Negotiation as a Cooperative Game

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Abstract

Game theory provides us with a set of important methodologies for the study of group decisions as well as negotiation processes. Cooperative game theory is a subfield of game theory that focuses on interactions in which involved parties have the power to make binding agreements. Many group decision and negotiation processes (such as legal arbitrations) fall into this category, and as such, they have been central in the development of cooperative game theory. Particularly, an area of cooperative game theory, called bargaining theory, focuses on bilateral negotiations as well as negotiation processes where coalition formation is not a

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central concern. The object of study in bargaining theory is a (bargaining) rule, which provides a solution to each bargaining problem (or in other words, negotiation). Studies on bargaining theory employ the axiomatic method to evaluate bargaining rules. This chapter reviews and summarizes several such studies. After a discussion of the bargaining model, we present the important bargaining rules in the literature (including the Nash bargaining rule), as well as the central axioms that characterize them. Next, we discuss strategic issues related to cooperative bargaining, such as the Nash program, implementation of bargaining rules, and games of manipulating bargaining rules. We conclude with a discussion of the recent literature on ordinal bargaining rules.

Keywords

Negotiation · Game theory · Cooperative · Nash bargaining solution · Pareto-optimal · Bargaining theory · Axiom · Ordinal bargaining

Introduction

Negotiation is an important aspect of social, economic, and political life. People negotiate at home, at work, at the marketplace; they observe their team, political party, country negotiating with others; and sometimes, they are asked to arbitrate negotiations among others. Thus, it is no surprise that researchers from a wide range of disciplines have studied negotiation processes.

In this chapter, we present an overview of how negotiation and group decision processes are modeled and analyzed in cooperative game theory.[1] This area of research, typically referred to as **cooperative bargaining theory**, originated in a seminal paper by Nash (1950). There, Nash provided a way of modeling negotiation processes and applied an axiomatic methodology to analyze such models. In what follows, we will discuss Nash's work in detail, particularly in application to the following example.

Example 1 *(An Accession Negotiation) The European Union, E, and a candidate country, C, are negotiating on the tariff rate that C will impose on its imports from E during C's accession process to the European Union. In case of disagreement, C will continue to impose the status-quo tariff rate on import goods from E and the accession process will be terminated, that is, C will not be joining the European Union.*

[1]Cooperative game theory analyzes interactions where agents can make binding agreements and it inquires how cooperative opportunities faced by alternative coalitions of agents shape the final agreement reached. Cooperative games do not specify how the agents interact or the mechanism through which their interaction leads to alternative outcomes of the game (and in this sense, they are different than noncooperative games). Instead, as will be exemplified in this chapter, they present a reduced form representation of all possible agreements that can be reached by some coalition.

Nash's (1950) approach to modeling negotiation processes such as Example 1 is as follows. *First*, the researcher identifies the set of all alternative agreements.[2] (Among them, the negotiators must choose by *unanimous agreement,* that is, each negotiator has the right to reject a proposed agreement.) *Second*, the researcher determines the implications of disagreement. In our example, disagreement leads to the prevalence of the status-quo tariff rate coupled with the fact that *C* will not be joining the European Union. *Third*, the researcher determines how each negotiator values alternative agreements, as well as the disagreement outcome. Formally, for each negotiator, a payoff function that represents its preferences are constructed. In the above example, this amounts to an empirical analysis that evaluates the value of each potential agreement for the European Union and the candidate country. *Finally*, using the obtained payoff functions, the negotiation is reconstructed in the payoff space. That is, each possible outcome is represented with a payoff profile that the negotiating parties receive from it. The **feasible payoff set** is the set of all payoff profiles resulting from an agreement (i.e., it is the image of the *set of agreements* under the players' payoff functions), and the **disagreement point** is the payoff profile obtained in case of disagreement. Via this transformation, the researcher reduces the negotiation process into a set of payoff profiles and a payoff vector representing disagreement. It is this object in the payoff space that is called a **(cooperative) bargaining problem** in cooperative game theory. For a typical bargaining problem, please see Fig. 1.

The object of study in cooperative bargaining theory is a **(bargaining) rule.** It maps each bargaining problem to a payoff profile in the feasible payoff set. For example, the *Nash bargaining rule* (Nash 1950) chooses, for each bargaining problem, the payoff profile that maximizes the product of the bargainers' gains with respect to their disagreement payoffs.

There are two alternative interpretations of a *bargaining rule*. According to the *first interpretation*, which is proposed by Nash (1950), a bargaining rule *describes*, for each bargaining problem, the outcome that will be obtained as result of the interaction between the bargainers. According to Nash (1950), a rule is thus a **positive** construct and should be evaluated on the basis of how well a description of real-life negotiations it provides. The *second interpretation* of a bargaining rule is alternatively **normative**. According to this interpretation, a bargaining rule produces, for each bargaining problem, a *prescription* to the bargainers (very much like an arbitrator). It should thus be evaluated on the basis of how useful it is to the negotiators in obtaining desirable agreements.

Studies on cooperative bargaining theory employ the axiomatic method to evaluate bargaining rules. (A similar methodology is used for social choice and fair division problems, as discussed in chapter ► "The Notion of Fair Division in Negotiations" of this handbook.) An **axiom** is simply a property of a bargaining rule. For example, one of the best-known axioms, *Pareto optimality*, requires that the

[2]This set contains all agreements that are physically available to the negotiators, including those that are "unreasonable" according to the negotiators' preferences.

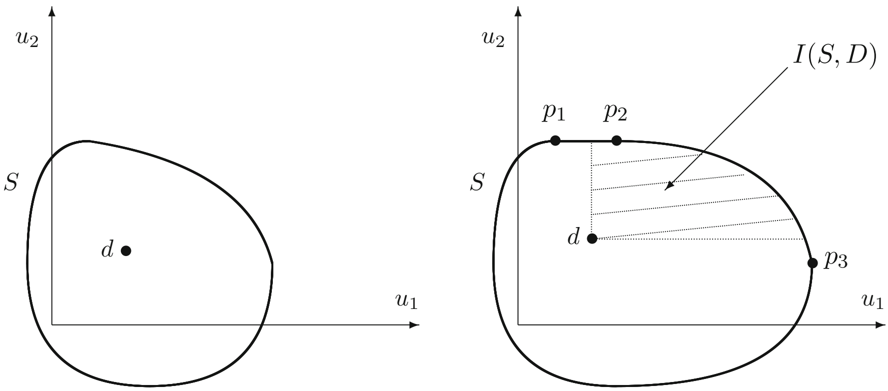

Fig. 1 The horizontal (respectively, vertical) axis represents the payoffs of Agent 1 (Agent 2). On the left: a strictly d-comprehensive bargaining problem. On the right: a weakly d-comprehensive bargaining problem, the individually rational set, the Pareto set (part of the north-east boundary between p_2 and p_3) and the weak Pareto set (part of the north-east boundary between p_1 and p_3)

bargaining rule choose a Pareto optimal agreement.[3] Researchers analyze implications of axioms that they believe to be "desirable." According to the positive interpretation of bargaining rules, a "desirable" axiom describes a common property of a relevant class of real-life negotiation processes. For example, Nash (1950) promotes the *Pareto optimality* axiom on the basis that the negotiators, being rational agents, will try to maximize their payoffs from the negotiation outcome and thus, will not terminate the negotiations at an agreement that is not optimal. According to the normative interpretation of a bargaining rule, an axiom is a normatively appealing property which we as a society would like arbitrations to a relevant class of negotiations to satisfy. Note that the *Pareto optimality* axiom can also be promoted on this basis.

It is important to note that an axiom need not be desirable in every application of the theory to real-life negotiations. Different applications might call for different axioms.

A typical study on cooperative bargaining theory considers a set of axioms, motivated by a particular application, and identifies the class of bargaining rules that satisfy them. An example is Nash (1950) which shows that the *Nash bargaining rule* uniquely satisfies a list of axioms including *Pareto optimality*. In the section "Bargaining Rules and Axioms," we discuss several such studies in detail.

As will be detailed in the section "The Bargaining Model," Nash's (1950) model analyzes situations where the bargainers have access to lotteries on a fixed and publicly known set of alternatives. It is also assumed that the bargainers' von Neumann-Morgenstern preferences are publicly known. While most of the following literature works on Nash's standard model, there also are many studies that analyze

[3]As will be formally introduced later, an agreement is *Pareto optimal* if there is no alternative agreement that makes an agent better-off without hurting any other agent.

the implications of dropping some of these assumptions. For example, in the section "Ordinal Bargaining," we discuss the recent literature on *ordinal bargaining* which analyzes cases where the agents do not necessarily have access to lotteries or do not have von Neumann-Morgenstern preferences.

It is important to mention that two negotiation processes who happen to have the same *feasible payoff set* and *disagreement point* are considered to be the same bargaining problem in Nash's (1950) model and thus, they have the same solution, independent of which bargaining rule is being used and how distinct the two negotiations are physically. This is sometimes referred to as the *welfarism axiom* and it has been a point of criticism of cooperative game theory. It should be noted that all the bargaining rules that we review in this chapter satisfy this property.

The chapter is organized as follows. In the section "The Bargaining Model," we present the bargaining model of Nash (1950). In the section "Bargaining Rules and Axioms," we present the main bargaining rules and axioms in the literature. In the section "Strategic Considerations," we discuss strategic issues related to cooperative bargaining, such as the Nash program, implementation, and games of manipulating bargaining rules (for more on strategic issues, see chapter ► "Non-cooperative Bargaining Theory" of this handbook). Finally, we present the more recent literature on ordinal bargaining in the section "Ordinal Bargaining."

For earlier surveys of cooperative bargaining theory, please see Peters (1992) and Thomson (1994), and the literature cited therein. These studies contain more detailed accounts of the earlier literature which we have summarized in the section "Bargaining Rules and Axioms." For a more recent treatment of distributive bargaining, see Binmore and Eguia (2017).

In the sections "Strategic Considerations" and "Ordinal Bargaining," we present a selection of the more recent contributions to cooperative bargaining theory, not covered by earlier surveys. Due to space limitations, we left out some important branches of the recent literature. For *nonconvex bargaining problems*, see Herrero (1989) or Zhou (1997) and the related literature. For *bargaining problems with incomplete information,* see De Clippel and Minelli (2004), as well as a literature review by Forges and Serrano (2013). For *rationalizability of bargaining rules*, see Peters and Wakker (1991) and the following literature. For extensions of the Nash model that focus on the *implications of disagreement*, see Kıbrıs and Tapk (2010, 2011) and the literature cited therein. For "semi-cooperative solutions" to noncooperative games, see Kalai and Kalai (2013) and the related literature. Finally, a more recent literature focuses on the empirical content of the Nash bargaining rule, as well as its applications (e.g., see Chiappori et al. (2012)).

Bargaining problems are cooperative games (called nontransferable utility games) where it is assumed that only the grand coalition or individual agents can affect the final agreement. This is without loss of generality for two-agent negotiations which are the most common type. However, for negotiations among three or more agents, the effect of coalitions on the final outcome might also be important. For more on this discussion, please see Bennett (1997) and Kıbrıs (2004b), and the literature cited therein.

The Bargaining Model

Consider a group of negotiators $N = \{1,...,n\}$. (While most real-life negotiations are bilateral, that is $N = \{1,2\}$, we do not restrict ourselves to this case.) A cooperative bargaining problem for the group N consists of a set, S, of payoff profiles (i.e., payoff vectors) resulting from every possible agreement and a payoff profile, d, resulting from the disagreement outcome. It is therefore defined on the space of all payoff profiles, namely the n-dimensional Euclidian space $\mathbb{R}^N$. Formally, the **feasible payoff set** S is a subset of $\mathbb{R}^N$ and the **disagreement point** d is a vector in $\mathbb{R}^N$. In what follows, we will refer to each $x \in S$ as an **alternative (agreement)**.

There is an important asymmetry between an alternative $x \in S$ and the disagreement point d. For the negotiations to end at x, unanimous agreement of the bargainers is required. On the other hand, each agent can unilaterally induce d by simply disagreeing with the others.

The pair (S, d) is called a **(cooperative bargaining) problem** (Fig. 1, *left*) and is typically assumed to satisfy the following properties[4]:

(i) S is *convex*, *closed*, and *bounded.*
(ii) $d \in S$ and there is $x \in S$ such that $x > d$.
(iii) S is *d-comprehensive* (i.e., $d \leqq y \leqq x$ and $x \in S$ imply $y \in S$).

Let $\mathcal{B}$ be the set of all cooperative bargaining problems.

Convexity of S means that (i) the agents are able to reach agreements that are lotteries on other agreements and (ii) each agent's preferences on lotteries satisfy the von Neumann-Morgenstern axioms and thus, can be represented by an expected utility function. For example, consider a couple negotiating on whether to go to the park or to the movies on Sunday. The *convexity* assumption means that they could choose to agree to take a coin toss on the issue (or agree to condition their action on the Sunday weather), and that each agent's payoff from the coin toss is the average of his payoffs from the park and the movies. *Boundedness* of S means that the agents' payoff functions are bounded (i.e., no agreement can give them an infinite payoff). *Closedness* of S means that the set of physical agreements is closed and the agents' payoff functions are continuous.

In the section "Ordinal Bargaining," we will extend the basic model to allow situations where the bargainers do not have access to lotteries and they do not necessarily have von Neumann-Morgenstern preferences.

The assumption $d \in S$ means that the agents are able to agree to disagree and induce the disagreement outcome. Existence of an $x \in S$ such that $x > d$ rules out degenerate problems where no agreement can make all agents better-off than the disagreement outcome. Finally, *d-comprehensiveness* of S means that utility is freely disposable above d.[5]

[4]We use the following vector inequalities $x \geqq y$ for each $i \in N, x_i \geqq y_i$; $x \geqq y$ and $x \neq y$; and $x > y$ if for each $i \in N, x_i > y_i$.

[5]A stronger assumption called *full comprehensiveness* additionally requires utility to be freely disposable below d.

Two concepts play an important role in the analysis of a bargaining problem (S, d). The first is the Pareto optimality of an agreement: it means that the bargainers can not all benefit from switching to an alternative agreement. Formally, the **Pareto set** of (S, d) is defined as $P(S, d) = \{x \in S \mid y \geq x \Rightarrow y \notin S\}$ and the **Weak Pareto set** of (S, d) is defined as $\mathrm{WP}(S, d) = \{x \in S \mid y > x \Rightarrow y \notin S\}$. The second concept, individual rationality, is based on the fact that each agent can unilaterally induce disagreement. Thus, it requires that each bargainer prefer an agreement to disagreement. Formally, the **individually rational set** is $I(S, d) = \{x \in S \mid x \geqq d\}$. Like Pareto optimality, individual rationality is desirable as both a positive and a normative property. On Fig. 1, *right*, we present the sets of Pareto optimal and individually rational alternatives.

We will occasionally consider a subclass $\mathcal{B}_{sc}$ of bargaining problems $\mathcal{B}$ that satisfy a stronger property than *d-comprehensiveness*: the problem (S, d) is *strictly d-comprehensive* if $d \leqq y \leqq x$ and $x \in S$ imply $y \in S$ and $y \notin \mathrm{WP}(S, d)$ (please see Fig. 1; the left problem is *strictly d-comprehensive* while the right one is not).

We will next present examples of modeling the accession negotiation of Example 1.

Example 2 *(Modeling the Accession Negotiation) The set of bargainers is* $N = \{E, C\}$. *Let* $T = [0,1]$ *be the set of all tariff rates. As noted in the section "Introduction," the bargainers' payoffs from alternative agreements (as well as disagreement) need to be determined by an empirical study which (not surprisingly) we will not carry out here. However, we will next present four alternative scenarios for these payoff functions,* U_C *and* U_E. *In each scenario, we assume for simplicity that each bargainer i) receives a zero payoff in case of disagreement and (ii) prefers accession with any tariff rate to disagreement. Due to (ii), the individually rational set coincides with the feasible payoff set of the resulting bargaining problem in each scenario.*

In the first scenario, both bargainers' payoffs are linear in the tax rate. (Thus, both are risk-neutral.[6]*)*

Scenario 1. *Let* $U_E(t) = 1 - t$ *and* $U_C(t) = t$

In the second scenario, we change the candidate's payoff to be a strictly concave function. (Compared to Scenario 1, C is now more risk-averse than E.)

Scenario 2. *Let* $U_E(t) = 1 - t$ *and* $U_C(t) = t^{\frac{1}{2}}$

In the third scenario, E's payoff is also changed to be a strictly concave function. (Now, both bargainers have the same level of risk-aversion.)

Scenario 3. *Let* $U_E(t) = (1 - t)^{\frac{1}{2}}$ *and* $U_C(t) = t^{\frac{1}{2}}$

In the fourth scenario, both bargainers have linear payoff functions. That is, they are both risk-neutral. But, differently from Scenario 1, now E's marginal gain from a change in the tariff rate is twice that of C.

Scenario 4. *Let* $U_E(t) = 2(1 - t)$ *and* $U_C(t) = t$

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[6]A decision-maker is *risk-neutral* if he is indifferent between each lottery and the lottery's expected (sure) return.

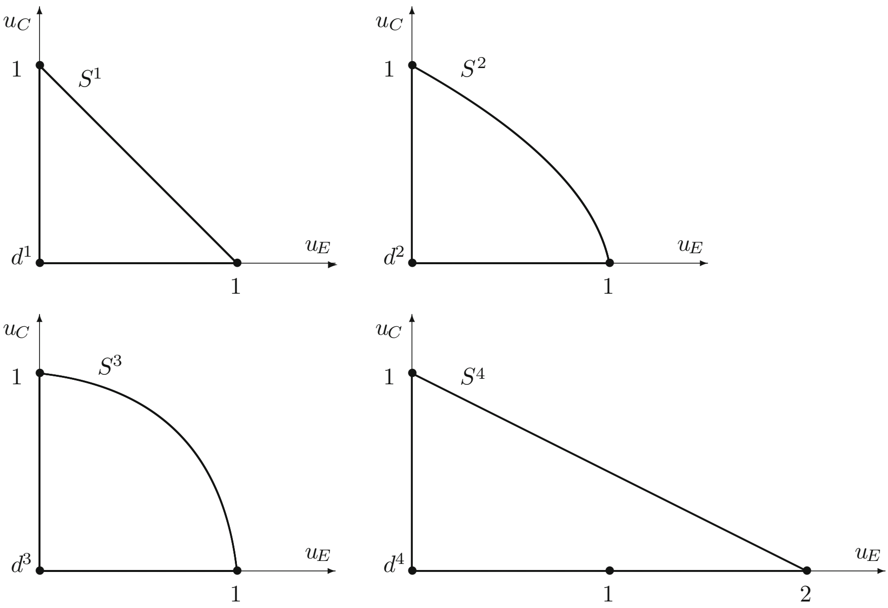

Fig. 2 The Accession Game: Scenario 1 (top left), Scenario 2 (top right), Scenario 3 (bottom left), and Scenario 4 (bottom right)

The resulting feasible payoff set and the disagreement point for each scenario is constructed in Fig. 2.

Since both bargainers prefer accession of C to its rejection from the European Union, the Pareto set *under all scenarios corresponds to those payoff profiles that result from accession with probability* 1. *The feasible payoff set is constructed by taking convex combinations of the* Pareto optimal *alternatives with the* disagreement point. *Thus, they represent payoff profiles of lotteries, including those between an accession agreement and disagreement.*

Bargaining Rules and Axioms

A **(bargaining) rule** $F : \mathcal{B} \to \mathbb{R}^n$ assigns each bargaining problem $(S, d) \in \mathcal{B}$ to a feasible payoff profile $F(S, d) \in S$. As discussed in the section "Introduction," F can be interpreted as either (i) a description of the negotiation process the agents in consideration are involved in (the positive interpretation) or (ii) a prescription to the negotiators as a "good" compromise (the normative interpretation).

In this section, we present examples of bargaining rules and discuss the main axioms that they satisfy. We also discuss these rules' choices for the four scenarios of Example 2.

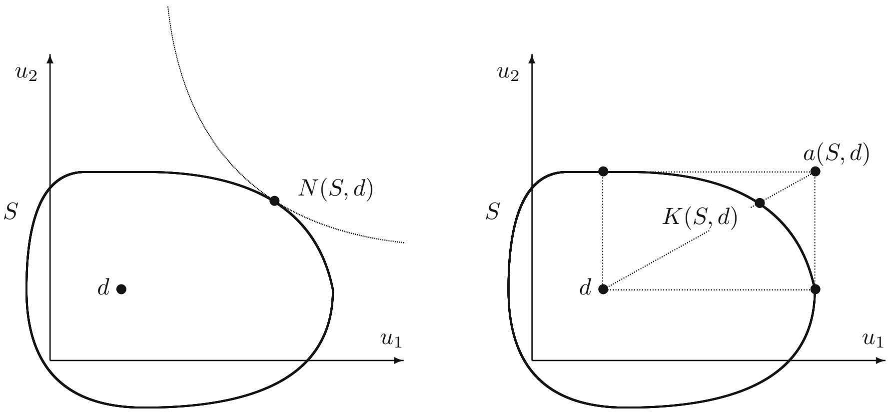

Fig. 3 The Nash (left) and the Kalai-Smorodinsky (right) solutions to a typical problem

The Nash Rule

The first and the best-known example of a bargaining rule is by Nash (1950). The **Nash rule** chooses, for each bargaining problem $(S, d) \in \mathcal{B}$ the individually rational alternative that maximizes the product of the agents' gains from disagreement (please see Fig. 3, *left*):

$$N(S,d) = \arg \max_{x \in I(S,d)} \prod_{i=1}^{n} (x_i - d_i).$$

Let us first check the Nash solutions to the accession negotiations of Example 2.

Example 3 *(Nash solution to the accession negotiations) For each of the four scenarios discussed in* Example 2*, the Nash rule proposes the following payoff profiles (the first payoff number is for E and the second is for C). For Scenario 1, $N(S^1, d^1) = \left(\frac{1}{2}, \frac{1}{2}\right)$. This payoff profile is obtained when the bargainers agree on accession at a tariff rate $t^1 = \frac{1}{2}$. For Scenario 2, $N(S^2, d^2) = \left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$, obtained at accession and the tariff rate $t^2 = \frac{1}{3}$. For Scenario 3, $N(S^3, d^3) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, obtained at accession and the tariff rate $t^3 = \frac{1}{2}$. For Scenario 4, $N(S^4, d^4) = \left(1, \frac{1}{2}\right)$, obtained at accession and the tariff rate $t^4 = \frac{1}{2}$.*

In Example 3, as *C* becomes more risk averse from *Scenario 1* to *Scenario 2,* the Nash solution changes in a way to benefit *E* (since the tariff rate decreases from $\frac{1}{2}$ to $\frac{1}{3}$). This is a general feature of the Nash bargaining rule: the Nash bargaining payoff of an agent increases as his opponent becomes more risk-averse (Kihlstrom et al. 1981).

Nash (1950) proposes four axioms and shows that his rule satisfies them. These axioms later play a central role in the literature. We will introduce them next.

The first axiom requires that the rule always choose a Pareto optimal alternative. Formally, a rule F is **Pareto optimal** if for each problem $(S, d) \in \mathcal{B}$, $F(S, d) \in P(S, d)$. As discussed in the section "Introduction," it is commonly agreed in the literature that negotiations result in a Pareto optimal alternative. Thus, most axiomatic analyses focus on *Pareto optimal* rules. In Example 3, *Pareto optimality* is satisfied since all four negotiations result in the accession of the candidate to the European Union.[7]

The second axiom, called **anonymity**, guarantees that the identity of the bargainers do not affect the outcome of negotiation. It requires that permuting the agents' payoff information in a bargaining problem should result in the same permutation of the original agreement. To formally introduce this axiom, let Π be the set of all permutations on N, $\pi : N \to N$. For $x \in \mathbb{R}^N$, let $\pi(x) = (x_{\pi(i)})_{i \in N}$ and for $S \subseteq \mathbb{R}^N$, let $\pi(S) = \{\pi(x) | \ x \in S\}$. Then, a rule F is *anonymous* if for each $\pi \in \Pi$, $F(\pi(S), \pi(d)) = \pi(F(S, d))$. Note that *anonymity* applies to cases where the bargainers have "equal bargaining power."

It is common practice in the literature to replace *anonymity* with a weaker axiom which requires that if a problem is symmetric (in the sense that all of its permutations result in the original problem), then its solution should be symmetric as well. Formally, a rule F is **symmetric** if for each $\pi \in \Pi$, $\pi(S) = S$ and $\pi(d) = d$ implies $F_1(S, d) = ... = F_n(S, d)$. Note that the bargaining problems under *Scenarios 1* and *3* are symmetric. Therefore, their Nash solutions are also *symmetric*.

The third axiom is based on the fact that a von Neumann-Morgenstern type preference relation can be represented with infinitely many payoff functions (that are positive affine transformations of each other) and the particular functions chosen to represent the problem should not affect the bargaining outcome. Formally, let Λ be the set of all $\lambda = (\lambda_1, \ldots, \lambda_n)$ where each $\lambda_i : \mathbb{R} \to \mathbb{R}$ is a positive affine function.[8] Let $\lambda(S) = \{\lambda(x) | \ x \in S\}$. Then, a rule F is **scale invariant** if for each $(S, d) \in \mathcal{B}$ and $\lambda \in \Lambda$, $F(\lambda(S), \lambda(d)) = \lambda(F(S, d))$. Note that in the accession negotiations, *Scenario 4* is obtained from *Scenario 1* by multiplying U_E by 2, which is a positive affine transformation. Thus, the Nash solutions to the two scenarios are related the same way (and the resulting tariff rates are identical).

The final axiom of Nash (1950) concerns the following case. Suppose the bargainers facing a bargaining problem (S, d) agree on an alternative x. However, they later realize that the actual feasible set T is smaller than S. Nash requires that if the original agreement is feasible in the smaller feasible set, $x \in T$, then the bargainers should stick with it. Formally, a rule F is **contraction independent** if for each $(S, d), (T, d) \in \mathcal{B}$ such that $T \subseteq S$, $F(S, d) \in T$ implies $F(T, d) = F(S, d)$. Nash

[7]This is *Pareto optimal* since both bargainers prefer accession to rejection. What they disagree on is the tariff rate.

[8]A function $\lambda_i : \mathbb{R} \to \mathbb{R}$ is positive affine if there is $a, b \in \mathbb{R}$ with $a > 0$ such that for each $x \in \mathbb{R}$, $\lambda_i(x) = ax + b$.

(1950) and some of the following literature alternatively calls this axiom **independence of irrelevant alternatives (IIA)**. However, the presumed irrelevance of alternatives in the choice of an agreement (as suggested by this name) is a topic of controversy in the literature. In fact, it is this controversy that motivates the bargaining rule of Kalai and Smorodinsky (1975) as will be discussed in the next subsection.

Nash (1950) shows that his bargaining rule uniquely satisfies these four axioms. We will next prove this result for two-agent problems.

Theorem 4 *(*Nash 1950*) A bargaining rule satisfies* Pareto optimality, symmetry, scale invariance*, and* contraction independence *if and only if it is the Nash rule.*

Proof It is left to the reader to check that the Nash rule satisfies the given axioms. Conversely, let F be a rule that satisfies them. Let $(S, d) \in \mathcal{B}$ and $N(S, d)=x$. We would like to show that $F(S, d)=x$.

By *scale invariance* of both rules, it is without loss of generality to assume that $d=(0,0)$ and $x=(1,1)$.[9] Then, by definition of N, the set $P(S, d)$ has slope -1 at x. Also, by boundedness of S, there is $z \in \mathbb{R}^N$ such that for each $x \in S, x \geqq z$. Now let $T = \{y \in \mathbb{R}^N | \sum_N y_i \leqq \sum_N x_i \text{ and } y \geqq z\}$. Then, $S \subseteq T$ and $(T, d) \in \mathcal{B}$ is a symmetric problem. Thus, by *symmetry* and *Pareto optimality* of F, $F(T, d) = x$. This, by *contraction independence* of F, implies $F(S, d) = x$, the desired conclusion. ■

It is useful to note that the following class of weighted Nash rules uniquely satisfy all of Nash's axioms except *symmetry*. These rules extend the Nash bargaining rule to cases where agents differ in their "bargaining power." Formally, let $p = (p_1,\ldots,p_n) \in 0,1]^N$ satisfy $\sum_N p_i = 1$. Each p_i is interpreted as the bargaining power of Agent i. Then the **p-weighted Nash bargaining rule** is defined as

$$N^p\ (S, d) = \arg \max_{x \in I(S,d)} \prod_{i=1}^{n} (x_i - d_i)^{p_i}.$$

The symmetric Nash bargaining rule assigns equal weights to all agents, that is, $p = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$.

The literature contains several other characterizations of the Nash bargaining rule. For example, see Peters (1986), Dagan et al. (2002), Anbarci and Sun (2011, 2013), and Rachmilevitch (2015).

The Kalai-Smorodinsky Rule

The Kalai-Smorodinsky rule (Raiffa 1953; Kalai and Smorodinsky 1975) makes use of each agent's aspiration payoff, that is, the maximum payoff an agent can get at an

[9] Any (S, d) can be "normalized" into such a problem by choosing $\lambda_i(x_i) = \frac{x_i - d_i}{N_i(S,d) - d_i}$ for each $i \in N$.

individually rational agreement. Formally, given a problem $(S, d) \in \mathcal{B}$, the **aspiration payoff** of Agent i is $a_i(S, d) = \arg_{\max_{x \in I(S,d)} x_i}$. The vector $a(S, d) = (a_i(S, d))_{i=1}^n$ is called the **aspiration point**.

The **Kalai-Smorodinsky rule**, K, chooses the maximum individually rational payoff profile at which each agent's payoff gain from disagreement has the same proportion to his aspiration payoff's gain from disagreement (please see Fig. 3, *right*). Formally,

$$K(S, d) = \arg \max_{x \in I(S,d)} \left(\min_{i \in \{1,\ldots,n\}} \frac{x_i - d_i}{a_i(S, d) - d_i} \right).$$

Geometrically, $K(S, d)$ is the intersection of the line segment $[d, a(S, d)]$ and the northeast boundary of S.

Example 5 *(Kalai-Smorodinsky solution to the accession negotiations) For each of the four scenarios discussed in* Example 2, *the Kalai-Smorodinsky rule proposes the following payoff profiles (the first payoff number is for E and the second is for C). For Scenario 1,* $K(S^1, d^1) = \left(\frac{1}{2}, \frac{1}{2}\right)$. *This payoff profile is obtained when the bargainers agree on accession at a tariff rate* $t^1 = \frac{1}{2}$. *For Scenario 2,* $K(S^2, d^2) = (0.62, 0.62)$, *obtained at accession and the tariff rate* $t^2 = 0.38$. *For Scenario 3,* $K(S^3, d^3) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, *obtained at accession and the tariff rate* $t^3 = \frac{1}{2}$. *For Scenario 4,* $K(S^4, d^4) = \left(1, \frac{1}{2}\right)$, *obtained at accession and the tariff rate* $t^4 = \frac{1}{2}$.

In Example 5, as C becomes more risk averse from *Scenario 1* to *Scenario 2,* the Kalai-Smorodinsky solution changes in a way to benefit E (since the tariff rate decreases from $\frac{1}{2}$ to 0.38). This is a general feature of the Kalai-Smorodinsky bargaining rule: the Kalai-Smorodinsky bargaining payoff of an agent increases as his opponent becomes more risk-averse (Kihlstrom et al. 1981).

As can be observed in Example 5, the Kalai-Smorodinsky rule is *Pareto optimal* for all two-agent problems. With more agents, however, it satisfies a weaker property: a rule F is **weakly Pareto optimal** if for each problem $(S, d) \in \mathcal{B}$, $F(S, d) \in WP(S, d)$. Example 5 also demonstrates that the Kalai-Smorodinsky rule is *symmetric* and *scale invariant*. Due to *weak Pareto optimality* and *symmetry*, the Kalai-Smorodinsky solutions to (S^1, d^1) and (S^3, d^3) are equal to the Nash solutions. Due to *scale invariance*, the two rules also coincide on (S^4, d^4). For the problem (S^2, d^2), however, the two rules behave differently: the Kalai-Smorodinsky rule chooses equal payoffs for the agents while the Nash rule favors E.

The Kalai-Smorodinsky rule violates Nash's *contraction independence* axiom. Kalai and Smorodinsky (1975) criticize this axiom and propose to replace it with a monotonicity notion which requires that an expansion of the feasible payoff set (and thus an increase in the cooperative opportunities) should benefit an agent if it does not affect his opponents' aspiration payoffs. Formally, a rule F satisfies **individual monotonicity** if for each $(S, d), (T, d) \in \mathcal{B}$ and $i \in N$, if $S \subseteq T$ and

$a_j(S, d) = a_j(T, d)$ for each $j \neq i$, then $F_i(S, d) \leqq F_i(T, d)$. The Nash rule violates this axiom.

Kalai and Smorodinsky (1975) present the following characterization of the Kalai-Smorodinsky rule. We will next prove this result for two-agent problems.

Theorem 6 *(*Kalai and Smorodinsky 1975*) A bargaining rule satisfies* Pareto optimality, symmetry, scale invariance*, and* individual monotonicity *if and only if it is the Kalai-Smorodinsky rule.*

Proof It is left to the reader to check that the Kalai-Smorodinsky rule satisfies the given axioms. Conversely, let F be a rule that satisfies them. Let $(S, d) \in \mathcal{B}$ and $K(S, d) = x$. We would like to show that $F(S, d) = x$.

By *scale invariance* of both rules, it is without loss of generality to assume that $d=(0,0)$ and $a(S, d) = (1,1)$.[10] Then, by definition of K, $x_1=x_2$. Now let $T = \text{conv}\{x, d, (1,0),(0,1)\}$. Then, $T \subseteq S$ and $(T, d) \in \mathcal{B}$ is a symmetric problem. Thus, by *symmetry* and *Pareto optimality* of F, $F(T, d) = x$. Since $T \subseteq S$, $x \in P(S, d)$, and $a(S, d) = a(T, d)$, *individual monotonicity* implies that $F(S, d)=x$, the desired conclusion. ■

Roth (1979) notes that the above characterization continues to hold under a weaker monotonicity axiom which only considers expansions of the feasible set at which the problem's aspiration point remains unchanged. Formally, a rule F satisfies **restricted monotonicity** if for each $(S, d), (T, d) \in \mathcal{B}$ and $i \in N$, if $S \subseteq T$ and $a(S, d) = a(T, d)$ then $F(S, d) \leqq F(T, d)$.The Nash rule violates this weaker monotonicity axiom as well.

The literature contains several other characterizations of the Kalai-Smorodinsky bargaining rule. For example, see Dubra (2001) and Karos et al. (2018).

The Egalitarian Rule

The **Egalitarian rule**, E, (Kalai 1977) chooses for each problem $(S, d) \in \mathcal{B}$ the maximum individually rational payoff profile that gives each agent an equal gain from his disagreement payoff (please see Fig. 4, *left*). Formally, for each $(S, d) \in \mathcal{B}$,

$$E(S, d) = \arg \max_{x \in I(S,d)} \left(\min_{i \in \{1,\ldots,n\}} (x_i - d_i) \right).$$

Geometrically, $E(S, d)$ is the intersection of the boundary of S and the half line that starts at d and passes through $d + (1,\ldots,1)$.

Example 7 *(Egalitarian solution to the accession negotiations) For each of the four scenarios discussed in* Example 2*, the Egalitarian rule proposes the following payoff*

[10] Any (S, d) can be "normalized" into such a problem by choosing $\lambda_i(x_i) = \frac{x_i - d_i}{a_i(S,d) - d_i}$ for each $i \in N$.

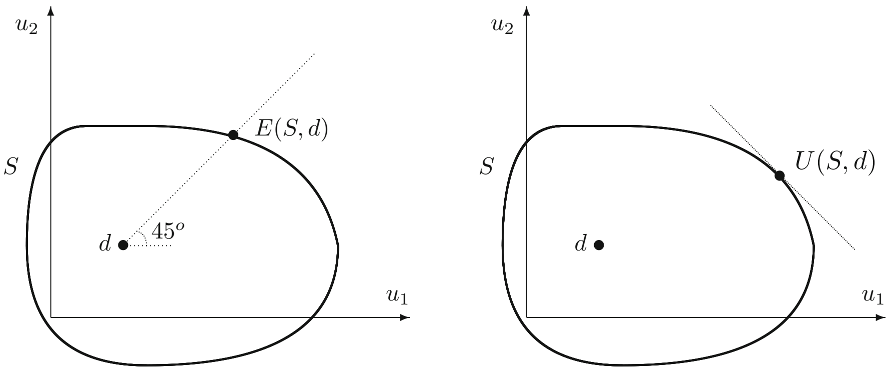

Fig. 4 The Egalitarian (left) and the Utilitarian (right) solutions to a typical problem

profiles (the first payoff number is for E and the second is for C). For Scenario 1, $E(S^1,d^1) = (\frac{1}{2},\frac{1}{2})$. *This payoff profile is obtained when the bargainers agree on accession at a tariff rate* $t^1 = \frac{1}{2}$. *For Scenario 2,* $E(S^2,d^2) = (0.62, 0.62)$, *obtained at accession and the tariff rate* $t^2 = 0.38$. *For Scenario 3,* $E(S^3,d^3) = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, *obtained at accession and the tariff rate* $t^3 = \frac{1}{2}$. *For Scenario 4,* $E(S^4,d^4) = (\frac{2}{3},\frac{2}{3})$, *obtained at accession and the tariff rate* $t^4 = \frac{2}{3}$.

The Egalitarian rule satisfies *Pareto optimality* only on the class of strictly *d*-comprehensive problems $\mathcal{B}_{SC}$. On $\mathcal{B}$, it only satisfies *weak Pareto optimality.*[11]

As observed in Example 7, the Egalitarian rule is *weakly Pareto optimal* and *symmetric*. Due to these two axioms, the Egalitarian solutions to (S^1,d^1) and (S^3,d^3) are equal to the Nash and Kalai-Smorodinsky solutions. Also, since the aspiration point of problem (S^2,d^2) is symmetric, $a(S, d) = (1,1)$, the Egalitarian and the Kalai-Smorodinsky rules pick the same solution.

Unlike the Nash and the Kalai-Smorodinsky rules, the Egalitarian rule violates *scale invariance*. This can be observed in Example 7 by comparing the Egalitarian solutions to (S^1,d^1) and (S^4,d^4).[12] The Egalitarian rule however satisfies the following weaker axiom: a rule F satisfies **translation invariant** if for each $(S,d) \in \mathcal{B}$ and $z \in \mathbb{R}^N$, $F(S + \{z\}, d + z) = F(S, d) + z$.[13]

[11]On problems that are not *d*-comprehensive, the Egalitarian rule can also violate *weak Pareto optimality.*

[12]For a *scale invariant* rule, (S^1,d^1) and (S^4,d^4) are alternative representations of the same physical problem. (Specifically, *E*'s payoff function has been multiplied by 2 and thus, still represents the same preferences.) For the Egalitarian rule, however, these two problems (and player *E*'s) are distinct. Since it seeks to equate absolute payoff gains from disagreement, the Egalitarian rule treats agents' payoffs to be comparable to each other. As a result, it treats payoff functions as more than mere representations of preferences.

[13]This property is weaker than *scale invariance* because, for an agent i, every translation $x_i + z_i$ is a positive affine transformation $\lambda_i(x_i) = 1x_i + z_i$.

On the other hand, the Egalitarian rule satisfies a very strong monotonicity axiom which requires that an agent never loose in result of an expansion of the feasible payoff set. Formally, a rule F satisfies **strong monotonicity** if for each $(S, d), (T, d) \in \mathcal{B}$, if $S \subseteq T$, then $F(S, d) \leqq F(T, d)$. This property is violated by the Kalai-Smorodinsky rule since this rule is sensitive to changes in the problem's aspiration point. The Nash rule violates this property since it violates the weaker individual monotonicity property.

The following characterization of the Egalitarian rule follows from Kalai (1977). We present it for two-agent problems.

Theorem 8 *(*Kalai 1977*) A bargaining rule satisfies* weak Pareto optimality, symmetry, translation invariance*, and* strong monotonicity *if and only if it is the Egalitarian rule.*

Proof It is left to the reader to check that the Egalitarian rule satisfies the given axioms. Conversely, let F be a rule that satisfies them. Let $(S, d) \in \mathcal{B}$ and $E(S, d) = x$. We would like to show that $F(S, d) = x$.

By *translation invariance* of both rules, it is without loss of generality to assume that $d = (0,0)$.[14] Then, by definition of E, $x_1 = x_2$. Now let $T = \text{conv}\{x, d, (x_1, 0), (0, x_2)\}$. Then, $T \subseteq S$ and $(T, d) \in \mathcal{B}$ is a symmetric problem. Thus, by *symmetry* and *weak Pareto optimality* of F, $F(T, d) = x$. Since $T \subseteq S$, *strong monotonicity* then implies $F(S, d) \geqq x$.

Case 1: $x \in P(S, d)$. Then $F(S, d) \geq x$ implies $F(S, d) \notin S$. Thus, $F(S, d) = x$, the desired conclusion.

Case 2: $x \in WP(S, d)$. Suppose $F_i(S, d) > x_i$ for some $i \in N$. Let $\delta > 0$ be such that $x_i + \delta < F_i(S, d)$, let $x' = x + (\delta, \delta)$, $x'' = \left(d_i, x'_{-i}\right)$ and $S' = \text{conv}\{x', x'', S\}$. Then $E(S', d) = x' \in P(S', d)$ and by Case 1, $F(S', d) = x'$. Since $S \subseteq S'$, by *strong monotonicity*, $F(S', d) = x' \geqq F(S, d)$. Particularly, $x_i + \delta \geqq F_i(S, d)$, a contradiction. Thus, $F(S, d) = x$. ■

The literature contains several other characterizations of the Egalitarian bargaining rule. For example, see Chun and Thomson (1990), Myerson (1981), Peters (1986), Anbarci and Sun (2011, 2013), Rachmilevitch (2011), and Karos et al. (2018).

Other Rules

In this section, we will present some of the other well-known rules in the literature.

The first is the **Utilitarian rule** which chooses for each bargaining problem $(S, d) \in \mathcal{B}$ the alternatives that maximize the sum of the agents' payoffs (please see Fig. 4, *right*):

[14] Any (S, d) can be "normalized" into such a problem by choosing $\lambda_i(x_i) = x_i - d_i$ for each $i \in N$.

$$U\ (S, d) = \arg_{\substack{\max \\ x \in S}} \sum_{i=1}^{n} x_i.$$

The *Utilitarian rule* is not necessarily single-valued, except when the feasible set is strictly convex. However, it is possible to define single-valued refinements (such as choosing the midpoint of the set of maximizers). Also, the *Utilitarian solution* to (S, d) is independent of d. Thus, the *Utilitarian rule* violates *individual rationality.* Restricting the choice to be from the individually rational set remedies this problem. Finally, the *Utilitarian rule* violates *scale invariance*. However, a variation which maximizes a weighted sum of utilities satisfies the property (e.g., see Dhillon and Mertens 1999).

The Utilitarian rule is *Pareto optimal*, *anonymous contraction independent*, and *translation invariant* even though it violates *restricted monotonicity.* For more on this rule, see Myerson (1981) and Thomson (1981). Blackorby et al. (1994) introduce a class of *Generalized Gini rules* that are mixtures of the Utilitarian and the Egalitarian rules.

The second rule represents extreme cases where one agent has all the "bargaining power." The **Dictatorial rule for Agent i** chooses the alternative that maximizes Agent i's payoff among those at which the remaining agents receive their disagreement payoffs (please see Fig. 5*, right*):

$$D^i\ (S, d) = \arg \max_{\substack{x \in I(S,d) \\ s.t. x_{-i} = d_{-i}}} x_i.$$

This rule is only *weakly Pareto optimal*, though on *strictly d-comprehensive* problems it is *Pareto optimal*. The following rule does not suffer from this problem: the *Serial Dictatorial rule* is defined with respect to a fixed order of agents and it first maximizes the payoff of the first ordered agent, then among the maximizers, maximizes the payoff of the second and so on.

Both the dictatorial and serial dictatorial rule violate *symmetry* (and thus *anonymity*). Otherwise, they are very well-behaved. Both rules are *scale invariant.* In fact, they satisfy an even stronger property, *ordinal invariance*, that we introduce and discuss in the section "Ordinal Bargaining." These rules also satisfy *contraction independence* and *strong monotonicity* (and thus, all weaker monotonicity properties).

The next class of rules, introduced by Yu (1973), is based on minimizing a measure of the distance between the agreement and the problem's aspiration point (defined in the subsection "The Kalai-Smorodinsky Rule"). Formally, for $p \in (1,\infty)$, the **Yu rule associated with p** is

$$Y^p(S, d) = \arg_{\substack{\max \\ x \in S}} \left(\sum_{i=1}^{n} |a_i(S, d) - x_i|^p \right)^{\frac{1}{p}}.$$

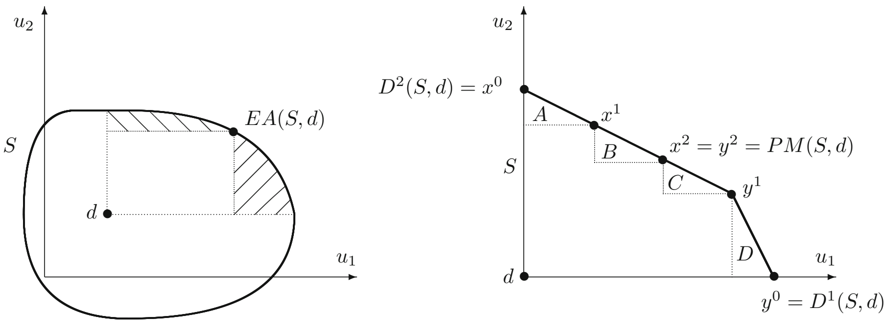

Fig. 5 The Equal Area solution to a typical problem equates the two shaded areas (left); the Perles-Maschler solution to a polygonal problem is the limit of the sequences $\{x^k\}$ and $\{y^k\}$ which are constructed in such a way that (i) $x^0 = D^2(S, d)$, $y^0 = D^1(S, d)$ are the two Dictatorial solutions and (ii) the areas A, B, C, and D are maximal and they satisfy $A = D$ and $B = C$

The Yu rules are *Pareto optimal*, *anonymous*, and *individually monotonic*. However, they violate *contraction independence*, *strong monotonicity*, and *scale invariance*.

The final two rules are defined for two-agent problems. They both are based on the idea of equalizing some measure of the agents' sacrifices with respect to their aspiration payoffs. The first, **Equal Area rule**, *EA*, chooses the *Pareto optimal* alternative at which the area of the set of better *individually rational* alternatives for Agent 1 is equal to that of Agent 2 (please see Fig. 5, *left*). This rule violates *contraction independence* but satisfies *anonymity*, *scale invariance*, and an "area monotonicity" axiom (e.g., see Calvo and Peters 2000). The second rule is by Perles and Maschler (1981). For problems (S, d) whose Pareto set $P(S, d)$ is polygonal, the **Perles-Maschler rule**, *PM,* chooses the limit of the following sequence. (The Perles-Maschler solution to any other problem (S, d) is obtained as the limit of Perles-Maschler solutions to a sequence of polygonal problems that converge to (S, d)). Let $x^0 = D^2(S, d)$ and $y^0 = D^1(S, d)$. For each $k \in \mathbb{N}$, let $x^k, y^k \in P(S, d)$ be such that (i) $x_1^k \leqq y_1^k$, (ii) $[x^{k-1}, x^k] \subset P(S, d)$, (iii) $[y^{k-1}, y^k] \subset P(S, d)$, (iv) $\left|\left(x_1^{k-1} - x_1^k\right)\left(x_2^{k-1} - x_2^k\right)\right| = \left|\left(y_1^{k-1} - y_1^k\right)\left(y_2^{k-1} - y_2^k\right)\right|$, and $\left|\left(x_1^{k-1} - x_1^k\right)\left(x_2^{k-1} - x_2^k\right)\right|$ is maximized (please see Fig. 5, *right*). The Perles-Maschler rule is *Pareto optimal*, *anonymous*, and *scale invariant*. It, however, is not *contraction independent* or *restricted monotonic*. For extending this rule to more than two agents, see Calvo and Gutiérrez (1994) and the literature cited therein.

Strategic Considerations

As noted in the section "Introduction," Nash (1950) interprets a bargaining rule as a description of a (noncooperative) negotiation process between rational agents. Nash (1953) furthers this interpretation and proposes what is later known as the **Nash program**: to relate choices made by cooperative bargaining rules to equilibrium outcomes of underlying noncooperative games. Nash argues that "the two approaches to the (bargaining) problem, via the (noncooperative) negotiation model or via the axioms, are complementary; each helps to justify and clarify the other."

Nash (1953) presents the first example of the Nash program. Given a bargaining problem (S, d), he proposes a two-agent noncooperative **Demand Game** in which each player i simultaneously declares a payoff number s_i. If the declared payoff profile is feasible (i.e., $s \in S$), players receive their demands. Otherwise, the players receive their disagreement payoffs with a probability p and their demands with the remaining probability. Nash shows that, as p converges 1, the equilibrium of the *Demand Game* converges to the Nash solution to (S, d).

Van Damme (1986) considers a related noncooperative game where, given a bargaining problem (S, d), each agent simultaneously declares a bargaining rule.[15] If the solutions proposed by the two rules conflict, the feasible payoff set is contracted in a way that an agent cannot receive more than the payoff he asks for himself. The two rules are now applied to this contracted problem and if they conflict again, the feasible set is once more contracted. Van Damme (1986) shows that for a large class of rules, the limit of this process is well-defined and the unique Nash equilibrium of this noncooperative game is both agents declaring the Nash bargaining rule.

Another well-known contribution to the Nash program is by Binmore et al. (1986) who relate the Nash bargaining rule to equilibrium outcomes of the following game. The **Alternating Offers Game** (Rubinstein 1982) is an infinite horizon sequential move game to allocate one unit of a perfectly divisible good between two agents. The players alternate in each period to act as "proposer" and "responder." Each period contains two sequential moves: the proposer proposes an allocation and the responder either accepts or rejects it. The game ends when a proposal is accepted. Rubinstein (1982) shows that the *Alternating Offers Game* has a unique subgame perfect Nash equilibrium in which the first proposal, determined as a function of the players' discount factors, is accepted. Binmore et al. (1986) show that, as the players' discount factors converge to 1 (i.e., as they become more patient), the equilibrium payoff profile converges to the *Nash bargaining solution* to the associated cooperative bargaining game. For more recent work on the Nash program, see Anbarci and Boyd (2011), Abreu and Pearce (2015), Binmore and Eguia (2017), and Karagözoğlu and Rachmilevitch (2018).

Another strategic issue arises from that fact that each negotiator, by misrepresenting his private information (e.g., about his preferences, degree of risk aversion, etc.), might be able to change the bargaining outcome in his favor. Understanding the "real" outcome of a bargaining rule then requires taking this kind of strategic behavior into account. A standard technique for this is to embed the original problem into a noncooperative game (in which agents strategically "distort" their private information) and to analyze its equilibrium outcomes. This is demonstrated in the following example.

[15]Thus, as in Nash (1953), each agent demands a payoff. But now, they have to rationalize it as part of a solution proposed by an "acceptable" bargaining rule.

Example 9 *(A noncooperative game of manipulating the Nash rule) Suppose that agents C and E in* Example 2 *have private information about their true payoff functions and that they play a noncooperative game where they strategically declare this information to an arbitrator who uses the Nash rule. Using the four scenarios of* Example 2*, fix the strategy set of C as* $\left\{t, t^{\frac{1}{2}}\right\}$ *and the strategy set of E as* $\left\{1-t, (1-t)^{\frac{1}{2}}, 2(1-t)\right\}$*. The resulting tariff rate is determined by the Nash bargaining rule calculated in* Example 3 *except for the profile* $\left(t, (1-t)^{\frac{1}{2}}\right)$*. The following table summarizes, for each strategy profile, the resulting tariff rate.*

$C \setminus E$	$1-t$	$(1-t)^{\frac{1}{2}}$	$2(1-t)$
t	0.5	0.66	0.5
$t^{\frac{1}{2}}$	0.33	0.5	0.33

Note that this is a competitive game: C is better-off and E is worse-off in response to an increase in the tariff rate t. Also note that, for C, declaring t strictly dominates declaring $t^{\frac{1}{2}}$ *(that is, he gains from acting less risk-averse). Similarly, for E, declaring* $(1-t)$ *strictly dominates declaring* $(1-t)^{\frac{1}{2}}$ *and, since the Nash bargaining rule is scale invariant, declaring* $(1-t)$ *and* $2(1-t)$ *are equivalent. The game has two equivalent dominant strategy equilibria:* $(t, 1-t)$ *and* $(t, 2(1-t))$ *where both players act to be risk-neutral.*

In some cases, such as Example 9, it is natural to assume that the agents' ordinal preferences are publicly known. (In the example, it is common knowledge that *C* prefers higher tariff rates and *E* prefers lower tariff rates.) Then, manipulation can only take place through misrepresentation of cardinal utility information (such as the degree of risk-aversion). In two-agent bargaining with the Nash or the Kalai-Smorodinsky rules, an agent's utility increases if his opponent is replaced with another that has the same preferences but a more concave utility function (Kihlstrom et al. 1981). On allocation problems, this result implies that an agent can increase his payoff by declaring a less concave utility function (i.e., acting to be less risk-averse). For the Nash bargaining rule, it is a dominant strategy for each agent to declare the least concave representation of his preferences. For a single good, the equilibrium outcome is equal division.

If ordinal preferences are not publicly known, however, their misrepresentation can also be used for manipulation. The resulting game does not have dominant strategy equilibria. Nevertheless, for a large class of two-agent bargaining rules applied to allocation problems, the set of allocations obtained at Nash equilibria in which agents declare linear utilities is equal to the set of "constrained" Walrasian allocations from equal division with respect to the agents' true utilities (Sobel 1981, 2001; Gómez 2006). Under a mild restriction on preferences, a similar result holds for pure exchange and public good economies with an arbitrary number of agents and for all *Pareto optimal* and *individually rational* bargaining rules (Kıbrıs 2002).

Ordinal Bargaining

Nash (1950) and most of the following literature restricts the analysis to bargaining processes that take place on lotteries and assumes that the bargainers' preferences on lotteries satisfy the von Neumann-Morgenstern assumptions (thus, they are representable by expected utility functions). This assumption has two important consequences. First, in a bargaining problem (S, d), the feasible payoff set S is then *convex*. Second, the *scale invariance* axiom of Nash (1950) is sufficient to ensure the invariance of the physical bargaining outcome with respect to the particular utility representation chosen.

In this section, we drop these assumptions and analyze bargaining in **ordinal environments**, where the agents' *complete*, *transitive*, and *continuous* preferences do not have to be of von Neumann-Morgenstern type. For ordinal environments, (i) the payoff set S is allowed to be nonconvex and (ii) *scale invariance* needs to be replaced with the following stronger axiom.[16] Formally, let Φ be the set of all $\phi = (\phi_1,\ldots,\phi_n)$ where each $\phi_i : \mathbb{R} \to \mathbb{R}$ is an *increasing* function. Let $\phi(S) = \{\phi(x) | \ x \in S\}$. Then, a rule F is **ordinal invariant** if for each $(S, d) \in \mathcal{B}$ and $\phi \in \Phi$, $F(\phi(S), \phi(d)) = \phi(F(S, d))$. Note that every *ordinal invariant* rule is also *scale invariant* but not vice versa.

If there are a finite number of alternatives, many *ordinal invariant* rules exist (e.g., see Kıbrıs and Sertel 2007). With an infinite number of alternatives, however, *ordinal invariance* is a very demanding property. Shapley (1969) shows that for two-agent problems, only dictatorial bargaining rules and the rule that always chooses disagreement satisfy this property. This result is due to the fact that the Pareto optimal set of every two-agent problem can be mapped to itself via a nontrivial increasing transformation $\phi = (\phi_1,\phi_2)$. In the following example, we demonstrate the argument for a particular bargaining problem.

Example 10 *Consider the problem* (S^1, d^1) *in Scenario 1 of* Example 2 *(represented in* Fig. 2, upper left*). Note that the Pareto set of* (S^1, d^1) *satisfies* $u_C + u_E = 1$. *Let* $\phi_C(u_C) = u_C^{\frac{1}{2}}$ *and* $\phi_E(u_E) = 1 - (1 - u_E)^{\frac{1}{2}}$ *and note that* $\phi_C(u_C) + \phi_E(u_E) = 1$. *Thus, the Pareto set of the transformed problem* $(\phi(S^1), \phi(d^1))$ *is the same as* (S^1, d^1). *In fact,* $S^1 = \phi(S^1)$ *and* $d^1 = \phi(d^1)$. *To summarize,* ϕ *maps* (S^1, d^1) *to itself* via *a nontrivial transformation of the agents' utilities. Now let F be some ordinally invariant bargaining rule. Since the two problems are identical,* $F(\phi(S^1), \phi(d^1)) = F(S^1, d^1)$. *Since F is ordinally invariant, however, we also have* $F(\phi(S^1), \phi(d^1)) = \phi(F(S^1, d^1))$. *For both requirements to be satisfied, we need* $\phi(F(S^1, d^1)) = F(S^1, d^1)$. *Only three payoff profiles in* (S^1, d^1) *satisfy this property:* (0,0), (1,0), *and* (0,1). *Note that they are the disagreement point and the two dictatorial solutions, respectively. So, F should coincide with either one of these rules on* (S^1, d^1).

[16]This is due to the following fact. Two utility functions represent the same *complete* and *transitive* preference relation if and only if one is an increasing transformation of the other.

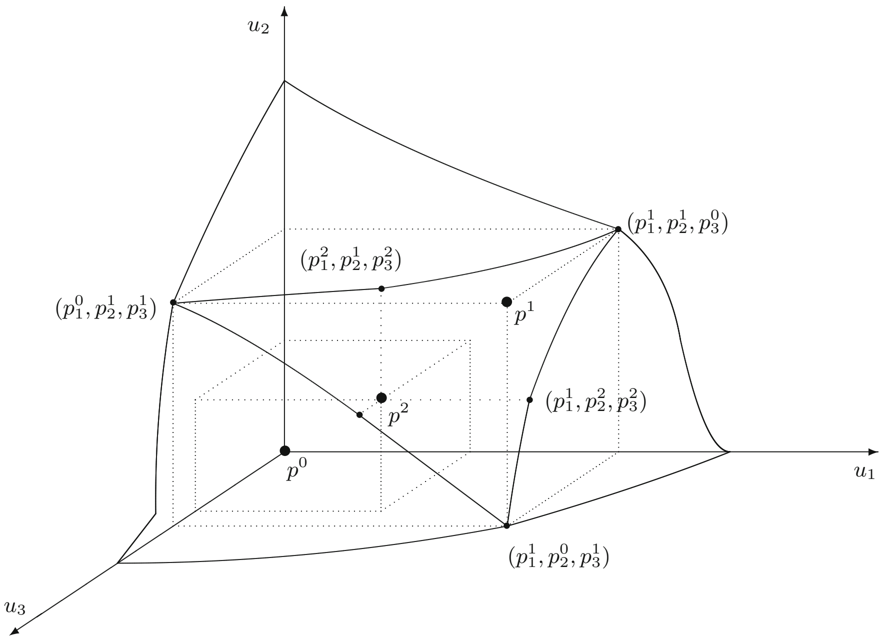

Fig. 6 The Shapley-Shubik solution to (S, d) is the limit of the sequence $\{p^k\}$

The construction of Example 10 is not possible for more than two agents (Sprumont 2000). For three agents, Shubik (1982) presents an *ordinally invariant* and *strongly individually rational* bargaining rule which we will refer to as the **Shapley-Shubik rule**.[17] The Shapley-Shubik solution to a problem (S, d) is defined as the limit of the following sequence. Let $p^0 = d$ and for each $k \in \{1,\ldots\}$, let $p^k \in \mathbb{R}^3$ be the unique point that satisfies

$$\left(p_1^{k-1}, p_2^k, p_3^k\right) \in P(S, d), \left(p_1^k, p_2^{k-1}, p_3^k\right) \in P(S, d), \text{and} \left(p_1^k, p_2^k, p_3^{k-1}\right) \in P(S, d).$$

The Shapley-Shubik solution is then $\text{Sh}(S, d) = \lim_{k\to\infty} p^k$. The construction of the sequence $\{p^k\}$ is demonstrated in Fig. 6.

Kıbrıs (2004a) shows that the *Shapley-Shubik rule* uniquely satisfies *Pareto optimality*, *symmetry*, *ordinal invariance*, and a weak monotonicity property. Kıbrıs (2012) shows that it is possible to replace monotonicity in this characterization with a weak contraction independence property. Samet and Safra (2005) propose

[17]There is no reference on the origin of this rule in Shubik (1982). However, Thomson attributes it to Shapley. Furthermore, Roth (1979) (pp. 72–73) mentions a three-agent ordinal bargaining rule proposed by Shapley and Shubik (1974, Rand Corporation, R-904/4) which, considering the scarcity of ordinal rules in the literature, is most probably the same bargaining rule.

generalizations of the Shapley-Shubik rule to an arbitrary number of agents. Vidal-Puga (2015) analyzes a noncooperative game whose subgame perfect Nash equilibrium coincides with the Shapley-Shubik rule.

The literature following Shapley (1969) also analyze the implications of weakening the *ordinal invariance* requirement on two-agent bargaining rules. Myerson (1977) and Roth (1979) show that such weakenings and some basic properties characterize Egalitarian type rules. Calvo and Peters (2005) analyze problems where there are both ordinal and cardinal players. There is also a body of literature which demonstrates that in alternative approaches to modeling bargaining problems, ordinality can be recovered (e.g., see Rubinstein et al. 1992; O'Neill et al 2004; Kıbrıs 2004b). Finally, there is a body of literature that allows nonconvex bargaining problems but does not explicitly focus on ordinality (e.g., see Herrero (1989), Zhou (1997), and the following literature).

Conclusion

In the last 60 years, a very large literature on cooperative bargaining formed around the seminal work of Nash (1950). In this chapter, we tried to summarize it, first focusing on some of the early results that helped shape the literature, and then presenting a selection of more recent studies that extend Nash's original analysis. An overview of these results suggests an abundance of both axioms and rules. We would like to emphasize that this richness comes out of the fact that bargaining theory is relevant for and applicable to a large number and wide variety of real-life situations including, but not limited to, international treaties, corporate deals, labor disputes, pre-trial negotiations in lawsuits, decision-making as a committee, or the everyday bargaining that we go through when buying a car or a house. Each one of these applications bring out new ideas on what the properties of a good solution should be and thus, lead to the creation of new axioms. It is our opinion that there are many more of these ideas to be explored in the future.

Cross-References

▶ Group Decisions: Choosing a Winner by Voting
▶ Group Decisions: Choosing Multiple Winners by Voting
▶ Non-cooperative Bargaining Theory
▶ Sharing Profit and Risk in a Partnership
▶ The Notion of Fair Division in Negotiations

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