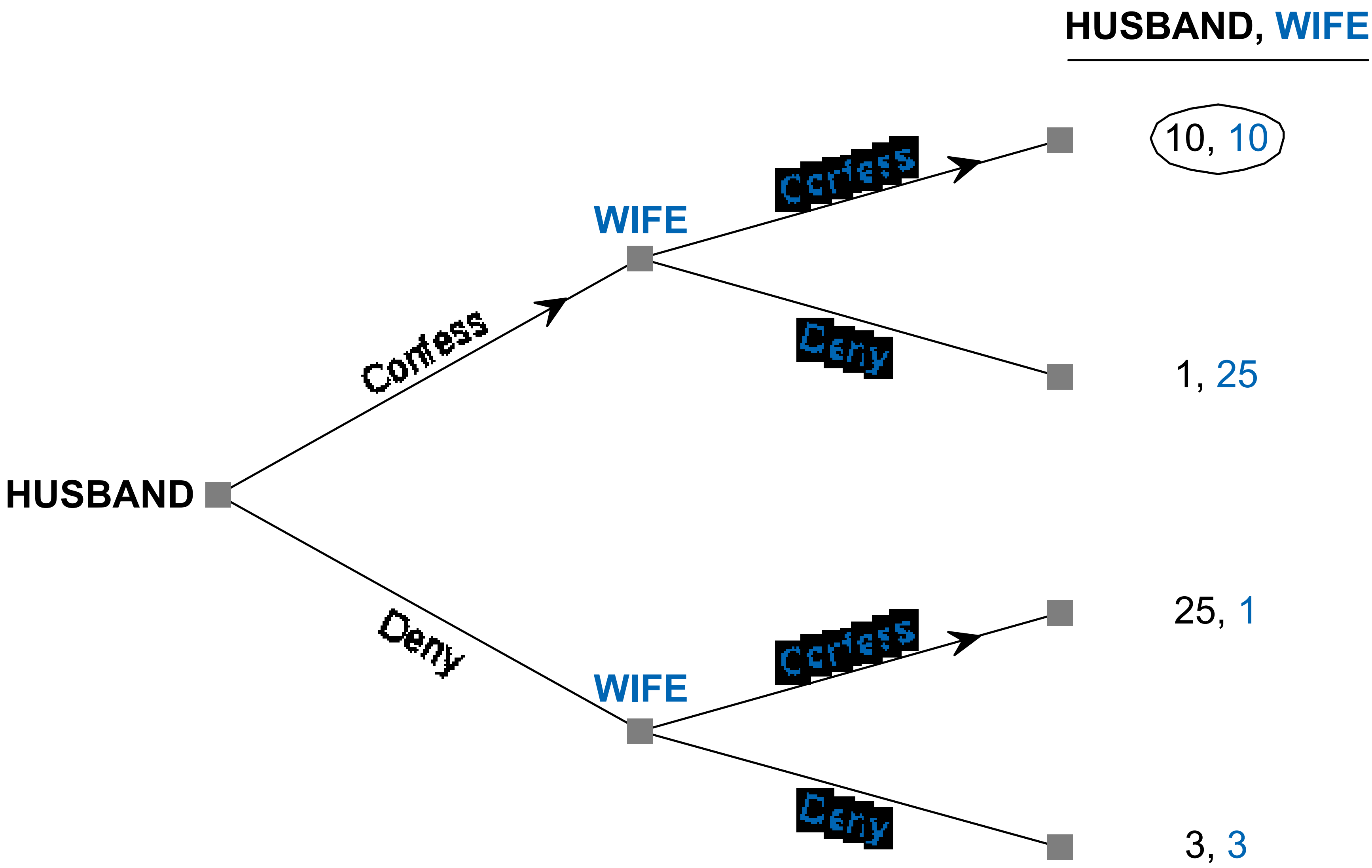

FIGURE 6.12 B Three Versions of the Prisoners' Dilemma Game

(c) Sequential play—Wife moves first

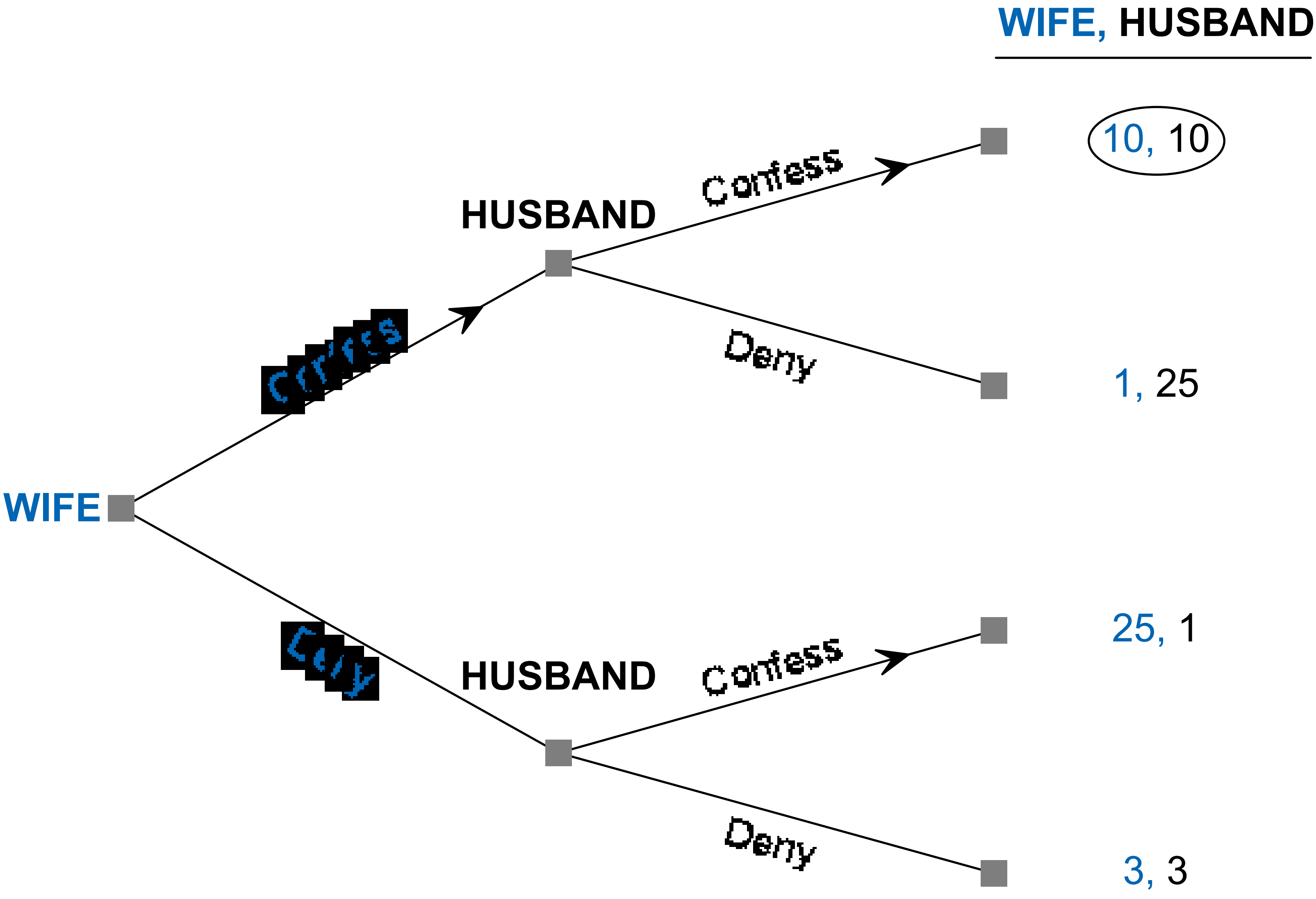

FIGURE 6.12 C Three Versions of the Prisoners' Dilemma Game

(a) Simultaneous play

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	–1, 1
	Straight	1, –1	–2, –2

FIGURE 6.14 A Chicken in Simultaneous- and Sequential-Play Versions

(b) Sequential play—James moves first

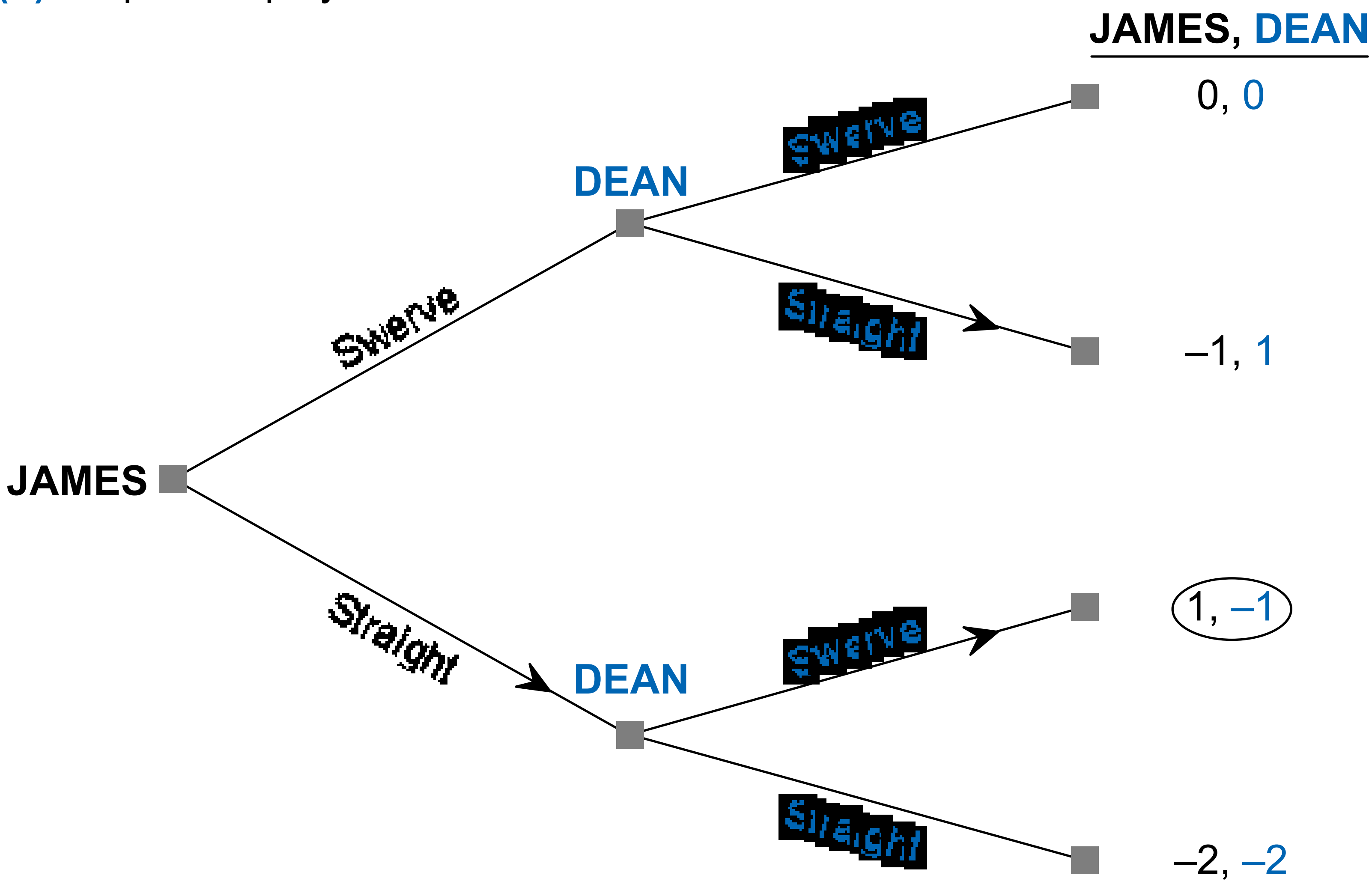

FIGURE 6.14 B Chicken in Simultaneous- and Sequential-Play Versions

(c) Sequential play—Dean moves first

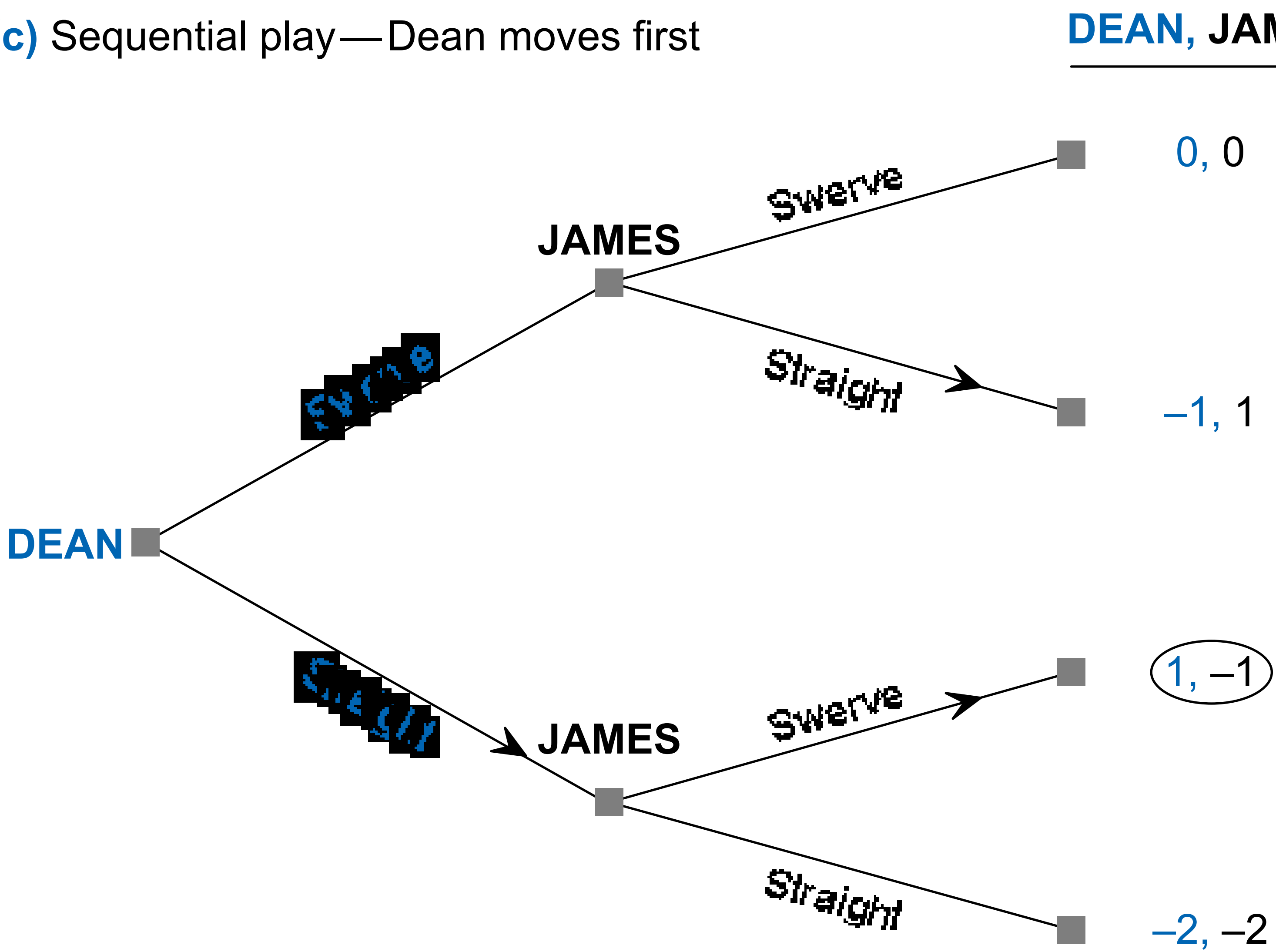

FIGURE 6.14 C Chicken in Simultaneous- and Sequential-Play Versions

(a) Simultaneous play

		HINGIS	
		DL	CC
SELES	DL	50	80
	CC	90	20

FIGURE 6.15 A Tennis Game in Simultaneous and Sequential Versions

(b) Sequential play—Seles moves first

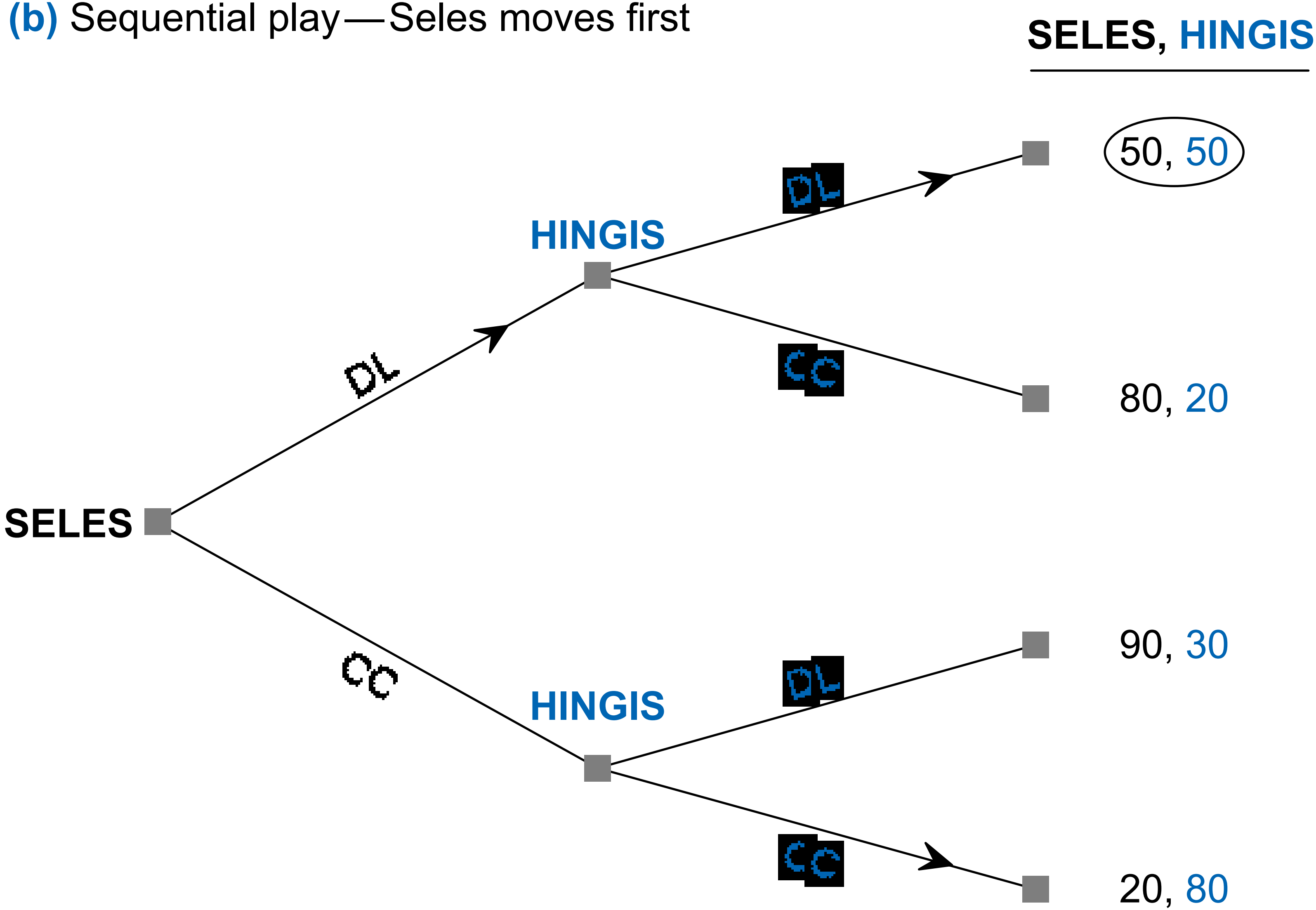

FIGURE 6.15 B Tennis Game in Simultaneous and Sequential Versions

(c) Sequential play—Hingis moves first

HINGIS, SELES

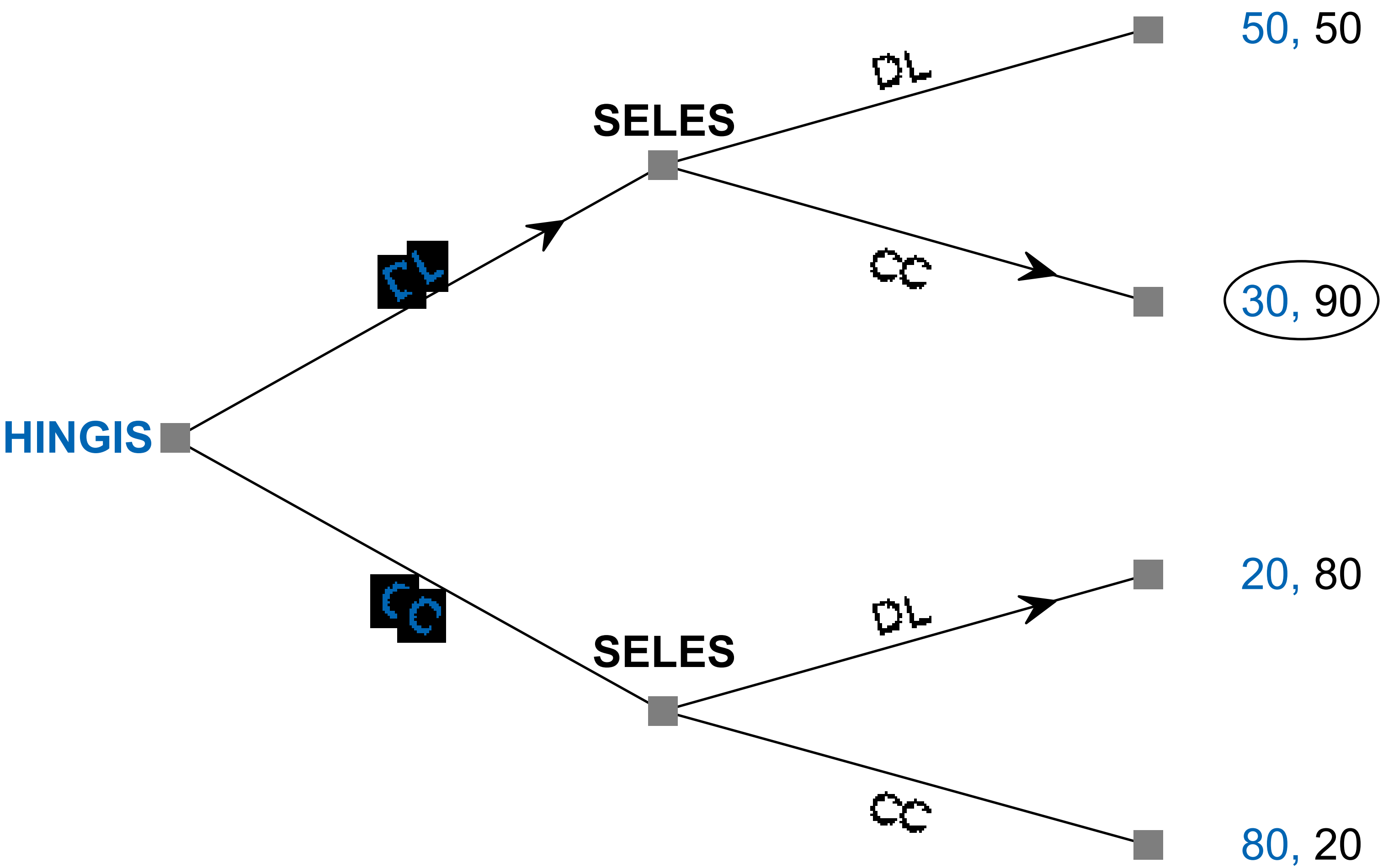

FIGURE 6.15 C Tennis Game in Simultaneous and Sequential Versions