A Random Reference Model

By Özgür Kibris, Yusufcan Masatlioglu and Elchin Suleymanov*

We provide two nested models of random reference-dependent choice in which the reference point is endogenously determined by random processes. Random choice behavior is due to random reference points, even though, from the decision maker's viewpoint, choices are deterministic. Through a revealed preference exercise, we establish when and how one can identify the reference-dependent preferences and the random reference rule from observed choice data. We also present behavioral postulates that characterize the empirical content of our models. Lastly, we investigate an application of our model to Bertrand competition with differentiated products.
JEL: D01, D11
Keywords: Random Choice, Random Reference, Reference Point Formation, Reference Dependence, Revealed Preference

Reference dependence is widely accepted as a fundamental feature of decision-making. In response to mounting evidence, a number of theories were proposed on reference-dependent choice. Their common feature is that, for each decision problem, a single alternative serves as a reference point. However, many real-life situations present a multitude of possible candidates for a reference point (e.g., see Kahneman, 1992, March and Shapira, 1992, Baucells, Weber and Welfens, 2011, Koop and Johnson, 2012, Baillon, Bleichrodt and Spinu, 2020). Indeed, the seminal work of Kahneman (1992) emphasizes that for each decision problem, there might be multiple potential reference points:

> "There are many situations in which people are fully aware of the multiplicity of relevant reference points, and the question of how they experience such outcomes and think about them must be raised. There appears to have been little discussion of this issue in behavioral decision research."

In an environment where the reference point is not observable, the existence of multiple reference points poses an additional challenge for predicting people's behavior. First, as an outside observer, we must know not only which reference points are used by the decision maker but also how frequently they are employed. Second, we need to identify how these reference points affect preferences. In this

* Kıbrıs: Faculty of Arts and Social Sciences, Sabancı University, ozgur@sabanciuniv.edu; Masatlioglu: Department of Economics, University of Maryland, yusufcan@umd.edu; Suleymanov: Department of Economics, Purdue University, esuleyma@purdue.edu. Parts of this research were conducted while Masatlioglu was visiting the Ca' Foscari University of Venice. The paper also immensely benefited from the inputs of the editor and four careful reviewers.

paper, we tackle these questions by using revealed preference techniques. Our study illustrates when and how one can identify from observed choice data the reference-dependent preferences as well as the distribution of reference points. Identifying the model's parameters is important for a policymaker who would like to make out-of-sample predictions and evaluate the impact of different policy interventions.

In the presence of multiple reference points, March and Shapira (1992) is the first to hypothesize that a decision maker probabilistically shifts her focus of attention between different reference points. In line with this idea, we consider a simple probabilistic attention rule where each alternative becomes the reference point with a certain (unobservable) probability. While the decision maker allocates her attention between multiple reference points probabilistically, the key assumption is that she always attends to one reference point at a time.[1] Given a reference point, the agent chooses the alternative which maximizes the corresponding reference-dependent preference. An important feature of this model is that, from the decision maker's viewpoint, choices are deterministic. Yet, due to changes in her reference point, the individual might pick different alternatives in repetitions of a decision problem.[2] Hence, the analyst observes probabilistic choice data due to shifts in the reference point. If the attention on the reference point is deterministic (always one alternative attracts all attention), the model becomes a classical deterministic reference-dependent model.

Without imposing further structure, this model cannot make any prediction. For better predictive power and identification, we first consider a simple, though non-trivial, attention rule which is tractable enough to use in applications. In addition, our attention rule is both alternative specific and context-dependent. Since reference points are alternatives that are simply salient to the decision maker (Bhatia and Golman, 2019), it is reasonable to assume that the reference probability of an alternative is an increasing function of its salience. For example, Tesla, being the most salient electric car brand, might have the highest reference probability among all electric car brands. To capture this idea, we utilize the well-known Logit formulation for the attention rule. The probability of an alternative being the reference point is determined by its own salience parameter relative to the total salience parameters of all available alternatives. It is important to note that the salience parameters are not observable and must be revealed from observed choices.[3]

[1] Kahneman (1992, p. 306) and March and Shapira (1992) both argue that only one reference point should be used at any given point in time.

[2] Repeated decisions are observable in naturally occurring data, such as scanner data from supermarkets and online data from digital platforms as well as experimental data.

[3] The salience of a product may depend on features of a product potentially irrelevant for its valuation, such as the size and/or color of its package (Milosavljevic et al., 2012), its shelf position in a traditional store (Liang and Lai, 2002), or its screen placement on an online store (Breugelmans, Campo and Gijsbrechts, 2007). The analyst does not directly observe how all these features affect the salience of an option for each decision maker. Hence, the salience parameters must be endogenously derived from observed probabilistic choice behavior.

We now define our model formally. The decision maker is endowed with reference-dependent preference relations $\{\succ_x\}_{x\in X}$, where $\succ_x$ reflects her preference when x is the reference point. A well-known Status Quo Bias (SQB) property, which simply requires that an alternative is more desirable when it is the reference point, interconnects the reference-dependent preferences.[4] The second key ingredient of our model is the attention rule which identifies the shifts between reference points. Each alternative x is endowed with a salience parameter s_x, assumed to be strictly positive, measuring the salience of the alternative as a reference point. The probability of an alternative being the reference point is then determined by the Logit attention rule. Overall, the probability of x being chosen in S is expressed as

$$p(x|S) = \sum_{y\in S} \underbrace{\left(\frac{s_y}{\sum\limits_{z\in S} s_z}\right)}_{\substack{\text{probability of } y \text{ being} \\ \text{the reference point}}} \underbrace{\mathbb{1}(x \text{ is } \succ_y\text{-best in } S)}_{x \text{ is the maximizer of } \succ_y} .$$

We call this the *Logit Random Reference Model (L-RAR)*. L-RAR is a canonical model in the sense that it applies to any individual choice problem, such as choice among consumption bundles, lotteries, acts, consumption streams, and distributions of wealth. Consequently, we focus on an abstract domain where the distribution of the reference points is only determined by the choice problem.[5]

We now illustrate when and how one can identify the reference-dependent preferences and the salience parameters from probabilistic choice data. The inference about reference-dependent preferences relies on three different observations. The first one is an alternative being chosen with probability 1. In this case, the unique choice must be revealed to be preferred to any other alternative in the choice set, independent of the reference point. The second is an alternative being chosen with positive probability in some choice problem. In this case, we can say that it is the best alternative in the choice problem when it is the reference point. The final observation uses the existence of regularity violations in the data, that is, cases where the elimination of an alternative x from a set S reduces the choice probability of another alternative y. In this case, we deduce that y must be better than any other alternative in S when x is the reference point. We show that these three observations completely characterize all the revealed preference implications

[4]More precisely, the status quo bias property says that if x is preferred to y when some z is the reference point, then x must also be preferred to y when x itself is the reference point. This restriction makes the deterministic models well-behaved (Masatlioglu and Ok, 2005, Sagi, 2006, Kőszegi and Rabin, 2006). Indeed, in the absence of this condition, unwanted behavioral patterns such as cyclical choice and status quo *aversion* emerge (e.g., see Sagi, 2006, Masatlioglu and Ok, 2014).

[5]While we consider an arbitrary domain of alternatives, in applying our model to specific domains, the reference probability could be a function of different attributes of an alternative. For example, if the objects are risky prospects, then the reference probability could depend on the prize dimension only or on both the prize and the probability dimensions.

of our model. That is, they are not only necessary but also sufficient.

We now discuss the identification of the salience parameters from observed choice data. Note that our model has a trivial non-uniqueness: multiplying all salience parameters by the same positive constant leads precisely to the same reference points and hence, choice behavior. Therefore, we can only identify the salience parameters relative to each other. In our model, the relative choice probabilities of two alternatives depend on the availability of other alternatives (*context-dependence*). This makes it difficult to reveal relative salience parameters. Nevertheless, we show that all "relevant" relative salience parameters can be revealed by looking at choice sets of sizes two and three. Hence, our model can be identified with limited data. There are three different observations through which the salience parameter of x relative to y can be revealed. First, the relative choice probabilities of x and y in $\{x, y\}$ reveal the salience parameter of x relative to y unless one of the alternatives is chosen with probability one. Second, if x is not chosen from $\{x, y, z\}$ and if its elimination induces a regularity violation on z, the difference between the relative choice probabilities of z and y in $\{x, y, z\}$ versus $\{y, z\}$ reveals the salience parameter of x relative to y. The third and final way the salience parameter of x relative to y can be revealed is when x and y are consecutive members of a cycle in which the salience parameters for every other consecutive pair are known.

So far, we have shown that the parameters of L-RAR are "almost" fully identified as long as they affect choice behavior. We next inquire whether these strong (reference-dependent) preference revelations are due to L-RAR's parametric structure in reference point formation. One might envision other probabilistic attention rules that determine reference point formation, and it is unclear whether our revelations will still survive under different reference point formation processes. To this end, we consider a general model of random reference where the reference formation process is only assumed to satisfy two basic conditions: the reference probabilities (*i*) are strictly positive, and (*ii*) satisfy strict regularity (that is, the reference probability of an alternative is decreasing as the menu becomes larger). We call this the *Random Reference Model (RAR).* Notice that L-RAR is a special case of RAR since L-RAR satisfies both conditions. Additionally, many other reference formation processes satisfy them.[6]

Surprisingly, we show that the revealed (reference-dependent) preferences of RAR are exactly the same as that of L-RAR. This result has two implications. First, the revealed preferences in L-RAR are not driven by our parametric modeling choice of reference probabilities. Second, any stochastic model of reference dependence satisfying these two assumptions will have exactly the same revealed preference as L-RAR.

Both RAR and L-RAR also offer a new perspective for probabilistic choices.

[6]For example, random utility representations with full support on preference rankings, the random consideration model of Manzini and Mariotti (2014), and the weighted linear stochastic choice model of Chambers et al. (2021) satisfy both conditions. In Section III, we provide more examples of reference point formation rules that satisfy both assumptions.

The previous literature interpreted stochastic choices of a single individual as the outcome of fluctuating tastes (Thurstone, 1927, Luce, 1959, Marschak, 1960)[7], random attention (Manzini and Mariotti, 2014, Brady and Rehbeck, 2016, Aguiar, 2017, Cattaneo et al., 2020), learning (Baldassi et al., 2020), random stopping (Dutta, 2020), imperfect information (Natenzon, 2019), random attributes (Gul, Natenzon and Pesendorfer, 2014), or deliberate randomization (Machina, 1985, Fudenberg, Iijima and Strzalecki, 2015, Cerreia-Vioglio et al., 2019). In our model, on the other hand, the source of randomness is probabilistic attention on potential reference points.

While L-RAR resembles the classical random utility model (RUM) where there are multiple preferences, these two models are completely different in terms of observed data. First, L-RAR violates the well-known regularity condition of RUM. Second, the intersection of L-RAR and RUM only contains the deterministic rational-choice model and the Logit model. In addition, while the set of "rationalizing" preferences in RUM is not unique in general, our reference-dependent preference is almost unique. This distinction is due to two important factors. First, the multiple preferences in L-RAR are related through the status quo bias condition. On the other hand, in RUM the set of preferences is arbitrary. Second, in L-RAR the set of reference-dependent preferences is context-dependent and the number of preferences applied to a choice set is bounded by the number of alternatives in it. In RUM, on the other hand, the set of preferences is independent of context.

It is well-known that deterministic models of reference dependence can accommodate the attraction effect (Kőszegi and Rabin, 2006, Ok, Ortoleva and Riella, 2015, Kıbrıs, Masatlioglu and Suleymanov, 2021), the finding that the relative choice proportion of two alternatives is affected by the availability of a third option that is asymmetrically dominated by one of the alternatives (Huber, Payne and Puto, 1982). Although the attraction effect has often been demonstrated in the marketing and economics literature using between-subjects designs, recently Berkowitsch, Scheibehenne and Rieskamp (2014) and Mohr, Heekeren and Rieskamp (2017) show that the attraction effect may also be observed for the same individual when making repeated decisions. The existing models of reference dependence, due to their deterministic nature, are not capable of explaining these experimental observations. On the other hand, L-RAR can accommodate the attraction effect in choice data involving repeated decisions.

Our framework includes interesting special cases. In one extreme, all the reference-dependent preferences are identical, and hence it is as if there is a single *reference-free preference*. In this case, RAR and L-RAR reduce to the classical model of deterministic rational choice. The other extreme is when each reference-dependent preference exhibits extreme bias towards its reference point, that is, when every reference-dependent preference ranks its reference point at the top.

[7]See also Apesteguia, Ballester and Lu (2017), Ahumada and Ulku (2018), Echenique and Saito (2019), Kovach and Tserenjigmid (2022), Filiz-Ozbay and Masatlioglu (2023), Horan (2021).

In this case, since an alternative is chosen only when it is the reference point, the choice probability of an alternative is equal to the probability of it being the reference point. While L-RAR becomes the Logit model in this extreme case, RAR becomes a model characterized by the regularity condition. This case also resembles the Kőszegi and Rabin (2006) idea of personal equilibrium: the distribution of reference points matches the distribution of choices. Note that this equivalence is independent of the parametric structure we have on reference point formation; it continues to hold for any stochastic reference formation process. In other words, personal equilibrium is equivalent to extreme status quo bias in our framework.

In Section IV, we provide a set of behavioral postulates that characterize the empirical content of RAR.[8] We first provide a set of ordinal axioms dealing with issues such as choice with zero probability or regularity violations. Our Axiom 1 imposes that if an alternative is chosen with zero probability in a binary comparison, adding new alternatives should not increase its choice probability. This statement is a significant weakening of the regularity axiom, which makes this requirement for all choice sets, and all alternatives regardless of their choice probability. The other ordinal axioms impose conditions under which a regularity violation might be observed. Axioms 2 and 3 state that a regularity violation can occur if and only if the eliminated alternative is chosen with zero probability. Axiom 4 imposes asymmetry on regularity violations. The axiom states that if removing an alternative z causes a regularity violation for another alternative x in the presence of y, then removing z cannot cause a regularity violation for y as long as x is available. Our final axiom, Axiom 5, is a cardinal axiom that relates revealed reference probabilities across choice sets.

The key assumption in our approach is that the reference-dependent preferences are related to each other via the SQB property (as discussed in footnote 4). While SQB is intuitive and widely accepted in the literature, one may wonder how much our strong preference identification results depend on it. In Section V, we investigate this question by replacing SQB with a weaker version of this property, which we call Weak Status Quo Bias (WSQB).[9] While SQB requires that being a reference point always unambiguously helps an alternative, WSQB only requires that being a reference point unambiguously helps an alternative in binary comparisons but may help or hurt it in general. WSQB is a natural weakening of the SQB property, as it requires that being a reference point helps an alternative in some but not necessarily all cases. While preference revelations under WSQB and SQB are different, it turns out that our strong preference identification result is still valid under WSQB. In particular, it is still true that the relative ranking of any two alternatives under any reference point can be revealed from observed choices as long as it matters for choice. In this section,

[8] Additionally, in the Appendix we present a set of behavioral postulates that characterize L-RAR.

[9] WSQB says that if x is preferred to y when y is the reference point, then x must also be preferred to y when x itself is the reference point.

we also discuss the distinguishing features of the WSQB property from SQB in terms of observed choice behavior.

In Section VI, we discuss an application of our model to Bertrand competition with differentiated products. Particularly, we discuss how the equilibrium prices depend on salience levels and reference-dependent preferences. First, we consider a simple model of duopoly where two firms engage in price competition on two products that are imperfect substitutes. While firms maximize their profits as in standard theory, all consumer choices are modeled as in L-RAR. Reference dependence is modeled as the consumer attaching an additional value to the reference point. This additional value, which we call the loyalty parameter, might depend on the product. There is additionally a salience parameter that determines the probability of a product being the reference point, as in L-RAR. For simplicity, we assume that all consumers have the same loyalty and salience parameters, but their valuations are different, and their reference points are determined independently. Firms in our application have different tools at their disposal to influence equilibrium prices and profits in contrast to the standard Bertrand equilibrium. For example, firms can influence the salience parameter via advertising and the loyalty parameter via reward programs. We show that an increase in the loyalty parameter of commodity i increases the equilibrium prices of both commodities. While it is expected that Firm i is now able to charge higher prices, it is interesting to note that this increase spills over to Firm j as well and increases its equilibrium price. On the other hand, an increase in the salience parameter of commodity i increases the equilibrium price of commodity i and decreases the equilibrium price of commodity j.

Our application also illustrates that L-RAR leads to a particular choice architecture: the introduction of additional products, seemingly irrelevant to the current consumption choice, might yet affect choice behavior. We study the implications of Firm 1 introducing a decoy product which, even though it does not create any demand for itself, increases consumers' loyalty to commodity 1. We assume that the decoy is not an attractive option even when it is the reference point. Hence, consumers never choose it. Therefore, Firm 1 does not directly profit from sales of the decoy but potentially benefits from the decoy by increasing the relative attractiveness of commodity 1. We first show that it is not always beneficial for Firm 1 to introduce the decoy and discuss the conditions under which the introduction of a decoy becomes profitable. This result sheds light on both the presence and absence of decoys under the market's competitive forces.

Our paper is foremost related to the growing literature on reference-dependent choice. The earliest strand of this literature treats the reference point as exogenous (Kahneman and Tversky, 1979, Tversky and Kahneman, 1991, Munro and Sugden, 2003, Sugden, 2003, Masatlioglu and Ok, 2005, Sagi, 2006, Salant and Rubinstein, 2008, Masatlioglu and Ok, 2014, Dean, Kıbrıs and Masatlioglu, 2017, Guney and Richter, 2018, Kovach and Suleymanov, 2021). Our paper is distinct from these papers in endogenizing the reference formation process. In addition, all of these papers except for Kovach and Suleymanov (2021) address deterministic choices. A second strand of the literature studies endogenous reference point formation. In models of Bodner and Prelec (1994), Kivetz, Netzer and Srinivasan (2004), Orhun (2009), Bordalo, Gennaioli and Shleifer (2012, 2013), and Tserenjigmid (2019), the reference point depends on the structure of the choice set, but it is independent of individual characteristics.[10] Kőszegi and Rabin (2006), Rubinstein and Salant (2006), Ok, Ortoleva and Riella (2015), Freeman (2017), Kıbrıs, Masatlioglu and Suleymanov (2021), and Lim (2020) consider models where the endogenous reference point might differ across individuals. Maltz (2020) considers a hybrid model which combines an exogenous status quo with an endogenous reference point. These papers, however, only address deterministic choice behavior.[11] To the best of our knowledge, L-RAR and RAR are the first stochastic models of endogenous reference dependence.

Among the deterministic models listed above, the closest to ours is Kıbrıs, Masatlioglu and Suleymanov (2021), where the reference point of a choice set is endogenously obtained from a maximization of a "conspicuity order" over alternatives. In L-RAR, we replace this maximization with a stochastic process in which the salience of an alternative now determines the probability that it serves as a reference point. Since L-RAR has more parameters than its deterministic counterpart, one might expect identification to be comparatively more difficult. However, the stochastic choice data L-RAR uses is richer than deterministic choice (in addition to being more realistic), and this richness more than compensates for the added parameters under stochastic reference points when it comes to identifying parameters.

Our paper (particularly the L-RAR model) is also related to a few recent papers that generalize the Luce model (Ahumada and Ulku, 2018, Echenique and Saito, 2019, Horan, 2021). These papers relax the Luce model's requirement that an alternative must be chosen from every choice problem once it is chosen from one choice problem. In these models, the DM first constructs a "consideration set" by eliminating dominated alternatives and then chooses alternatives within the consideration set via the Luce rule. While these papers have a completely different motivation from ours as they do not model reference-dependent choice, the L-RAR model can also be thought of as a generalization of the Luce model which allows an alternative to be chosen with positive probability from one choice set but with zero probability from another. However, the ways in which L-RAR

[10] Note that the notion of salience used in Bordalo, Gennaioli and Shleifer (2012, 2013) differs significantly from the one presented in this paper. In their framework, each product has distinct attributes, and the reference point (attribute-wise average) influences which attribute becomes salient (and receives more weight in the evaluation). Thus, their model uses the reference point to determine the salience of an *attribute*. Conversely, we define the salience of an *alternative* (rather than a specific attribute), which determines the random reference rule (without affecting preferences). Hence, the use of salience in the two approaches differ significantly.

[11] To clarify, what we refer to is deterministic choice behavior rather than the possibility of having alternatives that are lotteries. For example, Kőszegi and Rabin (2007) takes alternatives to be lotteries, yet studies deterministic choices from them.

and these models handle non-chosen alternatives are completely different. In these models, an alternative that is not chosen has no influence on the relative choice probabilities of chosen alternatives. In L-RAR, on the other hand, an alternative that is not chosen can still act as a reference point and influence the relative choice probabilities of other alternatives.

The paper is organized as follows. In Section I, we introduce L-RAR. In Section II, we show how the primitives of L-RAR are revealed from observed choices. In Section III, we generalize our model to RAR and discuss its identification. In Section IV, we introduce the behavioral postulates that characterize RAR and present our representation theorem. In Section V, we show that our strong preference identification results still hold under a weaker version of status quo bias. In Section VI, we present an application of our model to Bertrand competition. We conclude in Section VII. Lastly, the Appendix contains a characterization theorem for L-RAR and the proofs of all theorems.

I. Logit Random Reference Model

Let X be a non-empty finite set of alternatives, and let $\mathcal{X}$ be the set of all nonempty subsets of X. A *choice problem* is a set of alternatives $S \in \mathcal{X}$ from which the decision maker needs to make a choice. A *choice rule* is a map $p : X \times \mathcal{X} \to [0,1]$ such that for all $S \in \mathcal{X}$, $\sum_{x\in S} p(x|S) = 1$ and $p(x|S) > 0$ only if $x \in S$. The choice rule p represents data on the choice behavior of the *decision maker* (hereafter, *DM*). The expression $p(x|S)$ represents the probability of x being chosen from the choice problem S. Note that if $p(x|S) \in \{0,1\}$ for every x and S, then choices are deterministic. Hence, our formulation encompasses both stochastic and deterministic choice rules.

Our model has two components: (i) a family $\{\succ_x\}_{x\in X}$ of *reference-dependent preferences* where each $\succ_x$ is a strict linear order that represents the DM's preferences under the reference point x,[12] and (ii) a family $\{s_x\}_{x\in X}$ of *reference weights*, where each $s_x > 0$ measures the salience of alternative x as a reference point.[13]

We assume that the reference-dependent preferences $\{\succ_x\}_{x\in X}$ satisfy the following assumption. If x is preferred to y when z is the reference point, then x must also be preferred to y when x itself is the reference point. This assumption relates two reference-dependent preferences and, in line with the status quo bias, requires that being the reference point cannot hurt any alternative.[14]

Status Quo Bias (SQB). If $x \succ_z y$, then $x \succ_x y$.

In our model, the reference point is stochastically determined à la Luce (1959). That is, we assume that the probability of x being the reference point in S is

[12]A binary relation R on X is a strict linear order if it is (i) weakly connected: for every $x, y \in X$, $x \neq y$ implies either xRy or yRx, (ii) irreflexive: for every $x \in X$, it is not the case that xRx, and (iii) transitive: for every $x, y, z \in X$, xRy and yRz imply xRz.

[13]Throughout the paper, we refer to $\{s_x\}_{x\in X}$ as the reference weights, the salience weights or the salience parameters interchangeably.

[14]In Section V, we discuss the implications of a weaker assumption on preferences.

equal to its own reference weight relative to the total weight of all alternatives in S. Once a reference point x is determined from the choice problem S, the DM maximizes the associated reference-dependent preference $\succ_x$ in S to make a choice. The following definition formally states the choice process in our model.

DEFINITION 1: *A choice rule p is consistent with the* ***Logit Random Reference Model (L-RAR)*** *if there exist a family $\{\succ_x\}_{x\in X}$ of reference-dependent preferences satisfying SQB and a family $\{s_x\}_{x\in X}$ of reference weights, where $s_x > 0$ for each $x \in X$, such that for each $S \in \mathcal{X}$ and $x \in S$,*

$$p(x|S) = \sum_{y\in S}\left(\frac{s_y}{\sum\limits_{z\in S} s_z}\right)\mathbb{1}(x \textit{ is } \succ_y\textit{-best in } S).$$

We also say $\{\succ_x, s_x\}_{x\in X}$ represents p, or p admits an L-RAR representation.

L-RAR includes two well-known special cases. At one extreme, if all reference-dependent preferences are the same, then L-RAR reduces to the classical model of deterministic rational choice. At the other extreme, if each reference-dependent preference ranks its reference point at the top, then L-RAR coincides with the Luce model with weights $\{s_x\}_{x\in X}$.[15] Hence, the rational choice model and the Luce model are two extreme cases of L-RAR with no status quo bias and extreme status quo bias, respectively.

To illustrate the model, consider a simple example with three alternatives. Reference-dependent preferences are $x \succ_x y \succ_x z$, $y \succ_y x \succ_y z$, and $x \succ_z y \succ_z z$. The salience weights are 1, 2, and 3 for x, y, and z, respectively. Since x and y are the best options when they are the reference points, they will always be chosen with positive probability. Particularly, when only x and y are available, their relative choice probabilities will be equal to their relative reference probabilities, that is $(\frac{1}{3}, \frac{2}{3})$. As discussed later, this observation allows us to deduce reference probabilities from binary sets where both alternatives are chosen with positive probabilities. Alternatively, assume all alternatives are available. Then $(\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$ is the corresponding reference probability distribution. Since x is the best alternative when the reference point is either x or z, the choice probability of x is equal to the sum of $\frac{1}{6}$ and $\frac{3}{6}$ $(=\frac{2}{3})$. On the other hand, z is never chosen. Notice that removing z from the choice set $\{x, y, z\}$ lowers the probability that x is chosen from $2/3$ to $1/3$. This choice behavior violates the well-known regularity principle (Suppes and Luce, 1965), which is satisfied by the random utility models and many other stochastic choice models. An important feature of our model is that regularity violations are caused by the elimination of unchosen alternatives.

[15] Alternatively, if all alternatives are always chosen with positive probabilities in all choice sets, then we can conclude that reference points are always ranked at the top (see the discussion in the next section). Hence, L-RAR reduces to the Luce rule if no alternative is ever chosen with zero probability.

II. Identification of Parameters

In this section, we examine what can be inferred about the primitives of the model based on observed choices. This is important for understanding the underlying model and its predicted behavior, as well as for making out-of-sample predictions and welfare evaluations. We consider an analyst who observes stochastic choice data. The analyst posits that the decision maker has reference-dependent preferences and randomness in the choice data is due to random shifts in reference points. The analyst would like to answer the following two key questions: (i) What is the frequency with which each alternative serves as the decision maker's reference point in each choice set? (ii) What is the preference relation induced by each reference point? We show in this section how both questions can be answered within the framework of our model.

Since the analyst does not observe how often each alternative serves as the DM's reference point, identifying reference-dependent preferences might at first appear challenging. In addition, one might hypothesize that revealed reference-dependent preferences might often be non-unique. Surprisingly, within the framework of our model, we show that reference-dependent preferences can be revealed to a large extent. In particular, the relative ranking of any two alternatives under any reference point is revealed as long as at least one of the alternatives is at least as good as the reference point. Alternatively, the relative ranking of any two alternatives may not be revealed only if both alternatives are strictly inferior to the reference alternative. Since the latter cannot affect the decision maker's behavior (the DM can always choose the reference alternative rather than a strictly inferior alternative), identified preferences are "almost" unique. In Section III, we show that this result is not due to the logit random reference rule and can be generalized significantly. In Section V, we show that the result still holds even if one assumes a much weaker version of the status quo bias property.

To illustrate how reference weights and reference-dependent preferences can be identified, let us present two extreme examples. First, consider choice data satisfying Luce's IIA. It can have multiple L-RAR representations. In every one of them, the reference weight of an alternative x relative to an alternative y (that is, s_x/s_y) is uniquely determined as the choice probability of x relative to y in the grand set of alternatives (or, by Luce's IIA, in any set that contains them). In addition, each reference point must be ranked as the top alternative of its reference-dependent preferences in all these L-RAR representations. However, we cannot infer the relative ranking of two non-reference alternatives. This is an unavoidable non-uniqueness in our model. Note that non-reference alternatives are strictly inferior to reference alternatives in this case.

At the other extreme, consider deterministic choice data generated by the maximization of a single reference-independent preference relation. It also has multiple L-RAR representations. Unlike the other extreme case though, all reference-dependent preferences are uniquely identified. Indeed, they must be the same as the corresponding (reference-independent) preference relation generating the data. But unlike the other extreme case, reference weights cannot be identified: any positive vector of weights would work.

As will be demonstrated further in this section, the existence of multiple representations is not restricted to the extreme cases discussed above. Hence, in the presence of multiple representations, we need to be precise about how we define revealed preference and reference weights. Our approach will be conservative to ensure that we do not make any mistakes in revelations. In particular, if two representations disagree on the relative ranking of two alternatives under some reference point, then we do not make any conclusions regarding the ranking of these two alternatives. Similarly, if two representations disagree on the relative reference weights of alternatives, then we do not make any conclusions. Before making any conclusions regarding revealed preference and reference weights, we will require that all representations agree on these revelations.

Formally, assume p admits k L-RAR representations $\left(\{\succ^i_x, s^i_x\}_{x\in X}\right)_{i\in\{1,...,k\}}$. Then we say

1) x is revealed to be preferred to y under reference point z if $x \succ^i_z y$ for each $i \in \{1, ..., k\}$, and

2) the reference weight of x relative to y is revealed to be α_{xy} if $\alpha_{xy} = s^i_x/s^i_y$ for each $i \in \{1, ..., k\}$.

Note that the second item refers to the identification of the relative reference weights of two alternatives rather than the reference weight of a single alternative. This is because absolute reference weights are essentially non-unique: both $\{s_x\}_{x\in X}$ and $\{a * s_x\}_{x\in X}$ produce the same choice data whenever $a > 0$. Our definition bypasses this rather trivial non-uniqueness by considering relative rather than absolute weights.

According to our definition, to make any conclusions about revealed preference and reference weights, one needs to construct all possible L-RAR representations and verify the cases in which these representations agree. It is clearly not practical to use this method to obtain revelations. Thus, we next provide a practical method to obtain revealed reference-dependent preferences and reference weights.

Three types of observations reveal information about reference-dependent preferences. First, suppose an alternative is chosen with probability one in some choice set, that is, $p(y|S) = 1$. Then, clearly, the chosen alternative must be preferred to the reference alternative regardless of the reference point: $y \succ_z z$ for any $z \in S$. In addition, the SQB property implies that we must also have $y \succ_x z$ for any $z \in S$ and $x \in X$. Hence, y is the best alternative in S regardless of the reference point.

Next, consider the observation that an alternative is chosen with positive probability in some choice set: $p(y|S) > 0$. For an alternative to being chosen with positive probability, it must be that it is the best alternative under some reference point in that set: y is $\succ_z$-best in S for some $z \in S$. But then the SQB property implies that y must also be the best when it is itself the reference point. That is, $y \succ_y z$ for any $z \in S$.

Lastly, consider the observation that removing an alternative from a choice set decreases the probability that another alternative is chosen. Such regularity violations are allowed in our model and they give us information about the underlying reference-dependent preferences. In particular, suppose $p(y|S) > p(y|S \setminus x)$. This observation reveals that y must be the best alternative in S under the reference point x. To see this, let $A(y|S)$ denote the set of reference points in S under which y is the best alternative. Clearly, if x does not belong to $A(y|S)$, then we must have $A(y|S) \subseteq A(y|S \setminus x)$. In addition, given the logit reference rule, any alternative in $A(y|S)$ that is distinct from x must be the reference point in $S \setminus x$ with a higher probability. Combining these two observations, we would then have $p(y|S) \leq p(y|S \setminus x)$ if $x \notin A(y|S)$. Hence, y must be $\succ_x$-best in S.

Our discussion makes it clear that the above observations are necessary for revealed preference. An important question is whether these observations are also sufficient. To answer this question, for any $x \in X$ and $y \neq z$, we define

$$yP_x z \text{ if } \exists S \supseteq \{y, z\} \text{ such that one of the following is observed:} \quad \begin{array}{l} (i)\ p(y|S) = 1, \\ (ii)\ p(y|S) > 0 \text{ and } x = y, \text{ or} \\ (iii)\ p(y|S) > p(y|S \setminus x). \end{array}$$

We show that if p has an L-RAR representation, then P_x must be transitive. However, it might not be complete (see the first extreme case above). Since any L-RAR representation of p must be consistent with the above revelations, P_x must be part of the revealed reference-dependent preferences. Importantly, there is no other revelation: any family of reference-dependent preferences that respect $\{P_x\}_{x \in X}$ represents p. In other words, if two alternatives are not ranked according to P_x, then we can always find two representations where the relative ranking of these alternatives is opposite of each other. The proof of this result is analogous to the proof of Theorem 2, which provides a characterization result for an L-RAR representation.

PROPOSITION 1 (Revealed Preference): *Suppose p admits an L-RAR representation. Then y is revealed to be preferred to z under reference point x if and only if $yP_x z$.*

Proposition 1 establishes the empirical content of revealed preferences in our model. If $\{P_x\}_{x \in X}$ contained few revelations, this would be undesirable since it would be hard for policymakers to make welfare evaluations. However, it turns out that this is not the case. In particular, we show that the relative ranking of any two alternatives under any reference point can be revealed as long as it matters for choice. That is, if two alternatives y and z are not both strictly inferior to the reference alternative x, then the relative ranking of y and z under

x must be revealed.[16] Hence, revealed reference-dependent preferences in L-RAR are "almost" unique. The following proposition formally states this claim.

PROPOSITION 2: *Suppose p admits an L-RAR representation where at least one of $x \succ_x y$ and $x \succ_x z$ does not hold. Then, assuming $y \neq z$, either yP_xz or zP_xy must hold.*

PROOF:

Let $\{\succ_x, s_x\}_{x\in X}$ be an L-RAR representation of p such that at least one of $x \succ_x y$ and $x \succ_x z$ does not hold. In addition, without loss of generality, assume $y \succ_x z$. If $y = x$, then $p(y|\{y,z\}) > 0$, and hence yP_xz. If $z = x$, then $p(y|\{y,z\}) = 1$, and hence yP_xz again. So, suppose x, y, z are distinct and $y \succ_x z$. Since at least one of $x \succ_x y$ and $x \succ_x z$ is not true, we must have $y \succ_x x$, and by SQB, $y \succ_y x, z$. Now consider the choice set $\{x,y,z\}$. If $p(y|\{x,y,z\}) = 1$, then yP_xz and we are done. On the other hand, if $p(y|\{x,y,z\}) \in (0,1)$, then

$$p(y|\{x,y,z\}) = \frac{s_x + s_y}{s_x + s_y + s_z} > \frac{s_y}{s_y + s_z} = p(y|\{y,z\}).$$

Therefore, yP_xz again. This concludes the proof. □

We next discuss how we can reveal relative reference weights from choice data. To this end, we will only make use of binary and trinary sets, even though similar revelations hold for larger sets as well. Three types of observations reveal information about the reference weight of x relative to y. First, consider a binary choice set $\{x,y\}$ and suppose both x and y are chosen with positive probability. This reveals that both x and y are preferred to the other alternative when they are references. Hence, the relative choice probability of x and y reflects their relative reference weight.

Next, suppose we cannot reveal the relative reference weight of x and y from the binary choice set $\{x,y\}$. Suppose, however, there exists an alternative $z \in X$ such that $p(z|\{x,y,z\}) > p(z|\{y,z\})$. Notice that since p has an L-RAR representation, this implies $p(x|\{x,y,z\}) = 0$. This is due to the fact that $p(z|\{x,y,z\}) > p(z|\{y,z\})$ reveals z to be more preferred to x under $\succ_x$, and by SQB, z is better than x under any other reference. In addition, note that $p(z|\{x,y,z\}) > 0$ implies z is $\succ_z$-best in $\{x,y,z\}$, and $p(z|\{y,z\}) < 1$ implies y is $\succ_y$-best in $\{x,y,z\}$ (x cannot be $\succ_y$-best due to the previous observation). Therefore, we must have

$$\frac{p(z|\{x,y,z\})}{p(y|\{x,y,z\})} - \frac{p(z|\{y,z\})}{p(y|\{y,z\})} = \frac{s_x}{s_y},$$

[16] If y and z are both strictly worse than the reference alternative x, the reference point is always chosen against these alternatives. Hence, the relative ranking of these alternatives is irrelevant for choice behavior. Notice that even in this case, we might still reveal the relative ranking of y and z under x if either $y \succ_z z$ or $z \succ_y y$ holds. This revelation is due to the SQB property of reference-dependent preferences.

uniquely revealing the relative reference weights of x and y.

Lastly, suppose x and y are consecutive members of a cycle in which the relative reference weights for every other consecutive pair are known. That is, we have n alternatives $\{x_1, \ldots, x_n\}$, where we have already revealed the relative reference weights $s_{x_i}/s_{x_{i+1}}$ for $(n-1)$ pairs, with the abuse of notation $x_{n+1} = x_1$. Since reference weights are context-independent, the relative reference weights of any two alternatives in $\{x_1, \ldots, x_n\}$ and the corresponding binary choice set must coincide. This implies that the relative reference weights of a cycle multiply to 1, and knowing the values of $(n-1)$ pairs reveals the last one. This is evident when $n = 2$ as knowing s_{x_1}/s_{x_2} allows us to determine s_{x_2}/s_{x_1}. For arbitrary values of n, if only s_{x_n}/s_{x_1} is unknown in the chain $\{x_1, \ldots, x_n\}$, then it can be obtained from $\frac{s_{x_n}}{s_{x_1}} = \frac{s_{x_n}}{s_{x_{n-1}}} \frac{s_{x_{n-1}}}{s_{x_{n-2}}} \ldots \frac{s_{x_2}}{s_{x_1}}$, where the right-hand side of the expression is already known.

It is clear from the discussion that the above observations are necessary for revealed reference weights. In addition, our next proposition shows that they are also sufficient. To this end, for any x and y, let α_{xy} be defined as below:

1) if $p(x|\{x,y\}) \in (0,1)$,
$$\alpha_{xy} = \frac{p(x|\{x,y\})}{p(y|\{x,y\})},$$
2) if $p(z|\{x,y,z\}) > p(z|\{y,z\})$,
$$\alpha_{xy} = \frac{p(z|\{x,y,z\})}{p(y|\{x,y,z\})} - \frac{p(z|\{y,z\})}{p(y|\{y,z\})},$$
3) for any $\{x_1, ..., x_n\}$, if $(n-1)$ of the n alpha values $\{\alpha_{x_i x_{i+1}}\}_{i=1}^n$ are already known (with the abuse of notation $x_{n+1} = x_1$), the last one is defined through the equality
$$\prod_{i=1}^{n} \alpha_{x_i x_{i+1}} = 1.$$

Given choice data p that is consistent with the L-RAR model, let

$$xWy \text{ if } \alpha_{xy} \text{ is defined by one of the three patterns above.}$$

First, note that W is transitive and symmetric, as implied by the last observation. However, it may not be complete. For example, in the case of deterministic choice data generated by maximization of a single reference-independent preference relation, $W = \emptyset$. As we argued above, W must be part of the revealed relative reference weights. Furthermore, there is no other revelation: any vector $\{s_x\}_{x \in X}$ that respects α_{xy} for all $(x,y) \in W$ represents p. In other words, if $(x,y) \notin W$, then for any positive real number γ, we can find a representation where the reference weight of x relative to y is γ. The following proposition establishes this point. The proof is identical to the proof of Theorem 2.

PROPOSITION 3 (Revealed Reference Weights): *Suppose p admits an L-RAR representation. Then the reference weight of x relative to y is revealed to be α_{xy} if and only if xWy.*

The previous proposition fully characterizes revealed reference weights in L-RAR. As we noted before, W could be incomplete. However, similar to revealed preference, we can show that the relative weights are "almost" unique. That is, a relative reference weight can be revealed as long as it matters for choice. More specifically, we next show that the relative reference weight of any two alternatives x and y is revealed as long as there exists a choice set S containing these two alternatives such that no alternative from S is chosen with probability one. Note that if there exists no such S, then the relative reference weight of x and y can never influence choice behavior, and being irrelevant for choice, it cannot be revealed. The following proposition formally states this claim.

PROPOSITION 4: *Suppose p admits an L-RAR representation. Then, for any x and y, if there exists $S \supseteq \{x, y\}$ such that $p(z|S) = 1$ for no $z \in S$, then xWy.*

PROOF:

Since $p(z|S) = 1$ for no $z \in S$, there exist at least two alternatives that are chosen with positive probability. There are a few cases to consider.

Case 1: Both x and y are chosen with positive probability. This observation reveals to us that both x and y are the best in S when they are reference points. Hence, $p(x|\{x, y\}) > 0$ and $p(y|\{x, y\}) > 0$, and α_{xy} is defined.

Case 2: Either x or y is chosen with positive probability but not both. Without loss of generality, suppose x is chosen with positive probability. Since y is chosen with zero probability, there must be another alternative that is $\succ_y$-best. First, suppose x is $\succ_y$-best in S. Let z denote another alternative that is chosen with positive probability from S. Due to the representation, we must then have that $p(x|\{x, y, z\}) > p(x|\{x, z\})$. By using the second observation, we can then define α_{yz}. Since both $p(x|\{x, z\})$ and $p(z|\{x, z\})$ must be positive, α_{zx} is also defined. Hence, by using the third observation, $\alpha_{xy}\alpha_{yz}\alpha_{zx} = 1$, we can define α_{xy}. If, on the other hand, z is $\succ_y$-best in S, we must have $p(z|\{x, y, z\}) > p(z|\{y, z\})$, which defines α_{xy} through the second observation.

Case 3: Both x and y are chosen with zero probability. Then there exist z and t such that z is $\succ_x$-best in S and t is $\succ_y$-best. First, suppose z and t are distinct. By SQB, z is $\succ_z$-best in S and t is $\succ_t$-best. Hence, $p(z|\{z, t\})$ and $p(t|\{z, t\})$ are both positive. This defines α_{zt}. Next, note that by the representation, $p(z|\{x, z, t\}) > p(z|\{z, t\})$ and $p(t|\{y, z, t\} > p(t|\{z, t\})$. Using the second observation, we can define α_{xt} (and hence α_{tx}) and α_{yz}. But then, by using the last observation $\alpha_{xy}\alpha_{yz}\alpha_{zt}\alpha_{tx} = 1$, we can define α_{xy}. Lastly, suppose $z = t$. Since no alternative is chosen from S with probability one, there exists w such that

$p(w|S) > 0$. By the representation, $p(z|\{z,w\}) > 0$ and $p(w|\{z,w\}) > 0$. This defines α_{zw}. In addition, the representation implies $p(z|\{x,z,w\}) > p(z|\{z,w\})$ and $p(z|\{y,z,w\}) > p(z|\{z,w\})$. This defines α_{xw} (hence α_{wx}) and α_{yw} (hence α_{wy}) using the second observation. In addition, since $\alpha_{yz}\alpha_{zw}\alpha_{wy} = 1$, this defines α_{yz}. Lastly, since $\alpha_{xy}\alpha_{yz}\alpha_{zw}\alpha_{wx} = 1$, this defines α_{xy}, and we are done. □

An implication of Proposition 4 is that if no alternative is chosen with probability one in a choice set, then we can always uniquely reveal reference probabilities for that choice set. If there exists an alternative that is uniquely chosen, we might still be able to reveal reference probabilities by using choice data from other sets. Hence, in L-RAR, reference probabilities are uniquely revealed whenever it is possible to reveal information about them.

We next illustrate how to employ the previous propositions to reveal information. To this end, we revisit the choice data given at the end of Section I, presented in Table 1 for convenience. We show that all relative reference weights and reference-dependent preferences are uniquely identified and consistent with the primitives generating the data.

Table 1—A choice rule consistent with L-RAR. Preferences and relative reference weights can be fully revealed from the choice data.

$p(\cdot\|S)$	$\{x,y,z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$
x	2/3	1/3	1	-
y	1/3	2/3	-	1
z	0	-	0	0

Revealed Preference: To reveal P_z, note that $p(x|\{x,y,z\}) > p(x|\{x,y\})$ implies xP_zy, and $p(y|\{y,z\}) = 1$ implies yP_zz. Together, we have xP_zyP_zz. To reveal P_y, note that $p(y|\{x,y,z\}) > 0$ implies yP_yx and yP_yz, and $p(x|\{x,z\}) = 1$ implies xP_yz. Together, we have yP_yxP_yz. To reveal P_x, note that $p(x|\{x,y,z\}) > 0$ implies xP_xy and xP_xz, and $p(y|\{y,z\}) = 1$ implies yP_xz. Together, we have xP_xyP_xz.

Revealed Relative Reference Weights: From the set $\{x,y\}$, we observe $\alpha_{xy} = \frac{p(x|\{x,y\})}{p(y|\{x,y\})} = \frac{1}{2}$. Then, from the set $\{x,y,z\}$,

$$\alpha_{zy} = \frac{p(x|\{x,y,z\})}{p(y|\{x,y,z\})} - \frac{p(x|\{x,y\})}{p(y|\{x,y\})} = \frac{2}{1} - \frac{1}{2} = \frac{3}{2}.$$

Hence, $\alpha_{yz} = \frac{2}{3}$ and $\alpha_{xz} = \alpha_{xy}\alpha_{yz} = \frac{1}{3}$. For example, normalizing $s_x = 1$, this gives us $s_y = 2$ and $s_z = 3$.

III. Random Reference Model

In the previous section, we studied a specific model of reference point formation, namely the logit rule, and showed that L-RAR could be almost fully identified. While the logit rule has some appealing features, such as tractability, it is far from being a canonical model of reference point formation, as it is a particular parametric rule. In this section, we abstain from any specific parametric (stochastic) rule and instead consider a large class of non-parametric reference formation rules. Our general model imposes two intuitive conditions: (*i*) strict positivity and (*ii*) strict regularity (i.e., the reference probability of an alternative decreases as the menu becomes larger). We illustrate by means of examples that these conditions are satisfied by many plausible reference point formation rules, including logit. Given that the reference formation rule is not observable, this is crucial for the applicability of our revealed preference results by ensuring that our findings and empirical implications are valid under various rules that may be operating in the background. In other words, our revealed preference results are derived from non-parametric restrictions on the formation rule and hence are more robust to misspecification biases. In addition, this model allows us to investigate whether the strong revelations in the L-RAR model are due to its strong parametric structure in reference point formation.

Formally, the DM has context-dependent reference probabilities $\rho(\cdot|S)$. The expression $\rho(x|S)$ represents the likelihood that alternative x will be the reference point in S. Assume that ρ satisfies (*i*) $\rho(x|S) > 0$ for all $x \in S$ (strict positivity), and (*ii*) $\rho(x|S) < \rho(x|T)$ for all $x \in T \subsetneq S$ (strict regularity). Notice that the reference rule in L-RAR given by $\rho(x|S) = \frac{s_x}{\sum_{y \in S} s_y}$ satisfies both properties. To further demonstrate the richness of the framework and motivate the analysis to follow, we discuss several other examples of reference point formation models satisfying the two above conditions.

1) (Captive Reference Point) This example generalizes the logit reference-point formation. Each alternative x is now endowed with a captivity parameter $\theta_x \geq 0$ and $s_x > 0$ represents the salience of x as before. The captivity parameter reflects the attachment to a particular reference point in the sense that the DM picks this alternative as her reference point regardless of salience. Formally, let

$$\rho(x|S) = \frac{1}{1 + \sum_{y \in S} \theta_y} \frac{s_x}{\sum_{y \in S} s_y} + \frac{\theta_x}{1 + \sum_{y \in S} \theta_y}.$$

The second term is independent of the salience parameters and represents a lower bound for $\rho(x|S)$. The larger θ_x is, the more likely it is that x becomes the reference point. When $\theta_x = 0$ for all x, we obtain the logit model. This model improves upon the unrealistic feature of the logit model, which requires that the odds ratio of being a reference point, $\frac{\rho(x|S)}{\rho(y|S)}$, is independent of other alternatives, and independent of additions to, and deletions from a choice set.

2) (Random Attribute Order) Consider an environment where alternatives have attributes, such as price or quality. The decision maker randomly focuses on one of these attributes and the alternative that is ranked highest according to this attribute becomes the DM's reference point. For example, a DM who is buying an airplane ticket may focus on the price of the ticket in some cases and the duration of the flight in other cases. Let A be the finite set of attributes. Let λ_j represent the probability that attribute j "draws attention to itself". This reflects the salience and/or importance of attribute j. Let R_j denote the ranking of alternatives under attribute j. The probability of x being the reference point in S is then given by

$$\rho(x|S) = \sum_{j \in A} \lambda_j \mathbb{1}(x \text{ is } R_j\text{-best in } S).$$

Assuming that each alternative is R_j-best for at least one attribute j with $\lambda_j > 0$, $\rho(\cdot|\cdot)$ satisfies both strict positivity and strict regularity. Notice that for any given attribute (e.g., price) we can define another attribute that is the inverse of the first one (e.g., $-$ price). Hence, this example allows both the cheapest and the most expensive item to act as a reference point.

3) (Random Attribute Intensity) This example is an adaptation of Gul, Natenzon and Pesendorfer (2014) to reference point formation. As in the Random Attribute Order, alternatives have several attributes. Let A denote the finite set of attributes again. Each attribute j is endowed with a salience parameter $v_j > 0$. We let $I(x, j) \geq 0$ denote the intensity of alternative x in attribute j. For example, if color is the relevant attribute, then a product with red color may have a higher intensity than the one with gray color. An alternative with high intensity in more salient attributes is more likely to act as the reference point. Formally, let A_x denote the subset of attributes alternative x has, i.e., $j \in A_x$ if and only if $I(x, j) > 0$. Let $A_S = \bigcup_{x \in S} A_x$. Then,

$$\rho(x|S) = \sum_{j \in A_x} \frac{v_j}{\sum_{k \in A_S} v_k} \cdot \frac{I(x, j)}{\sum_{y \in S} I(y, j)}.$$

Assuming that each alternative has at least one attribute, strict positivity and strict regularity are satisfied. Note that this model reduces to the logit rule if no pair of alternatives share common attributes.

4) (Random Category) This is an adaptation of the nested logit model (Ben-Akiva, 1973, McFadden, 1978) to reference point formation. The consumer groups alternatives into categories $C_1, \ldots, C_n$ that partition the set of all alternatives X. The consumer first chooses a category, and then the reference

point is chosen from the category via the Luce rule with weights $\{s_x\}_{x\in X}$. Formally, if x is in C_i, then the probability of x being the reference point in S is given by

$$\rho(x|S) = \frac{\Big(\sum\limits_{x\in S\cap C_i} s_x\Big)^{\mu_i}}{\sum\limits_j \Big(\sum\limits_{z\in S\cap C_j} s_z\Big)^{\mu_j}} \frac{s_x}{\sum\limits_{y\in S\cap C_i} s_y},$$

where $\mu_i \in (0,1]$ for all i.

5) (Dual System) Our last example is an adaptation of the process "thinking fast and slow," an idea popularized by Kahneman (2011). In this model, there are two distinct systems in the mind of the consumer: system 1 and system 2. The consumer employs system 1 in familiar environments, while system 2 is employed when the consumer faces an unfamiliar environment. Assume that the consumer faces all the products most of the time. In this familiar environment, the consumer only activates system 1, which dictates that the probability of x acting as the reference point is at least $v_1(x)$. When some of the alternatives are not available and the choice set is S, there will be an excess measure of $(1 - \sum_{y\in S} v_1(y))$ of probabilities. In these cases, the consumer activates system 2 to allocate the excess probability among the feasible reference points. The excess probability is distributed to available products with respect to their relative salience according to system 2's evaluation, which is represented by v_2. Hence, the reference probabilities in this example can be written as the sum of the base probability (system 1) plus the fraction of the excess probability due to the unavailability of some products (system 2).[17] Formally,

$$\rho(x|S) = v_1(x) + \Big(1 - \sum_{y\in S} v_1(y)\Big)\frac{v_2(x)}{\sum\limits_{y\in S} v_2(y)}.$$

Assuming $v_1(x) > 0$ for all x, this reference rule satisfies strict positivity and strict regularity.

In what follows, the choice procedure of our agent is the same as in L-RAR, except that now reference probabilities are only required to satisfy strict positivity and strict regularity, which are satisfied by a wide range of models, as illustrated above. As before, we continue to assume that reference-dependent preferences $\{\succ_x\}_{x\in X}$ satisfy SQB. The following definition formally describes the choice process.

[17]This formulation also appears in Chambers et al. (2021) in the context of stochastic choice data.

DEFINITION 2: *A choice rule p is consistent with the* ***random reference model (RAR)*** *if there exist a family $\{\succ_x\}_{x\in X}$ of reference-dependent preferences satisfying SQB and context-dependent reference probabilities ρ satisfying strict positivity and strict regularity such that for each $S \in \mathcal{X}$ and $x \in S$,*

$$p(x|S) = \sum_{y\in S} \rho(y|S)\, \mathbb{1}(x \text{ is } \succ_y\text{-best in } S).$$

When the above definition is satisfied, we also say $(\{\succ_x\}_{x\in X}, \rho)$ represents p, or p admits a RAR representation.

As in Section II, we say that y is revealed preferred to z under x if for every RAR representation $(\{\succ_x\}_{x\in X}, \rho)$ of p, we have $y \succ_x z$. In addition, let P_x be defined as in the previous section. Our main result in this section states that the revealed preference in RAR is exactly the same as in L-RAR. In other words, any reference point formation rule that satisfies the two assumptions of RAR (e.g., any of the other examples in this section) shares the same exact preference revelation as in L-RAR.

PROPOSITION 5 (Revealed Preference): *Suppose p admits a RAR representation. Then y is revealed to be preferred to z under reference point x if and only if yP_xz.*

The formal proof of this result is analogous to the proof of Theorem 1, which provides a characterization result for RAR. Here we illustrate why this result must hold. The argument for the necessity is similar to the one for Proposition 1. First, if an alternative is chosen with probability one in some choice set, then this alternative must be the best alternative regardless of the reference point in that set. In addition, the SQB property guarantees that it also has to be the best, even for reference points outside the set. For the second revelation, if an alternative is chosen with a positive probability in some choice set, then it must be the best under at least one reference point in that set. But then the SQB property guarantees that it is also the best when it is itself the reference point. For the last revelation, assume we observe the probability of y being chosen to be strictly larger in S than $S \setminus x$. Notice that removing x from S does not shrink the set of alternatives that place y as the top-ranked alternative in S. In addition, since reference probabilities satisfy strict regularity, this observation is possible only if y is the best in S under x.

To see why these revelations are sufficient, assume that we have not revealed the relative ranking of $y, z \in X$ under the reference x using our definition for P_x. Then, by an argument along the lines of Proposition 2, we can show that $x \succ_x y$, $x \succ_x z$, $y \succ_y z$ and $z \succ_z y$. But if all these conditions are satisfied, the relative ranking of y and z under x cannot impose any restrictions on observed choices. Hence, our definition reveals to us everything that can be revealed about the preference, so it has to be sufficient. Notice that this also implies that the added structure of L-RAR cannot give us any new preference revelations. All the information that can be learned about preferences can already be learned under RAR.

The result in this section is important for two reasons. First, it shows that the strong preference revelations in L-RAR are not driven by its parametric structure. The same strong revelations also hold under a much more general non-parametric restriction. Second, for any other reference point formation rule that satisfies the two assumptions of RAR, we do not need to perform further revealed preference analysis. Our results in this section are directly applicable to all such reference point rules.

IV. Behavioral Postulates for RAR

In this section, we discuss the behavioral postulates that characterize the empirical content of RAR.[18] Our postulates can be classified into two groups. Axioms 1-4 consider ordinal properties of the observed choice rule, dealing with issues such as choice with zero probability or regularity violations. Axiom 5, on the other hand, puts a cardinal structure for reference probabilities.

The following terminology will be helpful. We say x *is chosen from* S if $p(x|S) > 0$. If $p(x|S) = 0$, we say x is not chosen from S. Our first axiom states that if x is not chosen against y in a binary comparison, it cannot be chosen from any choice problem that contains y. Hence, this axiom is a significant relaxation of the well-known regularity condition, which requires $p(x|S) \leq p(x|T)$ whenever $x \in T \subset S$.

AXIOM 1: *If $p(x|\{x, y\}) = 0$, then $p(x|S) = 0$ for every S that contains y.*

In our model, if x is not chosen against y, that means $y \succ_x x$. Since reference-dependent preferences that we consider satisfy the SQB property, it must be the case that y is ranked above x regardless of the reference point. Hence, x can never be chosen in the presence of y.

Our model allows for regularity violations. The following three axioms regulate what type of regularity violations can be observed. The next axiom states that if an alternative x is not chosen in S, there must be an alternative y which beats x in a binary comparison, and if this y does not beat all other alternatives in S, then eliminating x must induce a regularity violation for y.

AXIOM 2: *If $p(x|S) = 0$, then there is $y \in S$ such that $p(y|\{x, y\}) = 1$ and either $p(y|S) = 1$ or $p(y|S) > p(y|S \setminus x)$.*

This axiom necessarily holds in our model. If x is not chosen from S, that means another alternative y must be $\succ_x$-best in S, and by SQB, must also be $\succ_y$-best. Since y is better than x under both reference-dependent preferences, it must be

[18] We provide a characterization for L-RAR in the Appendix.

that x is not chosen from $\{x, y\}$. Furthermore, unless y similarly beats every other alternative in S (in which case y must be the only chosen alternative in S), it must be that elimination of x from S decreases the choice probability of y, inducing a regularity violation. To see why, let $A(y|S)$ be the set of reference points in S according to which y is the best alternative in S. By $p(y|\{x, y\}) = 1$, we know that x belongs to that set. With the elimination of x from S, $A(y|S \setminus x)$ becomes smaller than $A(y|S)$. In addition, for any $z \notin A(y|S)$, $\rho(z|S \setminus x) > \rho(z|S)$. Hence, if $A(y|S) \neq S$, this induces $p(y|S) > p(y|S \setminus x)$. If there are no other chosen alternatives (that is, if $A(y|S) = S$), on the other hand, $p(y|S) = p(y|S \setminus x) = 1$ remains the same.

Axiom 2 states that eliminating an unchosen alternative should induce a regularity violation unless only one alternative is chosen from S. The following axiom completes this picture by stating that eliminating a chosen alternative cannot induce a regularity violation. Hence, in our model, regularity violations happen only due to zero probability choices.

AXIOM 3: *If $p(x|S) > 0$, then $p(y|S) \leq p(y|S \setminus x)$ for any $y \in S \setminus x$.*

To see why this axiom holds in our model, note that if x is chosen in S, it must be $\succ_x$-best in S. Hence, if y is $\succ_z$-best for some alternative $z \in S$, then z is distinct from x, and y is also $\succ_z$-best in $S \setminus x$. That is, the set of reference points for which y is the best in S can not get smaller as x is eliminated: $A(y|S) \subseteq A(y|S \setminus x)$. Since $p(y|S) = \sum_{z \in A(y|S)} \rho(z|S)$ and ρ satisfies regularity, this implies $p(y|S) \leq p(y|S \setminus x)$ for every $y \in S \setminus x$.

The next axiom imposes a form of asymmetry on regularity violations. It considers a situation where eliminating z induces a regularity violation on x when y is available. Our axiom then states that, in the presence of x, eliminating of z cannot induce a regularity violation for y.

AXIOM 4: *If $p(x|S) > p(x|S \setminus z)$ and $x, y, z \in T \cap S$, then $p(y|T) \leq p(y|T \setminus z)$.*

To see why Axiom 4 is satisfied by our model, note that z induces a regularity violation on x in a set that contains y only if $x \succ_z y$. Due to the asymmetry of $\succ_z$, there cannot be another case where z induces a regularity violation on y when x is available.

Axiom 4 is related to the single reversal axiom on deterministic choice (Kıbrıs, Masatlioglu and Suleymanov, 2021). Their axiom states that if elimination of x induces a choice reversal in a choice set containing y (i.e., $c(S) \neq c(S \setminus x)$ when $x \neq c(S)$ and $y \in S$), then there cannot be another choice set containing x in which, now, elimination of y induces a choice reversal. The stochastic analog of the single reversal axiom would require that in any choice set, there is at most one alternative elimination which can cause a regularity violation. While this does not hold in our model, Axiom 4 restricts the number of possible regularity violations in another way by stating that eliminating an alternative can cause a regularity violation for at most one alternative.

Unfortunately, Axioms 1-4 are not sufficient, as illustrated in our next example. The choice rule presented in Table 2 satisfies Axioms 1-4, but cannot be represented by RAR. To see this, suppose not. By the previous discussion on revealed preferences, we must have $x \succ_y y \succ_y z$, $x \succ_x y, z$ and $z \succ_z x, y$. Since $p(y|\{y,z\}), p(z|\{y,z\}) \in (0,1)$, $\rho(y|\{y,z\}) = p(y|\{y,z\}) = 0.1$. Similarly, $\rho(x|\{x,z\}) = p(x|\{x,z\}) = 0.5$. By strict regularity, we must have $\rho(x|\{x,y,z\}) < 0.5$ and $\rho(y|\{x,y,z\}) < 0.1$. This is a contradiction since $p(x|\{x,y,z\}) > 0.1+0.5$.

Table 2—A choice rule that satisfies Axioms 1-4 but cannot be represented by RAR.

$p(\cdot, S)$	$\{x,y,z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$
x	0.7	1	0.5	-
y	0	0	-	0.1
z	0.3	-	0.5	0.9

The example in Table 2 illustrates that, for choice data to be representable by RAR, the size of regularity violations it exhibits must be bounded. The following axiom guarantees this is the case. Together with the previous axioms, it characterizes RAR. This axiom is closely related to Motzkin's transposition theorem (Motzkin, 1936), a member of the well-known theorems of the alternatives. It guarantees that the reference point formation rule satisfies strict positivity and strict regularity.

Before stating the axiom, we introduce the following notation. For any $x, y \in S \subseteq X$, let $\lambda_{xy}(S)$ denote a positive constant. Let λ stand for the vector consisting of all constants $\lambda_{xy}(S)$. For notational simplicity, let $\lambda_x(S)$ stand for $\lambda_{xx}(S)$. For any such λ, we define

$$V(\lambda, p) = \sum_{S\in\mathcal{X}} \sum_{x\in S\subseteq X} \lambda_x(S) p(x,S) + \sum_{S\in\mathcal{X}} \sum_{x,y\in S\subseteq X \text{ s.t. } x\neq y} \lambda_{xy}(S)(p(x, S\backslash y) - p(x,S)).$$

For any $x \in S$, let

$$\Gamma_\lambda(x,S) = \lambda_x(S) + \sum_{y\notin S} \lambda_{xy}(S\cup y) - \sum_{z\in S\backslash x} \lambda_{xz}(S).$$

Notice that if p satisfies the regularity condition, then $V(\lambda, p)$ must be positive for any vector $\lambda \geq 0$. On the other hand, since our model allows for regularity violations, $V(\lambda, p)$ may not always be positive for all possible positive λ. However, if λ is chosen in a way that satisfies certain constraints, as made precise in the axiom, then $V(\lambda, p)$ must be positive. This ensures that if we observe a regularity violation, and hence $p(x|S\backslash y) - p(x, S)$ is negative for some $x, y \in S$, the regularity violation is bounded so that $V(\lambda, p)$ is still positive.

AXIOM 5: *For any* $\lambda \geq 0$ *that satisfies* $\Gamma_\lambda(x,S) = \Gamma_\lambda(y,S)$ *for all* $x, y \in S$ *such that either* $p(x|S) = 1$ *or* $p(x,S) > p(x, S \setminus y)$*, we have* $V(\lambda, p) \geq 0$*, with strict inequality if* $\lambda \neq 0$*.*

The necessity of Axiom 5 is shown in the proof of Theorem 1. To see how the axiom rules out the example in Table 2, let $\lambda \geq 0$ be such that $\lambda_{xy}(\{x,y,z\}) = \lambda_{yx}(\{x,y,z\}) = 1$, and every other term in λ is zero. Note that $\Gamma_\lambda(x, \{x,y,z\}) = -1 = \Gamma_\lambda(y, \{x,y,z\})$ and $\Gamma_\lambda(x, \{x,y\}) = 0 = \Gamma_\lambda(y, \{x,y\})$, and hence the if part of the axiom is satisfied. On the other hand,

$$V(\lambda,p) = p(x|\{x,z\}) - p(x|\{x,y,z\}) + p(y|\{y,z\}) - p(y|\{x,y,z\}) = -0.1 < 0,$$

violating the axiom.

Similar to other studies in stochastic choice theory, our proof identifies the RAR representation as a solution to a finite linear system. Earlier studies such as Scott (1964), McFadden and Richter (1991), McFadden (2005) follow a similar strategy. Non-negativity of the Block-Marschak polynomials (BM) and the Axiom of Revealed Stochastic Preference (ARSP) of McFadden and Richter (1991) are both used to characterize the behavioral content of RUM. We think of Axiom 5 as belonging to the same category of axioms as BM and ARSP. We now state the characterization result.

THEOREM 1: *A random choice rule* p *satisfies Axioms 1-5 if and only if it has a RAR representation.*

The proof of the theorem is in the Appendix. In the proof, we first show that Axioms 1-4 guarantee P_x, as defined in Section II, is well-defined. Hence, we can construct reference-dependent preferences $\{\succ_x\}_{x\in X}$ by considering arbitrary completions of $\{P_x\}_{x\in X}$. Lastly, Axiom 5 ensures that there exists ρ satisfying strict positivity and strict regularity such that $(\{\succ_x\}_{x\in X}, \rho)$ represents p.

V. Weak Status Quo Bias

In the previous sections, our preference identification results make use of the SQB property. One may then wonder to what extent our results can be generalized when a weaker version of SQB is assumed. We answer this question in this section. It turns out that our preference identification results continue to hold in that case as well.

Our original SQB property requires that if x is preferred to y under some reference point, then x must also be preferred to y when x is the reference point. A weaker requirement, called Weak Status Quo Bias (WSQB), is that if x is preferred to y when y is the reference point, then x must also be preferred to y when x is the reference point. While under SQB being a reference point unambiguously helps an alternative, under WSQB being a reference point unambiguously helps an alternative in binary comparisons, but may help or hurt it in general.

Weak Status Quo Bias (WSQB). If $x \succ_y y$, then $x \succ_x y$.

Direct confirmations of WSQB can be found in the experimental literature on individual choice (see Knetsch, 1989, Masatlioglu and Uler, 2013). Motivated by these findings, WSQB has commonly been employed by reference-dependent choice models, seeking to capture status quo bias and loss aversion phenomena (see Sugden, 2003, Munro and Sugden, 2003, Sagi, 2006, Apesteguia and Ballester, 2009, Masatlioglu and Nakajima, 2013).[19]

For the sake of brevity, we do not pursue full axiomatic characterization of the model with WSQB in this paper. Nevertheless, we would like to highlight two main properties that distinguish the model with WSQB from the model with SQB. First, under SQB, if x is never chosen in a menu, then x will still never be chosen if we enlarge the menu with new alternatives, that is, $p(x|T) = 0$ implies $p(x|S) = 0$ as long as $T \subseteq S$. (Note that for binary T, this is the statement of Axiom 1.) This statement is no longer true under WSQB. Hence, the model with WSQB allows for choice overload type of phenomenon where previously unchosen alternatives can be chosen with positive probability when new alternatives are added to the menu.

Second, under SQB, removing alternatives that are chosen with positive probability cannot cause a regularity violation (Axiom 3): if $p(x|S) > 0$, then $p(y|S) \leq p(y|S \setminus x)$. In other words, regularity violations can happen only if zero probability alternatives are removed from the menu. On the other hand, under WSQB, we can have regularity violations even when non-zero probability alternatives are removed. The model with WSQB, hence, allows for a much richer set of choice behavior where regularity violations can occur under a variety of circumstances.

To illustrate both points, consider the example in Table 3. Notice that choices from binary menus exhibit a cycle: x is always chosen from $\{x, y\}$, y is always chosen from $\{y, z\}$, and z is always chosen from $\{x, z\}$. In the choice set $\{x, y, z\}$, each alternative is chosen with 1/3 probability. Now, since all alternatives are chosen with positive probabilities in $\{x, y, z\}$, but x is never chosen from $\{x, z\}$, y is never chosen from $\{x, y\}$, and z is never chosen from $\{y, z\}$, this example violates Axiom 1. In addition, removing x, y or z from $\{x, y, z\}$ causes a regularity violation, even though all three alternatives are chosen with positive probabilities in $\{x, y, z\}$, in violation of Axiom 3. Since this example violates both Axiom 1 and Axiom 3, it cannot be explained by RAR under SQB. On the other hand, consider the preferences $z \succ_x x \succ_x y$, $x \succ_y y \succ_y z$, and $y \succ_z z \succ_z x$. These preferences satisfy WSQB, and they can explain the observed choice behavior if

[19] Alternatively, one may choose to impose no assumptions on preferences at all. This may not necessarily be desirable since such a model allows "status quo aversion" type of choice patterns, which are against the existing experimental findings. Furthermore, preference identification results under this most general model are not as sharp as the results under SQB and WSQB. As a simple example, suppose there are only two alternatives and both of them are chosen with a positive probability. Under WSQB, we learn that each alternative is ranked top when it is the reference point. If we do not make any assumption on preferences, this is not necessarily true, and multiple reference-dependent preferences are consistent with the data.

we choose $\rho(a|\{x,y,z\}) = 1/3$ for $a \in \{x,y,z\}$. Reference probabilities in binary menus can be arbitrary as long as strict positivity and regularity are satisfied.

Table 3—A choice rule that is accommodated by RAR under WSQB but not SQB.

$p(\cdot|S)$	$\{x,y,z\}$	$\{x,y\}$	$\{x,z\}$	$\{y,z\}$
x	1/3	1	0	-
y	1/3	0	-	1
z	1/3	-	1	0

The two properties we highlight above can then be thought of as the additional behavioral implications of SQB. If one assumes that observed random choice behavior is driven by the unobserved randomness in reference points, where reference probabilities satisfy strict positivity and regularity properties, then a violation of either Axiom 1 or 3 leads one to conclude that the underlying reference-dependent preferences violate SQB while they may still satisfy WSQB.

Interestingly, it turns out that our sharp identification result is still true under WSQB. We first define the revealed preference under WSQB. Since some of the revelations under SQB no longer hold under WSQB, we modify our revealed preference as follows. For any $x \in X$ and $y \neq z$, we define

yP_xz if one of the following is observed:

(i) $p(y|S) = 1$ for some $S \supseteq \{x,y,z\}$,
(ii) $p(y|S) = 1$ for some $S \supseteq \{x,y\}$ and $p(x|T) = 1$ for some $T \supseteq \{x,z\}$,
(iii) $p(y|\{y,z\}) > 0$ and $x = y$, or
(iv) $p(y|S) > p(y|S \setminus x)$ for some $S \supseteq \{x,y,z\}$.

The first condition states that if y is chosen with probability one in some choice set containing x, y, and z, then y is revealed preferred to z under x. Unlike the revelation under SQB, we now need to require that the choice set also contains x, not just y and z. The second condition states that if y is chosen over x and x is chosen over z, then y must be preferred to z under x.[20] The third condition states that if x is chosen over z with a positive probability in a binary comparison, then x must be preferred to z under x. Unlike the case with SQB, this is no longer true for arbitrary choice sets. For example, in Table 3, x is chosen with positive probability from $\{x,y,z\}$, but x cannot be preferred to z under x, as x is never chosen from $\{x,z\}$. The last condition uses regularity violations to infer preferences and is the same as the revelation under SQB.

One can show that (i) if reference-dependent preferences satisfy WSQB and ref-

[20]Notice that this revelation is indirectly implied by (i). We only have (ii) in our definition of P_x to ensure that P_x as defined is transitive.

erence probabilities satisfy strict positivity and regularity, then P_x is transitive; (ii) analogous to Proposition 1, P_x as defined in this section captures all that can be learned about preferences under WSQB; and (iii) analogous to Proposition 5, if either $x \succ_x y$ or $x \succ_x z$ is not true, then we can reveal the relative ranking of y and z under x (in other words, we can always reveal the relative ranking of any alternatives as long as this ranking matters for observed choice behavior).[21] Hence, while we reveal less about preferences under WSQB than SQB, the revelations are still rich enough for us to learn all the information about preferences that matter for choice behavior. This also shows that revealed preference in RAR and L-RAR are equivalent not just under SQB but also under WSQB.[22]

VI. Bertrand Competition with Differentiated Products

In this section, we discuss an application of our model to Bertrand competition with differentiated products. Consider a simple model of duopoly where two substitute products, x_1 and x_2, are respectively sold by Firm 1 and Firm 2. (We will later extend this model by allowing Firm 1 to sell a "decoy" product additionally.) The firms compete by simultaneously choosing their selling prices, p_1 and p_2. For simplicity, assume that both firms have zero marginal costs. Also, assume that the two products are not perfect substitutes; that is, a consumer can value the two differently. The consumers' valuations, denoted by (v_1, v_2), are uniformly and independently distributed on $[0,1]^2$. Each consumer either buys one unit of product from one of the firms or leaves the market. The total volume of consumers is normalized to one.

All consumer choices are consistent with RAR. Let us first introduce reference-dependent preferences. If x_i is the reference point of a consumer with original valuation v_i, she attaches x_i an additional value of $a_i \in [0,1]$. In this case, the consumer compares her net utility from x_i (i.e., $v_i + a_i - p_i$) with that of x_j (i.e., $v_j - p_j$) and chooses the product that provides higher positive net utility. If both net utilities are negative, the consumer leaves the market without purchasing either product (i.e., chooses her outside option). Since a_i creates an affinity towards the reference point, we call it the *loyalty parameter of* x_i. In line with L-RAR, further, assume that each product i has a *salience level* $s_i > 0$ and becomes the reference point with probability $s_i/(s_i + s_j)$. Note that if $a_1 = a_2 = 0$, consumers have standard reference-free preferences and make a consumption decision by comparing $v_1 - p_1$ (buying from Firm 1), $v_2 - p_2$ (buying from Firm 2), and 0 (outside option). In this case, our model boils down to a standard Bertrand competition with differentiated products.

[21] We omit these results as well as the characterization for RAR under WSQB for the sake of brevity.

[22] Interestingly, this equivalence breaks once we impose no assumptions on preferences. That is, it is possible to provide an example where one can reveal strictly more information about preferences in L-RAR than RAR if one imposes no assumptions on preferences.This highlights that the equivalence of revealed preferences in L-RAR and RAR is not obvious ex ante. It is the (weak) status quo bias assumption in our model that delivers this equivalence.

For each consumer on $[0,1]^2$, her consumption decision as a function of p_1 and p_2 is demonstrated in Figure 1. On the left, we present the standard case where $a_1 = a_2 = 0$. In this case, there is no probabilistic choice, and consumers are divided into three groups: (i) consumes x_1, (ii) consumes x_2, or (iii) takes the outside option. The demand for Firm i at given prices (p_1, p_2) is

$$D_i(p_1,p_2) = \int_{p_i}^{1} \int_{0}^{\min\{v_i - p_i + p_j, 1\}} dv_j dv_i.$$

The profit of each firm is then $\Pi_i(p_1,p_2) = p_i D_i(p_1,p_2)$. Under reference-free preferences ($a_1 = a_2 = 0$), the price setting game has a unique symmetric equilibrium where both firms set their price equal to $p^* = \sqrt{2} - 1 > 0$ and receive profit $(\sqrt{2}-1)^2$.

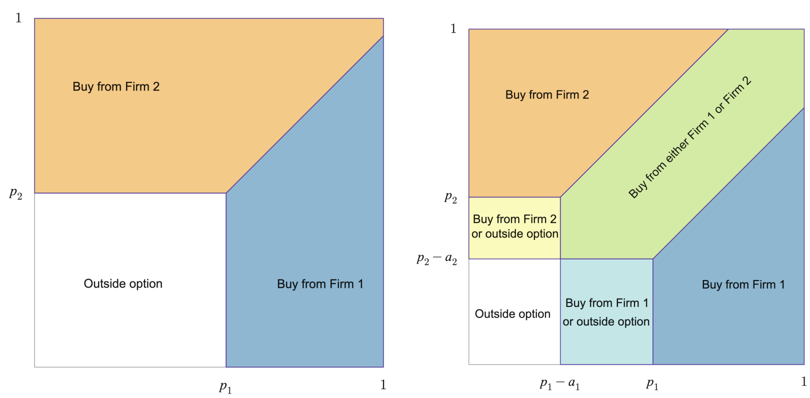

FIGURE 1. CONSUMERS' CHOICES AS A FUNCTION OF THEIR VALUATIONS. THE STANDARD CASE: $a_1 = a_2 = 0$ (LEFT PANEL) AND L-RAR: $a_1, a_2 > 0$ (RIGHT PANEL).

The right panel in Figure 1 represents consumer decisions under RAR with reference-dependent preferences ($a_1, a_2 > 0$). It shows that when $a_1, a_2 > 0$, there are now three additional groups of consumers. Each group randomizes between two of the three possibilities, purchasing from Firm 1 or 2 or taking the outside option. For each such consumer, consumption decision depends on the random reference point. The aggregate demand for x_i can then be written as

$$D_i^{RAR}(p_1,p_2) = \frac{s_i}{s_i + s_j} D_i(p_i - a_i, p_j) + \frac{s_j}{s_i + s_j} D_i(p_i, p_j - a_j).$$

The first part of this expression captures the demand when x_i is the reference point, which happens with probability $s_i/(s_i + s_j)$. Here, the valuation of the marginal individual purchasing from Firm i is $p_i - a_i$ instead of p_i due to the additional value the consumers get from their reference point. The second term gives the demand for Firm i when the other product is the reference point, which happens with probability $s_j/(s_i+s_j)$. In this case, Firm j has more market power due to affinity towards commodity j ($a_j > 0$). Hence, this case can be written as if Firm j is charging $p_j - a_j$ while Firm i is charging p_i.

Aggregating individual decisions over consumers, we can obtain the market demand and the profit function of each firm. For simplicity, we assume $a_1 = a_2 = a$ and $s_1 = s_2 = s$. In this case, there exists a symmetric equilibrium of the price setting game where both firms set their price equal to $p^* = a - 1 + \sqrt{a^2 - 2a + 2}$. Furthermore, Firm i's equilibrium profits can be expressed as $\Pi_i(p^*, p^*) = (2 - a)(a - 1 + \sqrt{a^2 - 2a + 2})^2/2$. Note that both equilibrium prices and profits are increasing in the common loyalty parameter a. This is because an increase in a increases consumers' net valuations of their reference points, which implies that at any fixed price level, there are now more consumers who purchase from one of the firms rather than choose their outside option. This increased loyalty towards the reference point on the consumers' side allows firms to charge higher prices and make higher profits. A second point to note is that a simultaneous increase in the common salience parameter s has no effect on equilibrium prices and profits. This is simply because if both firms increase their salience level equally, there is no change in the resulting reference probabilities. Given that advertising is costly, a game of choosing salience levels through advertising resembles a prisoners' dilemma game where firms will choose sub-optimally high advertising levels in equilibrium.

A firm in our model can utilize different tools at its disposal to influence equilibrium prices and profits. It can either increase the salience level s_i of its product via advertising or increase its loyalty parameter a_i via strategies that improve brand loyalty. The following proposition provides an answer to how each of these strategies affects equilibrium prices and profits, taking the symmetric equilibrium as the starting point of the analysis.

PROPOSITION 6: *Given $a_1 = a_2 = a$ and $s_1 = s_2 = s$, the resulting symmetric equilibrium has the following comparative statics properties:*

1) *An increase in the salience parameter of commodity i, s_i, increases the equilibrium price of commodity i and decreases the equilibrium price of commodity j in the same amount. Additionally, it increases the equilibrium profits of Firm i and decreases the equilibrium profits of Firm j in the same amount.*

2) *An increase in the loyalty parameter of commodity i, a_i, increases the equilibrium prices of both commodities. But the increase in the price of commodity i is higher than that of commodity j. Additionally, it increases the equilibrium profits of Firm i and decreases the equilibrium profits of Firm j. Again, the increase in the equilibrium profits of Firm i is higher in absolute value than the decrease in Firm j's profits.*

3) An increase in the common loyalty parameter, a, increases the effect of commodity i's salience, s_i, on firms' equilibrium prices. On the other hand, an increase in the common salience parameter, s, does not change the effect of commodity i's loyalty parameter, a_i, on firms' equilibrium prices.

The first item in Proposition 6 considers a unilateral increase in the salience parameter of commodity i. As a result of this, commodity i becomes the reference point with a higher probability. This increases the market power of Firm i compared to Firm j. As a result, Firm i can now charge higher prices in equilibrium. To compete, Firm j responds by decreasing its equilibrium selling price. Hence, in asymmetric cases, we expect firms with more salient products to charge higher equilibrium prices. An increase in the salience of commodity i also increases the equilibrium profits of Firm i while decreasing the competitor's equilibrium profits.

The second item in Proposition 6 considers an increase in brand loyalty towards commodity i. As a result, a higher percentage of consumers whose reference point is x_i end up purchasing commodity i. On the other hand, consumers with reference point x_j do not alter their behavior. It is not surprising that Firm i can now charge higher prices as a result. It is more interesting to note that this increase spills over to Firm j as well, and Firm j also increases its equilibrium price, albeit to a lesser extent. Additionally, it is useful to note that the positive effect of a_i on p_j can only be guaranteed to start from the symmetric equilibrium. In other words, it is possible to find cases where $s_1 \neq s_2$ and an increase in brand loyalty for one firm leads to a decrease in the competitor's equilibrium price.[23]

The third item in Proposition 6 provides additional observations based on the analysis of the cross-partial derivatives. First, in environments where consumers have high brand loyalty for both products (high a), an increase in the salience of one of the products has a more significant effect on equilibrium prices. This is because when a is high, a firm's return from being the reference point of a consumer is also higher. This magnifies the effect of the salience parameters s_i and s_j on equilibrium prices. Second, in contrast to the first observation, environments where the consumers have high salience for both products (high s) are equivalent to those with a low s. This is because a change in the common salience s does not affect reference probabilities. So in real terms, it has no effect on the duopoly game, particularly on how much individual loyalty parameters affect equilibrium prices. The interaction between individual salience and loyalty parameters in determining equilibrium prices and profits is more involved and depends on the initial values of the loyalty and salience parameters in question.

A. *Competition via a Decoy Product*

Choice architecture, a term coined by Thaler and Sunstein (2008), reflects that there are many ways to present a choice problem to a decision maker and choices

[23] For example, if $a_1 = a_2 = 0.1$, $s_1 = 0.34$, $s_2 = 0.07$, the equilibrium prices are $p_1 = 0.46$ and $p_2 = 0.42$, and we have $\frac{\partial p_2}{\partial a_1} < 0$.

often depend upon this presentation. A rapidly growing literature shows that cleverly designed choice architecture can be used by planners to steer decision makers towards making better choices, or by firms towards more profitable ones (e.g., see Sunstein, 2017). In terms of choice architecture, RAR suggests that the presentation of additional products, seemingly irrelevant to the current consumption choice of a consumer, might yet affect her choice behavior in a desired manner. Casual observations in environments of repeated choice suggest that a consumer's choice behavior might be affected by the sheer existence of a product, even though she will never consume it. For example, imagine someone who goes to the same restaurant for lunch every day. Imagine that the restaurant serves pork chops on some days and not on others. Suppose the consumer does not eat pork and, therefore, never orders pork chops. However, her choice of food (say, between pasta and veal) can be affected if, for example, the smell of pork chops increases her appetite for meat. Relatedly, the literature on deterministic choice with endogenous reference dependence contains a variety of models that capture common behavioral patterns, such as the attraction or compromise effects (e.g., Kőszegi and Rabin, 2006, Ok, Ortoleva and Riella, 2015, Kıbrıs, Masatlioglu and Suleymanov, 2021). However, this literature is silent about stochastic choices with endogenous reference dependence. Our model fills this gap.[24]

In line with this motivation, we revisit the duopoly example in the previous section and discuss the implications of Firm 1 introducing a *decoy product* which, even though it does not create any demand for itself, increases consumers' loyalty for x_1 against x_2 (hence, potentially changing the relative ranking of the two products). We then discuss conditions under which such choice architecture will be profitable for Firm 1.

Let d denote the decoy product that is never chosen and s_d its salience level. When the decoy is the reference point, this increases the relative attractiveness of x_1 against x_2. The utility boost that the decoy provides for x_1 is denoted by $a_d \in [0,1]$. We interpret a_d as a second loyalty parameter for x_1, effective when the decoy serves as the reference point. We assume that the loyalty the decoy provides for x_1 is not stronger than that of x_1 towards itself, that is, $a_d \leq a_1$. Consumers in this extended model behave as before when their reference points are x_1 or x_2. However, the reference probabilities of these alternatives are now lowered to $s_i/(s_1+s_2+s_d)$ where $i \in \{1,2\}$. When a consumer's reference point is d, her payoff from x_1 becomes $v_1 + a_d - p_1$, while her payoff from x_2 and the outside option are not affected. Aggregating individual choices, we obtain the demand for x_1 as

$$D_1^{RAR_d}(p_1,p_2) = \frac{s_1 D_1(p_1-a_1,p_2) + s_d D_1(p_1-a_d,p_2) + s_2 D_1(p_1,p_2-a_2)}{s_1+s_2+s_d}.$$

[24]While there are other stochastic choice models capturing regularity violations, their explanations rely on different mechanisms, such as limited attention (e.g., see Cattaneo et al., 2020). For a review of empirical evidence on regularity violations and alternative theories explaining them, see Rieskamp, Busemeyer and Mellers (2006).

Figure 2 demonstrates how the demand for x_1 changes from the two commodities (left panel) to the three commodities (right panel) case. First, consumers who, in the two commodities, consume x_1 with probability 1 (that is, independent of the reference point) still do so after the introduction of the decoy. This is because their valuation of x_1 is sufficiently higher than that of x_2 as $v_1 - p_1 \geq v_2 + a_2 - p_2$. The second group of consumers on the left panel are those who consume x_1 only when it is their reference point. With the addition of the decoy, this group of consumers is divided into two on the right panel. One subgroup now consumes x_1 under two reference points: both x_1 and d. For this subgroup, the probability that they will purchase x_1 increases from $s_1/(s_1 + s_2)$ to $(s_1 + s_d)/(s_1 + s_2 + s_d)$, affecting Firm 1's profits positively. Consumers in the other subgroup continue to consume x_1 only when it is the reference point itself. However, due to the diminished probability of x_1 being the reference point, the probability that they will purchase x_1 decreases from $s_1/(s_1 + s_2)$ to $s_1/(s_1 + s_2 + s_d)$, affecting Firm 1's profits negatively. Overall, due to the above trade-off, it is not always beneficial for Firm 1 to introduce the decoy. If a_d (that is, the loyalty the decoy provides for x_1) is sufficiently high, then Firm 1 benefits from introducing the decoy. Otherwise, the addition of the decoy might hurt Firm 1's profits.

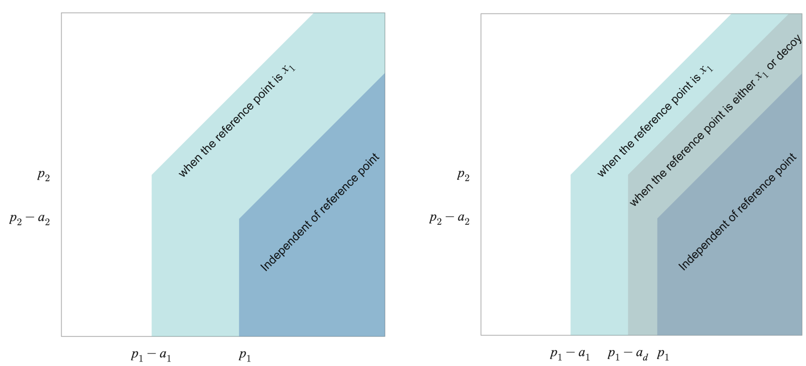

FIGURE 2. CONSUMERS WHO CHOOSE FIRM 1'S PRODUCT x_1 AS A FUNCTION OF THEIR VALUATIONS AND REFERENCE POINTS. THE TWO COMMODITY CASE (LEFT PANEL) AND THE THREE COMMODITIES (RIGHT PANEL).

Since consumers do not purchase the decoy, the two firms compete on the prices of the first two products as in the previous model. However, due to the decoy, it is no longer possible to focus on a symmetric equilibrium. Furthermore, the analysis of a closed-form asymmetric solution is too involved and outside the scope of this paper. Instead, we provide some numerical examples showing how the introduction of the decoy affects the firms' profits. For some parameter values, the introduction of the decoy increases Firm 1's profits while decreasing Firm 2's profits.[25] These parameter combinations correspond to cases where the loyalty the decoy provides for x_1 is sufficiently high with respect to the other parameters. In contrast, in cases where the loyalty the decoy provides is comparatively low, profit levels of both Firm 1 and Firm 2 fall in response to the decoy's introduction.[26] This is due to the previously mentioned trade-off where the decoy increases x_1's consumption probability for one subgroup while decreasing it for another. The size of a_d determines how big the first group is in comparison to the second and hence how profitable the introduction of the decoy is. Overall, since it may or may not be profitable to introduce a decoy, this might explain both the presence and absence of decoys in different markets.

VII. Conclusion

In this paper, we provide two simple models of random reference-dependent choice. The first, L-RAR, is a parametric model where the attention rule takes the logit form. The second model, RAR, is a non-parametric generalization of L-RAR. We demonstrate when and how the reference-dependent preferences and the reference weights in L-RAR can be identified. We show that the parameters of L-RAR can be almost fully identified from choice data. Surprisingly, we then show that the revealed preferences in RAR are exactly the same as in L-RAR. Hence, the strong identification of preferences in L-RAR is not due to the parametric structure of the reference formation process. We also investigate our revealed preference results under alternative assumptions on reference-dependent preferences and show that a much weaker status quo bias property ensures that preferences can be identified whenever they matter for choice. Lastly, we offer behavioral postulates that characterize the models and provide an application of our model to a duopoly price-setting game.

We mainly interpret RAR as a model of individual choice, focusing on repeated decisions such as scanner data from supermarkets or online data from digital platforms. Alternatively, we can also interpret RAR as a model of aggregate choice from a population where different agents might have different reference points, but any two agents with the same reference point share the same preference. In this interpretation, stochastic choice data arises not from the heterogeneity of preferences but from the heterogeneity of reference points in the population. Unfortunately, a model of aggregate choice where one allows for both heterogeneity in reference points and preferences imposes few restrictions on observed aggregate choices. Nevertheless, one can still perform revealed preference analysis in this general setup. For example, suppose we observe that $p^A(y|S) > p^A(y|S \setminus x)$

[25] For example, if $a_1 = a_2 = 0.2$ and $s_1 = s_2 = 0.1$, the addition of a decoy with $a_d = 0.1$ and $s_d = 0.1$ increases Firm 1's profits by more than 1% while decreasing Firm 2's profits by almost 8%. It is also interesting to note that the decoy decreases both equilibrium prices marginally for Firm 1 at 0.6% and more significantly for Firm 2 at 4.6%.

[26] If the loyalty parameter of the decoy is 0.01 instead of 0.1 in the previous footnote, its addition decreases equilibrium profits of both firms, by 5.6% for Firm 1 and by a higher 6.5% for Firm 2. Additionally, it decreases both equilibrium prices, 4.5% for Firm 1 and 5% for Firm 2.

where p^A represents the aggregate choice data. We can then conclude that there are at least some people in the population who prefer y over any other alternative in S when their reference point is x. If this were not the case, then for every agent i in the population, we would have $p_i(y|S) \leq p_i(y|S \setminus x)$, and hence $p^A(y|S) \leq p^A(y|S \setminus x)$. The complete analysis of this general model is left for future research.

One feature of our model that survives under various assumptions is that if all alternatives are chosen with positive probabilities, then the observed choices directly inherit any assumptions we make on reference probabilities. Hence, the model in its current form does not allow for non-standard reference-dependent choice behavior when all alternatives are always chosen with positive probability. One possible way to address this concern would be to make an assumption on reference-dependent preferences that is even weaker than the weak status quo bias property. Interestingly, an alternative way to address this concern would be by assuming an even stronger form of reference dependence. To illustrate, suppose the reference point affects the decision maker's choices via two channels. First, with a small but positive probability, the agent always sticks with her reference point, exhibiting an extreme status quo bias. This captures the idea that agents might choose to stick with their reference points to avoid making any comparisons, thus reducing cognitive or time costs. With the remaining probability, the agent makes her choices by maximizing her reference-dependent preferences. This captures the idea that even when the reference point is not necessarily chosen, it may influence the decision maker's preferences among other alternatives. Formally, consider the model

$$p(x|S) = \sum_{y \in S} \left(\frac{s_y}{\sum_{z \in S} s_z} \right) \big(\varepsilon \mathbb{1}(x = y) + (1 - \varepsilon) \mathbb{1}(x \text{ is } \succ_y\text{-best in } S) \big)$$

where $\varepsilon > 0$. Here, ε reflects the probability that the decision maker exhibits extreme status quo bias and sticks with her reference point. As $\varepsilon \to 0$, this model converges to L-RAR. As long as $\varepsilon > 0$, we do not observe zero probabilities, yet the model allows for interesting forms of reference-dependent choice behavior, such as the violations of the regularity property.

Throughout this paper, we assume that the analyst has access to stochastic choice data but does not observe the agent's reference point. The analyst attributes random choice behavior to randomness in reference points. Alternatively, suppose the analyst now observes not only the agent's stochastic choice data but also her exogenously given reference point. That is, the analyst observes a dataset $p_r(\cdot|S)$ where $r \in S \subseteq X$. To address this possibility, we can consider an extension of L-RAR that allows for an exogenous reference point r to probabilistically affect the reference formation process as follows:

$$p_r(x|S) = \sum_{y \in S} \left(\frac{s_y^r}{\sum_{z \in S} s_z^r} \right) \mathbb{1}(x \text{ is } \succ_y\text{-best in } S)$$

where $s_r^r = s_r + b_r$ and $s_z^r = s_z$ for all $z \neq r$. In this model, the exogenously given reference point r receives a boost $b_r > 0$ to its original reference weight, and hence it becomes the reference point with a higher probability. As the boost, $b_r \to \infty$, this model converges to a deterministic choice model where the DM maximizes $\succ_r$ when her reference point is r. In this case, the exogenous reference point is the same as the endogenous reference point, which drives the agent's choices. On the other hand, as $b_r \to 0$, the exogenous reference point matters less and less for the agent's choices. Hence, the magnitude of b_r captures the relative importance of exogenous and endogenous reference points. Further analysis of such extensions is left for future research.

REFERENCES

Aguiar, Victor H. 2017. "Random Categorization and Bounded Rationality." *Economics Letters*, 159: 46–52.

Ahumada, Alonso, and Levent Ulku. 2018. "Luce Rule with Limited Consideration." *Mathematical Social Sciences*, 93: 52–56.

Apesteguia, Jose, and Miguel A Ballester. 2009. "A Theory of Reference-Dependent Behavior." *Economic Theory*, 40(3): 427–455.

Apesteguia, Jose, Miguel A Ballester, and Jay Lu. 2017. "Single-Crossing Random Utility Models." *Econometrica*, 85(2): 661–674.

Baillon, Aurélien, Han Bleichrodt, and Vitalie Spinu. 2020. "Searching for the Reference Point." *Management Science*, 66(1): 93–112.

Baldassi, Carlo, Simone Cerreia-Vioglio, Fabio Maccheroni, Massimo Marinacci, and Marco Pirazzini. 2020. "A Behavioral Characterization of the Drift Diffusion Model and Its Multialternative Extension for Choice Under Time Pressure." *Management Science*, 66(11): 5075–5093.

Baucells, Manel, Martin Weber, and Frank Welfens. 2011. "Reference-Point Formation and Updating." *Management Science*, 57(3): 506–519.

Ben-Akiva, Moshe E. 1973. "Structure of Passenger Travel Demand Models." *Ph.D. Thesis, Massachusetts Institute of Technology.*

Berkowitsch, Nicolas AJ, Benjamin Scheibehenne, and Jörg Rieskamp. 2014. "Rigorously Testing Multialternative Decision Field Theory Against

Random Utility Models." *Journal of Experimental Psychology: General*, 143(3): 1331.

Bhatia, Sudeep, and Russell Golman. 2019. "Attention and Reference Dependence." *Decision*, 6(2): 145–170.

Bodner, Ronit, and Drazen Prelec. 1994. "The Centroid Model of Context Dependent Choice." *Unpublished Manuscript, MIT.*

Bordalo, Pedro, Nicola Gennaioli, and Andrei Shleifer. 2012. "Salience Theory of Choice Under Risk." *Quarterly Journal of Economics*, 127(3): 1243–1285.

Bordalo, Pedro, Nicola Gennaioli, and Andrei Shleifer. 2013. "Salience and Consumer Choice." *Journal of Political Economy*, 121(5): 803–843.

Brady, Richard L., and John Rehbeck. 2016. "Menu-Dependent Stochastic Feasibility." *Econometrica*, 84(3): 1203–1223.

Breugelmans, Els, Katia Campo, and Els Gijsbrechts. 2007. "Shelf Sequence and Proximity Effects on Online Grocery Choices." *Marketing Letters*, 18(1): 117–133.

Cattaneo, Matias D., Xinwei Ma, Yusufcan Masatlioglu, and Elchin Suleymanov. 2020. "A Random Attention Model." *Journal of Political Economy*, 128(7): 2796–2836.

Cerreia-Vioglio, Simone, David Dillenberger, Pietro Ortoleva, and Gil Riella. 2019. "Deliberately Stochastic." *American Economic Review*, 109(7): 2425–45.

Chambers, Chris, Yusufcan Masatlioglu, Paulo Natenzon, and Collin Raymond. 2021. "Weighted Linear Stochastic Choice." *Working paper.*

Dean, Mark, Özgür Kıbrıs, and Yusufcan Masatlioglu. 2017. "Limited Attention and Status Quo Bias." *Journal of Economic Theory*, 169: 93–127.

Dutta, Rohan. 2020. "Gradual Pairwise Comparison and Stochastic Choice." *Theoretical Economics*, 15(4): 1335–1364.

Echenique, Federico, and Kota Saito. 2019. "General Luce Model." *Economic Theory*, 68: 811–826.

Filiz-Ozbay, Emel, and Yusufcan Masatlioglu. 2023. "Progressive Random Choice." *Journal of Political Economy*, 131(3): 716–750.

Freeman, David J. 2017. "Preferred Personal Equilibrium and Simple Choices." *Journal of Economic Behavior & Organization*, 143: 165–172.

Fudenberg, Drew, Ryota Iijima, and Tomasz Strzalecki. 2015. "Stochastic Choice and Revealed Perturbed Utility." *Econometrica*, 83(6): 2371–2409.

Gul, Faruk, Paulo Natenzon, and Wolfgang Pesendorfer. 2014. "Random Choice as Behavioral Optimization." *Econometrica*, 82(5): 1873–1912.

Guney, Begum, and Michael Richter. 2018. "Costly Switching from a Status Quo." *Journal of Economic Behavior & Organization*, 156: 55–70.

Horan, Sean. 2021. "Stochastic Semi-Orders." *Journal of Economic Theory*, 192: 105171.

Huber, Joel, John W. Payne, and Christopher Puto. 1982. "Adding Asymmetrically Dominated Alternatives: Violations of Regularity and the Similarity Hypothesis." *Journal of Consumer Research*, 9(1): 90–98.

Kahneman, Daniel. 1992. "Reference Points, Anchors, Norms, and Mixed Feelings." *Organizational Behavior and Human Decision Processes*, 51(2): 296–312.

Kahneman, Daniel. 2011. *Thinking, Fast and Slow.* New York:Farrar, Straus and Giroux.

Kahneman, Daniel, and Amos Tversky. 1979. "Prospect Theory: An Analysis of Decision under Risk." *Econometrica*, 47(2): 263–292.

Kıbrıs, Özgür, Yusufcan Masatlioglu, and Elchin Suleymanov. 2021. "A Theory of Reference Point Formation." *Economic Theory*, 1–30.

Kivetz, Ran, Oded Netzer, and V Srinivasan. 2004. "Alternative Models for Capturing the Compromise Effect." *Journal of Marketing Research*, 41(3): 237–257.

Knetsch, Jack L. 1989. "The Endowment Effect and Evidence of Nonreversible Indifference Curves." *The American Economic Review*, 79(5): 1277–1284.

Koop, Gregory J., and Joseph G. Johnson. 2012. "The Use of Multiple Reference Points in Risky Decision Making." *Journal of Behavioral Decision Making*, 25(1): 49–62.

Kőszegi, Botond, and Matthew Rabin. 2006. "A Model of Reference-Dependent Preferences." *Quarterly Journal of Economics*, 121(4): 1133–1165.

Kőszegi, Botond, and Matthew Rabin. 2007. "Reference-Dependent Risk Attitudes." *American Economic Review*, 97(4): 1047–1073.

Kovach, Matthew, and Elchin Suleymanov. 2021. "Reference Dependence and Random Attention." *arXiv 2106.13350*.

Kovach, Matthew, and Gerelt Tserenjigmid. 2022. "Behavioral Foundations of Nested Stochastic Choice and Nested Logit." *Journal of Political Economy*, 130(9): 2411–2461.

Liang, Ting-Peng, and Hung-Jen Lai. 2002. "Effect of Store Design on Consumer Purchases: An Empirical Study of On-line Bookstores." *Information & Management*, 39(6): 431–444.

Lim, X. Z. 2020. "Ordered Reference Dependent Choice." *Working Paper.*

Luce, R. Duncan. 1959. *Individual Choice Behavior.* John Wiley and Sons, Inc.

Machina, Mark J. 1985. "Stochastic Choice Functions Generated from Deterministic Preferences over Lotteries." *The Economic Journal*, 95(379): 575–594.

Maltz, Amnon. 2020. "Exogenous Endowment - Endogenous Reference Point." *The Economic Journal*, 130(625): 160–182.

Manzini, Paola, and Marco Mariotti. 2014. "Stochastic Choice and Consideration Sets." *Econometrica*, 82(3): 1153–1176.

March, James G, and Zur Shapira. 1992. "Variable Risk Preferences and the Focus of Attention." *Psychological Review*, 99(1): 172.

Marschak, Jacob. 1960. "Binary Choice Constraints on Random Utility Indicators." Stanford University Press.

Masatlioglu, Yusufcan, and Daisuke Nakajima. 2013. "Choice by Iterative Search." *Theoretical Economics*, 8(3): 701–728.

Masatlioglu, Yusufcan, and Efe A Ok. 2005. "Rational Choice with Status Quo Bias." *Journal of Economic Theory*, 121(1): 1–29.

Masatlioglu, Yusufcan, and Efe A Ok. 2014. "A Canonical Model of Choice with Initial Endowments." *Review of Economic Studies*, 81(2): 851–883.

Masatlioglu, Yusufcan, and Neslihan Uler. 2013. "Understanding the Reference Effect." *Games and Economic Behavior*, 82: 403–423.

McFadden, Daniel. 1978. "Modeling the Choice of Residential Location." In *Spatial Interaction Theory and Planning Models.* , ed. A. Karlqvist, J. W. Weibull, L. Lundqvist and F. Snickars, 75–96. North Holland.

McFadden, Daniel, and Marcel K Richter. 1991. "Stochastic Rationality and Revealed Stochastic Preference." In *Preferences, Uncertainty, and Rationality.* , ed. J. Chipman, D. McFadden and K. Richhter, 161–186. Westview Press.

McFadden, Daniel L. 2005. "Revealed Stochastic Preference: A Synthesis." *Economic Theory*, 26(2): 245–264.

Milosavljevic, Milica, Vidhya Navalpakkam, Christof Koch, and Antonio Rangel. 2012. "Relative Visual Saliency Differences Induce Sizable Bias in Consumer Choice." *Journal of Consumer Psychology*, 22(1): 67–74.

Mohr, Peter NC, Hauke R Heekeren, and Jörg Rieskamp. 2017. "Attraction Effect in Risky Choice Can Be Explained by Subjective Distance Between Choice Alternatives." *Scientific Reports*, 7(1): 1–10.

Motzkin, Theodore Samuel. 1936. *Beiträge zur Theorie der linearen Ungleichungen.* Azriel Press.

Munro, Alistair, and Robert Sugden. 2003. "On the Theory of Reference-Dependent Preferences." *Journal of Economic Behavior & Organization*, 50(4): 407–428.

Natenzon, Paulo. 2019. "Random Choice and Learning." *Journal of Political Economy*, 127(1): 419–457.

Ok, Efe A, Pietro Ortoleva, and Gil Riella. 2015. "Revealed (P)reference Theory." *American Economic Review*, 105(1): 299–321.

Orhun, A. Yeşim. 2009. "Optimal Product Line Design When Consumers Exhibit Choice Set-Dependent Preferences." *Marketing Science*, 28(5): 868–886.

Rieskamp, Jörg, Jerome R. Busemeyer, and Barbara A. Mellers. 2006. "Extending the Bounds of Rationality: Evidence and Theories of Preferential Choice." *Journal of Economic Literature*, 44(3): 631–661.

Rubinstein, Ariel, and Yuval Salant. 2006. "Two Comments on the Principle of Revealed Preference." *Unpublished paper.*

Sagi, Jacob S. 2006. "Anchored Preference Relations." *Journal of Economic Theory*, 130(1): 283–295.

Salant, Yuval, and Ariel Rubinstein. 2008. "(A, f): Choice with Frames." *Review of Economic Studies*, 75(4): 1287–1296.

Scott, Dana. 1964. "Measurement Structures and Linear Onequalities." *Journal of Mathematical Psychology*, 1(2): 233–247.

Sugden, Robert. 2003. "Reference-Dependent Subjective Expected Utility." *Journal of Economic Theory*, 111(2): 172–191.

Sunstein, Cass R. 2017. "Nudges That Fail." *Behavioural Public Policy*, 1(1): 4–25.

Suppes, Patrick, and R. Duncan Luce. 1965. "Preference, Utility, and Subjective Probability." *Handbook of Mathematical Psychology*, 3: 249–410.

Thaler, Richard H, and Cass R. Sunstein. 2008. *Nudge: Improving Decisions About Health, Wealth, and Happiness.* Yale University Press.

Thurstone, Louis L. 1927. "A Law of Comparative Judgement." *Pschological Review*, 34: 273–286.

Tserenjigmid, Gerelt. 2019. "Choosing with the Worst in Mind: A Reference-Dependent Model." *Journal of Economic Behavior & Organization*, 157: 631–652.

Tversky, Amos, and Daniel Kahneman. 1991. "Loss Aversion in Riskless Choice: A Reference-Dependent Model." *Quarterly Journal of Economics*, 106(4): 1039–1061.

Appendix

A1. Proof of Theorem 1

The necessity of Axioms 1-4 is discussed in the main text. We next show the necessity of Axiom 5.

CLAIM 1: *If p has a RAR representation $(\{\succ_x\}_{x\in X}, \rho)$, then it satisfies Axiom 5.*

PROOF:

Let $X = \{x_1, \dots, x_N\}$ so that $|X| = N$. We first introduce a binary relation $\gg$ on $\mathcal{X}$. For any non-empty $S \subseteq X$, let $k(S)$ denote the smallest integer in $\{1, \dots, N\}$ such that $x_{k(S)} \in S$. Let $S \gg S'$ if $|S| < |S'|$ or $|S| = |S'|$ and $k(S \setminus S') > k(S' \setminus S)$. Note that $\gg$ is irreflexive, and since $S \setminus S' \neq \emptyset$, $S' \setminus S \neq \emptyset$, and $(S \setminus S') \cap (S' \setminus S) = \emptyset$ for any $S \neq S'$ with $|S| = |S'|$, it is weakly connected. In addition, suppose $|S| = |S'| = |S''|$, $k(S \setminus S') > k(S' \setminus S)$, and $k(S' \setminus S'') > k(S'' \setminus S')$. Then, we must have that either $k(S' \setminus S) > k(S'' \setminus S')$ or $k(S'' \setminus S') > k(S' \setminus S)$ (equality is not possible since $(S' \setminus S) \cap (S'' \setminus S') = \emptyset$). Since S and S' have the same elements from $\{x_1, \dots, x_{k(S' \setminus S)-1}\}$ with $x_{k(S' \setminus S)} \in S' \setminus S$, and S' and S'' have the same elements from $\{x_1, \dots, x_{k(S'' \setminus S')-1}\}$ with $x_{k(S'' \setminus S')} \in S'' \setminus S'$, in both cases above $k(S \setminus S'') > k(S'' \setminus S)$ follows. Hence, $\gg$ is transitive. Let $m : \mathcal{X} \to \{0, 1, 2, \dots, 2^N - 2\}$ be a numeric representation of $\gg$ that satisfies $m(S) > m(S')$ if and only if $S \gg S'$. Hence, $m(X) = 0$, $m(\{x_N\}) = 2^N - 2$, and etc.

We next define the vector of choice probabilities. For any $S \subseteq X$, let $p(\cdot|S)$ denote the vector of choice probabilities for the choice set S, where the probability corresponding to alternative x_i is placed higher in the vector (alternatively, has a lower row number) than alternative x_j if $i < j$. Note that if $x_i \notin S$, then $p(x_i|S) = 0$. Each $p(\cdot|S)$ is an $N \times 1$ vector. Let $\mathbf{p}$ denote the vector of choice probabilities $[p(\cdot, S)]_{S\subseteq X}$ stacked as follows: if $m(S) < m(S')$, then the $p(\cdot|S)$ is placed higher in the vector $\mathbf{p}$ than $p(\cdot|S')$. Hence, the rows between $m(S)N+1$ and $(m(S)+1)N$ in the vector $\mathbf{p}$ correspond to $p(\cdot|S)$. Note that $\mathbf{p}$ is a $(2^N-1)N\times 1$ vector.

We next define the vector of reference probabilities. Let $\rho(\cdot|S)$ denote the $N\times 1$ vector of reference probabilities, where $\rho(x_i|S)=0$ if $x_i\notin S$. We stack the reference probabilities $\rho(\cdot|\cdot)$ in the vector $\boldsymbol{\rho}$ in the same order as $\mathbf{p}$.

Next, we encode the preferences in a matrix as follows. For any $S\subseteq X$, let $A(S)$ denote an $N\times N$ matrix of zeros and ones such that

$$[A(S)]_{ij}=1 \text{ if and only if } x_i=\arg\max(\succ_{x_j},S) \text{ and } x_j\in S.$$

Note that if $p(\cdot|\cdot)$ has a RAR representation, then for any nonempty $S\subseteq X$,

$$A(S)\rho(\cdot|S)=p(\cdot|S).$$

We stack the matrices $A(S)$ in a matrix $\mathbf{A}$ as follows. For a matrix $\mathbf{A}$ and integers m,n,k,l such that $n\geq m$ and $l\geq k$, let $\mathbf{A}[m:n,k:l]$ denote the $(n-m+1)\times(l-k+1)$ matrix consisting of elements $\mathbf{A}_{ij}$ such that $i\in\{n,\ldots,m\}$ and $j\in\{k,\ldots,l\}$. Then, $\mathbf{A}$ is an $(2^N-1)N\times(2^N-1)N$ matrix given by

$$[\mathbf{A}_{ij}]=A(S) \quad \text{where} \quad i,j\in\{m(S)N+1,\ldots,(m(S)+1)N\} \text{ for some } S\in\mathcal{X}$$

and $\mathbf{A}_{ij}=0$ otherwise. If $p(\cdot|\cdot)$ has a RAR representation $(\{\succ_x\}_{x\in X},\rho)$, then, by construction,

$$\mathbf{A}\boldsymbol{\rho}=\mathbf{p}.$$

Our next step is to incorporate strict positivity and regularity constraints for reference probabilities in a matrix. It will be convenient to introduce the following function: for any $S\in\mathcal{X}$, let $n_S:\{1,\ldots,N\}\to\{0,1,\ldots,N\}$ be given by

$$n_S(i)=\begin{cases}|\{x_j\in S|j\leq i\}| & \text{if } x_i\in S,\\ 0 & \text{if } x_i\notin S.\end{cases}$$

Now, let $I(X)$ denote the $N\times N$ identity matrix. For any non-empty $S\subseteq X$, let $I(S)$ denote $|S|\times N$ matrix where the rows in $I(X)$ which correspond to elements in $X\setminus S$ are eliminated. That is,

$$[I(S)]_{ij}=1 \quad \text{if and only if} \quad n_S(j)=i.$$

We stack the matrices $I(S)$ in a $(\sum_{S\in\mathcal{X}}|S|)\times(2^N-1)N$ matrix $\mathbf{B_1}$ as follows:

$$[\mathbf{B_1}]_{ij}=1 \text{ if and only if } j\in\{m(S)N+1,\ldots,(m(S)+1)N\} \text{ for some } S\in\mathcal{X}$$
$$\text{and } i=n_S(j-m(S)N)+\sum_{S'\in\mathcal{X}:m(S')<m(S)}|S'|$$

The matrix $\mathbf{B_1}$ encodes the requirement that $\rho(x_i|S)>0$ for any $x_i\in S\subseteq X$.

We use the matrix $\mathbf{B_2}$ to encode the regularity requirement: $\rho(x_i|S)<\rho(x_i|S\setminus x_j)$ for any $x_i\in S\subseteq X$ and $x_j\in S\setminus x_i$. To this end, for any $x_i\in S\subseteq X$, let $B_2(x_i,S)$ denote the $|S\setminus x_i|\times(2^N-1)N$ matrix where

$$[B_2(x_i,S)]_{kl}=\begin{cases}-1 & \text{if } l=m(S)N+i,\\ 1 & \text{if } k=n_{S\setminus x_i}(j), l=m(S\setminus x_j)N+i \text{ for } x_j\in S\setminus x_i,\\ 0 & \text{otherwise.}\end{cases}$$

Stack the matrices $B_2(x_i,S)$ in the matrix $B_2(S)$ where the matrix corresponding to the element $x_i\in S$ is ranked higher in the matrix than element $x_j\in S$ if $i<j$. Lastly, create a matrix $\mathbf{B_2}$ consisting of matrices $B_2(S)$, where $B_2(S)$ is placed higher in $\mathbf{B}_2$ than $B_2(S')$ if $m(S)<m(S')$.

Now, let

$$\mathbf{B}=\begin{bmatrix}\mathbf{B_1}\\ \mathbf{B_2}\end{bmatrix}.$$

By construction, $p(\cdot|\cdot)$ has a RAR representation $(\{\succ_x\}_{x\in X},\rho)$ if and only if

$$\text{(A1)}\qquad \mathbf{A}\boldsymbol{\rho}=\mathbf{p}\quad\text{and}\quad\mathbf{B}\boldsymbol{\rho}>0.$$

Let

$$\mathbf{C}=\begin{bmatrix}\mathbf{A}\\ -\mathbf{A}\end{bmatrix}\quad\text{and}\quad\mathbf{b}=\begin{bmatrix}\mathbf{p}\\ -\mathbf{p}\end{bmatrix}.$$

Then, we can alternatively write equation A1 as

$$\text{(A2)}\qquad \mathbf{C}\boldsymbol{\rho}\leq\mathbf{b}\quad\text{and}\quad\mathbf{B}\boldsymbol{\rho}>0.$$

By Motzkin transposition theorem (Motzkin, 1936), the system described by equation A2 has a solution if and only if for all vectors $\mathbf{y}\geq 0$ and $\mathbf{z}\geq 0$,

$$\text{(A3)}\qquad \mathbf{C}^T\mathbf{y}=\mathbf{B}^T\mathbf{z}\quad\Rightarrow\quad\mathbf{b}^T\mathbf{y}\geq 0$$

with strict inequality if $\mathbf{z}\neq 0$. Note that $\mathbf{C}^T\mathbf{y}$ and $\mathbf{B}^T\mathbf{z}$ are $(2^N-1)N\times 1$ vectors.

Note that each row in the matrix $\mathbf{B}$ is associated either with a strict positivity or a regularity constraint, and for each row i in $\mathbf{B}$ there is a corresponding $z_i\geq 0$ in the vector $\mathbf{z}$. For row i such that $[\mathbf{B}]_i$ is associated with the positivity constraint

$\rho(x_k|S) > 0$, we use the notation $\lambda_{x_k}(S) = z_i$ to indicate that row i represents the strict positivity requirement for x_k in S. Similarly, for row j such that $[\mathbf{B}]_j$ is associated with the constraint $\rho(x_k|S) < \rho(x_k|S \setminus x_l)$, we use the notation $\lambda_{x_k x_l}(S) = z_j$. Let $\boldsymbol{\lambda} = \mathbf{z}$. Then, we can rewrite equation A3 as

$$\text{(A4)} \qquad \mathbf{C}^T\mathbf{y} = \mathbf{B}^T\boldsymbol{\lambda} \quad \Rightarrow \quad \mathbf{b}^T\mathbf{y} \geq 0$$

with strict inequality if $\boldsymbol{\lambda} \neq 0$.

Now, let $\kappa = (2^N - 1)N$. We can express $\mathbf{b}^T\mathbf{y}$ as

$$\mathbf{b}^T\mathbf{y} = \sum_{S\in\mathcal{X}} \sum_{i\in\{1,\dots,N\}} p(x_i|S)(y_{m(S)N+i} - y_{\kappa+m(S)N+i}).$$

For any $S \in \mathcal{X}$, let η_S denote the function $\eta_S : \{1,\dots,N\} \to \{1,\dots,N\}$ such that

$$\eta_S(i) = \{j \in \{1,\dots,N\} |\, x_j = \arg\max(\succ_{x_i}, S)\}.$$

Then, for $i \in \{1,\dots,N\}$ and $S \in \mathcal{X}$,

$$[\mathbf{C}^T\mathbf{y}]_{m(S)N+i} = \begin{cases} y_{m(S)N+\eta_S(i)} - y_{\kappa+m(S)N+\eta_S(i)} & \text{if } x_i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$[\mathbf{B}^T\boldsymbol{\lambda}]_{m(S)N+i} = \begin{cases} \lambda_{x_i}(S) + \sum_{x_k\notin S} \lambda_{x_i x_k}(S\cup x_k) - \sum_{x_j\in S\setminus x_i} \lambda_{x_i x_j}(S) & \text{if } x_i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $x_i \in S \subseteq X$, the constraint in equation A4 implies

$$\text{(A5)} \quad y_{m(S)N+\eta_S(i)} - y_{\kappa+m(S)N+\eta_S(i)} = \lambda_{x_i}(S) + \sum_{x_k\notin S} \lambda_{x_i x_k}(S\cup x_k) - \sum_{x_j\in S\setminus x_i} \lambda_{x_i x_j}(S).$$

In addition, since $p(x_i|S) = 0$ if and only if $\eta_S(i) \neq i$, using the above equations, we get

$$\begin{aligned} \mathbf{b}^T\mathbf{y} &= \sum_{S\in\mathcal{X}} \sum_{i\in\{1,\dots,N\}} p(x_i|S)\big[\lambda_{x_i}(S) + \sum_{x_k\notin S} \lambda_{x_i x_k}(S\cup x_k) - \sum_{x_j\in S\setminus x_i} \lambda_{x_i x_j}(S)\big] \\ &= \sum_{S\in\mathcal{X}} \sum_{x_i\in S} \lambda_{x_i}(S)p(x_i|S) + \sum_{S\in\mathcal{X}} \sum_{x_i\in S} \sum_{x_j\in S\setminus x_i} \lambda_{x_i x_j}(S)(p(x_i|S\setminus x_j) - p(x_i|S)) \end{aligned}$$

Lastly, given the above equation, the only implication of equation A5 is that for

$x_i, x_l \in S$ such that $\eta_S(i) = \eta_S(l)$, we have

$$\lambda_{x_i}(S) + \sum_{x_k \notin S} \lambda_{x_i x_k}(S \cup x_k) - \sum_{x_j \in S \setminus x_i} \lambda_{x_i x_j}(S) = \lambda_{x_l}(S) + \sum_{x_k \notin S} \lambda_{x_l x_k}(S \cup x_k) - \sum_{x_j \in S \setminus x_l} \lambda_{x_l x_j}(S).$$

Since for $x_i \neq x_l$ in S, $\eta_S(i) = \eta_S(l) = i$ if and only if $p(x_i|S) = 1$ or $p(x_i|S) > p(x_i|S \setminus x_l)$, letting

$$\Gamma_\lambda(x_i, S) = \lambda_{x_i}(S) + \sum_{x_k \notin S} \lambda_{x_i x_k}(S \cup x_k) - \sum_{x_j \in S \setminus x_i} \lambda_{x_i x_j}(S)$$

and

$$V(\lambda, p) = \sum_{S \in \mathcal{X}} \sum_{x_i \in S} \lambda_{x_i}(S) p(x_i|S) + \sum_{S \in \mathcal{X}} \sum_{x_i \in S} \sum_{x_j \in S \setminus x_i} \lambda_{x_i x_j}(S)(p(x_i|S \setminus x_j) - p(x_i|S))$$

yields the axiom. This concludes the proof of the claim. □

We next prove sufficiency. To this end, we first define reference-dependent preferences as in Section II.

DEFINITION 3: *For any x and $y \neq z$, let yP_xz if and only if there exists $S \supseteq \{y, z\}$ such that at least one of the following is satisfied:*

(i) $p(y|S) = 1$*,*

(ii) $p(y|S) > 0$ *and* $x = y$*,*

(iii) $p(y|S) > p(y|S \setminus x)$*.*

The next claim ensures that P_x is well-defined.

CLAIM 2: *If $p(\cdot|\cdot)$ satisfies Axioms 1-4, then P_x is transitive for all $x \in X$.*

PROOF:

Let yP_xzP_xt. We will show that yP_xt. There are nine cases (3 by 3) to consider. Each case is named according to the corresponding conditions in the definition.

(i)-(i): There exist $S \supseteq \{y, z\}$ and $T \supseteq \{z, t\}$ such that $p(y|S) = 1$ and $p(z|T) = 1$. By Axiom 2, $p(y|\{y, z\}) = 1$ and $p(z|\{z, t\}) = 1$. By Axiom 1, $p(y|\{y, z, t\}) = 1$, and yP_xt follows.

(i)-(ii): There exist $S \supseteq \{y, z\}$ and $T \supseteq \{z, t\}$ such that $p(y|S) = 1$ and $p(z|T) > 0$, where $x = z$. Axiom 2 implies $p(y|\{y, z\}) = 1$. Axiom 1 implies $p(z|\{z, t\}) > 0$ and $p(z|\{y, z, t\}) = 0$. Axiom 2 implies that either $p(y|\{y, z, t\}) =$

1 or $p(y|\{y,z,t\}) > p(y|\{y,t\})$. In both cases yP_zt follows, and since $x = z$, yP_xt follows.

(i)-(iii): There exist $S \supseteq \{y,z\}$ and $T \supseteq \{z,t\}$ such that $p(y|S) = 1$ and $p(z|T) > p(z|T \setminus x)$. Axiom 2 implies $p(y|\{y,z\}) = 1$. Hence, Axiom 1 implies $p(z|\{x,y,z,t\}) = 0$. Since $p(z|T) > p(z|T \setminus x)$, Axiom 1 implies $p(z|\{x,z\}) > 0$ and $p(z|\{z,t\}) > 0$. Then, Axiom 2 implies $p(z|\{x,z,t\}) > 0$. Moreover, since $p(z|T) > p(z|T \setminus x)$, Axiom 3 implies $p(x|T) = 0$, and hence Axioms 2 and 4 imply $p(x|\{x,z\}) = 0$. Then, Axiom 1 implies $p(x|\{x,y,z,t\}) = 0$. Note that, by Axioms 1 and 2, $p(t|\{x,y,z,t\}) < 1$ as $p(z|\{x,z,t\}) > 0$. In addition, since $p(z|T) > p(z|T \setminus x)$, Axiom 4 implies $p(t|\{x,y,z,t\}) \leq p(t|\{y,z,t\})$. Therefore, by Axiom 2, either $p(y|\{x,y,z,t\}) = 1$ or $p(y|\{x,y,z,t\}) > p(y|\{y,z,t\})$. In both cases yP_xt follows.

(ii)-(i): There exist $S \supseteq \{y,z\}$ and $T \supseteq \{z,t\}$ such that $p(y|S) > 0$ and $p(z|T) = 1$, where $x = y$. Axiom 1 implies $p(y|\{y,z\}) > 0$. Axiom 2 implies $p(z|\{z,t\}) = 1$. Therefore, Axiom 1 implies $p(t|\{y,z,t\}) = 0$. Since $p(y|\{y,z\}) > 0$ and $p(t|\{y,z,t\}) = 0$, by Axiom 2, we cannot have $p(y|\{y,z,t\}) = 0$. Hence, $p(y|\{y,z,t\}) > 0$ which implies yP_yt, and since $x = y$, yP_xt follows.

(ii)-(ii): There exist $S \supseteq \{y,z\}$ and $T \supseteq \{z,t\}$ such that $p(y|S) > 0$ and $p(z|T) > 0$, where $x = y$ and $x = z$. Since the definition of P_x requires that $y \neq z$ whenever yP_xz, this case is not possible.

(ii)-(iii): There exist $S \supseteq \{y,z\}$ and $T \supseteq \{z,t\}$ such that $p(y|S) > 0$ and $p(z|T) > p(z|T \setminus x)$, where $x = y$. Axiom 1 implies $p(y|\{y,z\}) > 0$. Axiom 3 implies $p(x|T) = 0$. Axioms 2 and 4 imply $p(x|\{x,z\}) = 0$. This contradicts $p(y|\{y,z\}) > 0$ as $x = y$. Hence, this case is not possible.

(iii)-(i): There exist $S \supseteq \{y,z\}$ and $T \supseteq \{z,t\}$ such that $p(y|S) > p(y|S \setminus x)$ and $p(z|T) = 1$. Axiom 1 implies $p(y|\{x,y\}) > 0$ and $p(y|\{y,z\}) > 0$, and hence Axiom 2 implies $p(y|\{x,y,z\}) > 0$. Since $p(z|T) = 1$, Axiom 2 implies $p(z|\{z,t\}) = 1$, and hence Axiom 1 implies $p(t|\{x,y,z,t\}) = 0$. Now since $p(y|S) > p(y|S \setminus x)$, Axiom 3 implies $p(x|S) = 0$, and hence Axioms 2 and 4 imply $p(x|\{x,y\}) = 0$. Therefore, Axiom 1 implies $p(x|\{x,y,z,t\}) = 0$. Note that, by Axioms 1 and 2, $p(z|\{x,y,z,t\}) < 1$ as $p(y|\{x,y,z\}) > 0$. Moreover, by Axiom 4, $p(z|\{x,y,z,t\}) \leq p(z|\{y,z,t\})$ as $p(y|S) > p(y|S \setminus x)$. Since $p(x|\{x,y,z,t\}) = 0$, by Axiom 2, either $p(y|\{x,y,z,t\}) = 1$ or $p(y|\{x,y,z,t\}) > p(y|\{y,z,t\})$. In both cases yP_xt follows.

(iii)-(ii): There exist $S \supseteq \{y,z\}$ and $T \supseteq \{z,t\}$ such that $p(y|S) > p(y|S \setminus x)$ and $p(z|T) > 0$, where $x = z$. Since $x = z$ and $p(y|S) > p(y|S \setminus x)$, Axiom 3 implies $p(z|S) = 0$, and hence Axioms 2 and 4 imply $p(z|\{y,z\}) = 0$. Now, Axiom 1 implies $p(z|\{y,z,t\}) = 0$ and $p(z|\{z,t\}) > 0$. Hence, by Axiom 2, $p(t|\{y,z,t\}) < 1$. Moreover, by Axiom 4, $p(t|\{y,z,t\}) \leq p(t|\{y,t\})$. Therefore, by Axiom 2, either $p(y|\{y,z,t\}) = 1$ or $p(y|\{y,z,t\}) > p(y|\{y,t\}) > 0$. In both cases yP_zt, and hence yP_xt, follows.

(iii)-(iii): There exist $S \supseteq \{y,z\}$ and $T \supseteq \{z,t\}$ such that $p(y|S) > p(y|S \setminus x)$ and $p(z|T) > p(z|T \setminus x)$. By Axiom 3, $p(x|S) = p(x|T) = 0$. Axioms 2 and 4 imply $p(x|\{x,y\}) = p(x|\{x,z\}) = 0$. Then, Axiom 1 implies $p(x|\{x,y,z,t\}) = 0$. Since $p(z|T) > p(z|T \setminus x)$, by Axiom 1, $p(z|\{x,z\}) > 0$ and $p(z|\{z,t\}) > 0$, and hence, by Axiom 2, $p(z|\{x,z,t\}) > 0$. Therefore, by Axioms 1 and 2, $p(t|\{x,y,z,t\}) < 1$. Moreover, by Axiom 4, $p(t|\{x,y,z,t\}) \leq p(t|\{y,z,t\})$. By the same argument, $p(y|S) > p(y|S \setminus x)$ implies that $p(z|\{x,y,z,t\}) < 1$ and $p(z|\{x,y,z,t\}) \leq p(z|\{y,z,t\})$. Hence, by Axiom 2, either $p(y|\{x,y,z,t\}) = 1$ or $p(y|\{x,y,z,t\}) > p(y|\{y,z,t\})$. In both cases yP_xt follows.

We have now shown that in all possible cases yP_xt follows. Hence, P_x is transitive. □

Now let $\succ_x$ be an arbitrary completion of P_x. The next claim shows that $\{\succ_x\}_{x\in X}$ satisfies the SQB property.

CLAIM 3: *If $y \succ_x z$, then $y \succ_y z$.*

PROOF:

Suppose $z \succ_y y$. Then it must be that there exists no $S \supseteq \{y,z\}$ with $p(y|S) > 0$. In particular, $p(z|\{y,z\}) = 1$. But then, by definition, zP_xy which contradicts $y \succ_x z$. Hence, $y \succ_y z$ must be true. □

By Claims 2 and 3, Axioms 1-4 guarantee that preferences are well-defined and satisfy the SQB property. We then define the vectors $\mathbf{p}$, $\boldsymbol{\rho}$, and matrices $\mathbf{A}$, $\mathbf{B}$ as in the proof of the necessity of Axiom 5. Here the vector $\boldsymbol{\rho}$ is unknown. Proving the representation is equivalent to showing that equation A1, and hence equation A2, holds for some $\boldsymbol{\rho}$. We can then use Motzkin transposition theorem to show that the representation holds if and only if equation A3, and hence equation A4 holds. Let $\eta_S(i)$ be defined as in the proof of the necessity. Notice that our definition of $\{\succ_x\}_{x\in X}$ and the axioms guarantee that (i) $\eta_S(i) \neq i$ if and only if $p(x_i|S) = 0$, and (ii) $\eta_S(i) = \eta_S(l) = i$ for $x_i \neq x_l$ in S if and only if $p(x_i|S) = 1$ or $p(x_i|S) > p(x_i|S \setminus x_l)$. The rest of the proof is the same as the proof of the necessity. □

A2. Characterization for L-RAR

Clearly, Axioms 1-4 are satisfied by L-RAR. We will replace Axiom 5 with four new axioms.

The next two axioms are related to Luce's well-known IIA axiom which states that the relative choice probability of two alternatives is independent of the choice problem they are considered in, that is, for $x, y \in S \cap T$,

$$\frac{p(x|S)}{p(y|S)} = \frac{p(x|T)}{p(y|T)}.$$

L-RAR may not satisfy IIA in general. First, L-RAR does not require that an alternative is chosen from every choice problem (that is, with a positive probability) once it is chosen from one choice problem. Second, even when the two alternatives are both chosen from the two choice problems, L-RAR can exhibit IIA violations. Hence, accommodating the type of choice behavior we are interested in requires formulation of more qualified versions of IIA. This is what we will do next.

Our first "IIA axiom" ensures that the weights of alternatives revealed from binary and trinary choice sets are consistent. First, note that for any binary choice problem $\{x, y\}$, if $p(x|\{x, y\}) \in (0, 1)$, then $p(x|\{x, y\})$ must reflect the reference probability of x in $\{x, y\}$. Hence, in any binary choice problem where both alternatives are chosen, reference probabilities are fully revealed. We use this to construct a function $q(\cdot|\cdot)$ where $q(x|S)$ reflects the probability that x is the reference point in S. For any $\{x, y\}$ with $p(x|\{x, y\}) \in (0, 1)$ and $a \in \{x, y\}$, let

$$q(a|\{x, y\}) = p(a|\{x, y\}).$$

Note that $q(\cdot|\cdot)$ cannot be defined for all binary choice sets, since we might have $p(x|\{x, y\}) \in \{0, 1\}$ for some $\{x, y\}$. However, when q is defined, it must be strictly between 0 and 1.

Next, consider a trinary choice set. First, if all alternatives are chosen, then it must be the case that observed choice probabilities correspond to reference probabilities. Hence, for any $\{x, y, z\}$ where $p(a|\{x, y, z\}) > 0$ for all $a \in \{x, y, z\}$, we have

$$q(a|\{x, y, z\}) = p(a|\{x, y, z\}).$$

Now suppose that $p(x|\{x, y, z\}) > 0$, $p(y|\{x, y, z\}) > 0$, and $p(z|\{x, y, z\}) = 0$. As discussed in Section II, we can also fully reveal $q(\cdot|\cdot)$ in this case. To see this, note that if p is consistent with L-RAR, we must have either $p(x|\{x, y, z\}) > p(x|\{x, y\})$ or $p(y|\{x, y, z\}) > p(y|\{x, y\})$. We can assume $p(x|\{x, y, z\}) > p(x|\{x, y\})$. This reveals that $z \in A(x|\{x, y, z\})$, and hence the observed choice probability of y from $\{x, y, z\}$ must be the same as its reference probability:

$$q(y|\{x, y, z\}) = p(y|\{x, y, z\}).$$

In addition, if p is consistent with L-RAR, we must also have $p(x|\{x, y\}) > 0$ and $p(y|\{x, y\}) > 0$ (Axiom 1). Hence, $q(x|\{x, y\})$ and $q(y|\{x, y\})$ are defined and equal to $p(x|\{x, y\})$ and $p(y|\{x, y\})$, respectively. Since reference probabilities satisfy IIA in our model, we should then have

$$\frac{q(x|\{x, y, z\})}{q(y|\{x, y, z\})} = \frac{q(x|\{x, y\})}{q(y|\{x, y\})}.$$

Therefore, we can define $q(x|\{x, y, z\})$ as

$$q(x|\{x, y, z\}) = \frac{p(x|\{x, y\})p(y|\{x, y, z\})}{p(y|\{x, y\})}.$$

Lastly, since reference probabilities add up to 1, we have $q(z|\{x,y,z\})$ as

$$q(z|\{x,y,z\}) = p(x|\{x,y,z\}) - \frac{p(x|\{x,y\})p(y|\{x,y,z\})}{p(y|\{x,y\})}.$$

Notice that for singleton choice sets $p(x|\{x\}) = q(x|\{x\}) = 1$ must hold. However, as discussed above, if S is not a singleton set, we cannot reveal $q(\cdot|S)$ when there is an alternative $x \in S$ that satisfies $p(x|S) = 1$. We will let $\mathcal{T}$ denote all singleton, binary, and trinary choice sets for which q is defined as above. The next axiom ensures that q is generated by the Luce rule on $\mathcal{T}$.

AXIOM 6: *For any $S_1, S_2, \ldots, S_N \in \mathcal{T}$ and any $x_1, \ldots, x_N \in X$ such that $\{x_i, x_{i+1}\} \subseteq S_i$ for $i < N$ and $\{x_1, x_N\} \subseteq S_N$,*

$$\frac{q(x_1|S_N)}{q(x_N|S_N)} = \prod_{i=1}^{N-1} \frac{q(x_i|S_i)}{q(x_{i+1}|S_i)}.$$

Since reference probabilities satisfy the IIA property in our model, we should expect that observed choices should also satisfy IIA under certain conditions. For example, if all alternatives are chosen with positive probability from every set, then our model reduces to the Luce rule and, hence, satisfies IIA. Our next axiom generalizes this observation. Consider a chosen alternative $z \in S$ such that elimination of no alternative in S induces a regularity violation for z. Axiom 7 then states that choices must satisfy IIA when such a "well-behaved" alternative z is eliminated from S.

AXIOM 7: *If $p(x|S)p(y|S)p(z|S) > 0$ and $p(z|S) \leq p(z|S \setminus t)$ for all $t \in S \setminus z$, then*

$$\frac{p(x|S)}{p(y|S)} = \frac{p(x|S \setminus z)}{p(y|S \setminus z)}.$$

To see why our model satisfies this axiom, note that $p(x|S)p(y|S)p(z|S) > 0$ implies that for these alternatives, the associated reference-dependent preference ranks the reference point as the best in S. Furthermore, elimination of no alternative in S induces a regularity violation for z. That means $A(z|S) = \{z\}$, that is, no other reference-dependent preference in S is maximized at z. Hence, elimination of z does not affect the sets of reference-dependent preferences maximized at x or y: $A(x|S) = A(x|S \setminus z)$ and $A(y|S) = A(y|S \setminus z)$. Thus, the relative choice probability of x and y remains unchanged.

It is useful to note that if all alternatives are chosen with positive probability in S, then Axiom 3 guarantees that observed choices satisfy regularity when any alternative is eliminated from S, and hence Axioms 3 and 7 jointly guarantee that IIA holds in this case.

The IIA violations allowed by L-RAR have a certain structure. Axiom 6 provides information about this structure in binary and trinary sets by imposing a condition on revealed reference probabilities. The next two axioms describe the structure of these violations in all other sets. Consider a choice problem S where x and y are chosen, and z and t are not chosen. Furthermore, suppose elimination of z from S induces a regularity violation on x. Since in our model z can induce a regularity violation for only one alternative, this guarantees that elimination of z from S will change the relative choice probabilities of x and y. Similarly, eliminating z from $S \setminus t$ will also change the relative choice probabilities of x and y. The following two axioms describe how these change are related between S and $S \setminus t$. It turns out that the relationship depends on whether elimination of t from S induces a regularity violation on x or not.

For the first axiom, assume that elimination of t from S does not induce a regularity violation for x. In this case, we look at the ratio

$$\frac{p(x|S)/p(y|S)}{p(x|S \setminus z)/p(y|S \setminus z)}$$

which measures how much more likely x is to be chosen relative to y from S compared to $S \setminus z$. Our axiom states that the above ratio must be the same on S and $S \setminus t$. Hence, z improves the likelihood that x is chosen in a consistent way.

AXIOM 8: *If* $p(x|S \setminus t) \geq p(x|S) > p(x|S \setminus z)$, $p(y|S) > 0$, *and* $p(t|S) = 0$, *then*

$$\frac{p(x|S)/p(y|S)}{p(x|S \setminus z)/p(y|S \setminus z)} = \frac{p(x|S \setminus t)/p(y|S \setminus t)}{p(x|S \setminus \{t,z\})/p(y|S \setminus \{t,z\})}.$$

To see that our model satisfies this axiom, note that under the given assumptions, $z \in A(x|S)$. Hence, $A(y|S) = A(y|S \setminus z)$. This implies

$$\frac{p(x|S)/p(y|S)}{p(x|S \setminus z)/p(y|S \setminus z)} = \left(\frac{\sum\limits_{A(x|S)} s_a}{\sum\limits_{A(y|S)} s_a}\right)\left(\frac{\sum\limits_{A(y|S\setminus z)} s_a}{\sum\limits_{A(x|S\setminus z)} s_a}\right) = \frac{\sum\limits_{A(x|S)} s_a}{\sum\limits_{A(x|S\setminus z)} s_a}.$$

Since $t \notin A(x|S)$, we have $A(x|S) = A(x|S \setminus t)$ and $A(x|S \setminus z) = A(x|S \setminus \{z,t\})$. Hence, the above ratio remains the same when we replace S with $S \setminus t$.

For the second axiom, now assume elimination of t from S does induce a regularity violation for x. In this case, we consider the difference

$$\frac{p(x|S)}{p(y|S)} - \frac{p(x|S \setminus z)}{p(y|S \setminus z)},$$

which is an alternative measure of how much more likely x is to be chosen relative to y from S compared to $S \setminus z$. Our axiom states that the above difference must be the same on S and $S \setminus t$.

AXIOM 9: *If* $p(x|S) > \max\{p(x|S \setminus z), p(x|S \setminus t)\}$ *and* $p(y|S) > 0$*, then*

$$\frac{p(x|S)}{p(y|S)} - \frac{p(x|S \setminus z)}{p(y|S \setminus z)} = \frac{p(x|S \setminus t)}{p(y|S \setminus t)} - \frac{p(x|S \setminus \{z,t\})}{p(y|S \setminus \{z,t\})}.$$

To see that our model satisfies this axiom, note that under the given assumptions, $z, t \in A(x|S)$. Hence, $A(y|S) = A(y|S \setminus z)$. This implies

$$\frac{p(x|S)}{p(y|S)} - \frac{p(x|S \setminus z)}{p(y|S \setminus z)} = \frac{\sum\limits_{A(x|S)} s_a}{\sum\limits_{A(y|S)} s_a} - \frac{\sum\limits_{A(x|S\setminus z)} s_a}{\sum\limits_{A(y|S\setminus z)} s_a} = \frac{s_z}{\sum\limits_{A(y|S)} s_a}.$$

Since $A(y|S) = A(y|S \setminus t)$, this expression remains the same when we replace S with $S \setminus t$.

To get an intuition for the last two axioms, consider an attraction effect example with four alternatives where z is dominated by x and t is either dominated by x or y. Axioms 8 and 9 impose consistency on how addition of z to a choice set will improve the relative likelihood that x is chosen. Clearly, the impact of adding z will depend on whether t is dominated by x or y. First, suppose t is dominated by y. Then, Axiom 8 states that adding z to either $\{x, y\}$ or $\{x, y, t\}$ must increase the relative choice probability of x and y by the same percentage. Note that since t is dominated by y, x must be chosen with a smaller probability from $\{x, y, t\}$ compared to $\{x, y\}$, and hence an increase by the same percentage implies a smaller increase in magnitude when z is added to $\{x, y, t\}$.

Now, suppose t is dominated by x. Intuitively, one might expect that in this case adding z to $\{x, y, t\}$ should make a smaller impact than adding z to $\{x, y\}$, since there is already an alternative dominated by x in the former choice set. In fact, it can be seen that in this case the relative choice probability of x and y will increase by a smaller percentage when z is added to $\{x, y, t\}$ compared to the case when z is added to $\{x, y\}$. Axiom 9 then imposes an alternative consistency requirement that the magnitude of the increase in the relative choice probabilities must be the same.

We can now state the characterization result.

THEOREM 2: *A random choice rule p satisfies Axioms 1-4 and 6-9 if and only if it has an L-RAR representation.*

PROOF:

Necessity of the axioms is clear from the preceding discussion. We prove sufficiency. As shown in the proof of Theorem 1, Axioms 1-4 guarantee that we can construct reference-dependent preferences $\{\succ_x\}_{x \in X}$ which satisfy the SQB property and respect revealed preferences $\{P_x\}_{x \in X}$. Next, we let q and $\mathcal{T}$ be defined as before. The next claim shows that, under Axiom 6, q is generated by the Luce rule.

CLAIM 4: *Suppose $p(\cdot|\cdot)$ satisfies Axioms 1-4. If q satisfies Axiom 6, then there exist weights $\{s_x\}_{x\in X}$ such that for any $S \in \mathcal{T}$ and $x \in S$,*

$$q(x|S) = \frac{s_x}{\sum_{y\in S} s_y}.$$

PROOF:

First, we construct a partition $\mathcal{P}$ of the set of alternatives as follows. Let x and y belong to the same partition element $P \in \mathcal{P}$ if there exist $S_1, \ldots, S_{N-1} \in \mathcal{T}$ and $\{x_1, \ldots, x_N\}$ such that $x_1 = x$, $x_N = y$, and $\{x_i, x_{i+1}\} \subseteq S_i$ for $i \in \{1, \ldots, N-1\}$. We let $P(x)$ denote the partition element corresponding to the alternative x.

Next, we construct the weights as follows. Pick an arbitrary element $x \in X$ and let $s_x = 1$. Choose $y \in P(x)$ and suppose $S_1, \ldots, S_{N-1} \in \mathcal{T}$ and $\{x_1, \ldots, x_N\}$ are such that $x_1 = x$, $x_N = y$, and $\{x_i, x_{i+1}\} \subseteq S_i$ for $i \in \{1, \ldots, N-1\}$. Then, we let

$$s_y = \prod_{i=1}^{N-1} \frac{q(x_{i+1}|S_i)}{q(x_i|S_i)}.$$

To see that s_y is well-defined, let $S'_1, \ldots, S'_{K-1} \in \mathcal{T}$ and $\{x'_1, \ldots, x'_K\}$ be such that $x'_1 = x$, $x'_K = y$, and $\{x_j, x_{j+1}\} \subseteq S'_j$ for $j \in \{1, \ldots, K\}$. Then, by Axiom 6, we get

$$1 = \frac{q(x_2|S_1)}{q(x_1|S_1)} \cdots \frac{q(x_N|S_{N-1})}{q(x_{N-1}|S_{N-1})} \frac{q(x'_{K-1}|S'_{K-1})}{q(x'_K|S'_{K-1})} \cdots \frac{q(x'_1|S'_1)}{q(x'_2|S'_1)},$$

since $x_N = x'_K$, $x_1 = x'_1 = x$, and $q(x|\{x\}) = 1$. Hence,

$$s_y = \prod_{i=1}^{N-1} \frac{q(x_{i+1}|S_i)}{q(x_i|S_i)} = \prod_{j=1}^{K-1} \frac{q(x'_{j+1}|S'_j)}{q(x'_j|S'_j)}.$$

If $X \setminus P(x)$ is empty, then we are done. Otherwise, pick an alternative $y \notin P(x)$ and repeat the procedure above until we are done.

Now let $S \in \mathcal{T}$ be given. If S is a singleton, the claim follows trivially. Hence, let $y, z \in S$ be given. Then, we know that $S \subseteq P(x)$ for some $x \in X$. Let $S_1, \ldots, S_{N-1}$ and $\{x_1, \ldots, x_N\}$ be such that $x_1 = x$, $x_N = y$, and $\{x_i, x_{i+1}\} \subseteq S_i$ for $i \leq N-1$, and let $S'_1, \ldots, S'_{K-1}$ and $\{x'_1, \ldots, x'_K\}$ be such that $x'_1 = x$, $x'_K = z$, and $\{x'_j, x'_{j+1}\} \subseteq S'_j$ for $j \leq K-1$. Now, by Axiom 6,

$$\frac{q(y|S)}{q(z|S)} = \underbrace{\frac{q(x_N|S_{N-1})}{q(x_{N-1}|S_{N-1})} \cdots \frac{q(x_2|S_1)}{q(x_1|S_1)}}_{s_y} \underbrace{\frac{q(x'_1|S'_1)}{q(x'_2|S'_1)} \cdots \frac{q(x'_{K-1}|S'_{K-1})}{q(x'_K|S'_{K-1})}}_{1/s_z},$$

since $x_N = y$, $x'_K = z$, and $x_1 = x'_1 = x$. Hence, we get that for any $S \in \mathcal{T}$ and any $y, z \in S$,

$$\frac{q(y|S)}{q(z|S)} = \frac{s_y}{s_z}.$$

It then follows that

$$q(y|S) = \frac{s_y}{\sum_{t \in S} s_t},$$

as desired. $\square$

We next show that Axioms 1-4 and 6 guarantee that the characterization theorem holds for all sets with at most three alternatives.

CLAIM 5: *Suppose $p(\cdot|\cdot)$ satisfies Axioms 1-4 and 6, and let $\{\succ_x\}_{x \in X}$ and $\{s_x\}_{x \in X}$ be defined as in the previous claims. Then, for any S with $|S| \leq 3$ and $x \in S$,*

$$p(x|S) = \sum_{y \in S} \frac{s_y}{\sum_{z \in S} s_z} \mathbb{1}(x = \arg\max(\succ_y, S)).$$

PROOF:

Let S with $|S| \leq 3$ and $x \in S$ be given. There are a few cases to consider.

Case 1: $p(x|S) = 1$ for some $x \in S$. Then, by our definition of $\succ_y$, $x \succ_y z$ for any $y, z \in S$. Since x is $\succ_y$-maximal element in S for all $y \in S$, the characterization follows.

Case 2: $p(x|S) > 0$ for all $x \in S$. B our definition of $\succ_x$, each x is $\succ_x$-best element in S. Moreover, by our definition of $q(\cdot|\cdot)$, we have that $p(x|S) = q(x|S)$. Since by Claim 4

$$q(x|S) = \frac{s_x}{\sum_{y \in S} s_y},$$

the characterization follows.

Case 3: $S = \{x, y, z\}$, $p(x|S) > 0$, $p(y|S) > 0$, and $p(z|S) = 0$. Note that by our definition of $\succ_x$ and $\succ_y$, x and y are $\succ_x$ and $\succ_y$ maximal elements in S, respectively. Moreover, by Axioms 2 and 4, exactly one of $p(x|S) > p(x|S \setminus z)$ and $p(y|S) > p(y|S \setminus z)$ must hold. Without loss, suppose $p(y|S) > p(y|S \setminus z)$. Then, we must have that y is $\succ_z$-maximal element in S, which shows that the representation holds for z. In addition, by our definition of $q(\cdot|\cdot)$, we have $p(x|S) = q(x|S)$. Since by Claim 4

$$q(x|S) = \frac{s_x}{s_x + s_y + s_z},$$

the representation follows for x. Since $p(y|S) = 1 - p(x|S)$, the representation also follows for y. This concludes the proof of Claim 5. $\square$

The next claim shows that if we assume Axioms 1-4 and 6-7, the characterization holds for all sets where all alternatives are chosen with positive probability.

CLAIM 6: *Suppose $p(\cdot|\cdot)$ satisfies Axioms 1-4 and 6-7, and let $\{\succ_x\}_{x \in X}$ and $\{s_x\}_{x \in X}$ be defined as in the previous claims. Then, for any S such that $p(y|S) >$*

0 *for all* $y \in S$, *and any* $x \in S$,

$$p(x|S) = \sum_{y \in S} \frac{s_y}{\sum_{z \in S} s_z} \mathbb{1}(x = \arg\max(\succ_y, S)).$$

PROOF:

Let S be such that $p(y|S) > 0$ for all $y \in S$ and let $x \in S$. By Axiom 1, $p(x|\{x,y\}) > 0$ for any $y \in S$. In addition, by our definition of $\succ_y$, each $y \in S$ is $\succ_y$-maximal element in S. Hence, by Claim 5, for any $y \in S$,

$$\frac{p(x|\{x,y\})}{p(y|\{x,y\})} = \frac{s_x}{s_y}.$$

Now let $z \in S$ be given. By Axiom 3, we get $p(z|S) \leq p(z|S \setminus t)$ for all $t \in S \setminus z$. Hence, by Axiom 7,

$$\frac{p(x|S)}{p(y|S)} = \frac{p(x|S \setminus z)}{p(y|S \setminus z)}.$$

for any $y \in S \setminus z$. Now note that if $\{x,y\} \subseteq T \subseteq S$, then Axioms 1 and 2 imply that $p(t|T) > 0$ for all $t \in T$. Hence, by repeatedly applying the above reasoning, we get that for any $y \in S \setminus x$,

$$\frac{p(x|S)}{p(y|S)} = \frac{p(x|\{x,y\})}{p(y|\{x,y\})} = \frac{s_x}{s_y}.$$

Hence,

$$\frac{1 - p(x|S)}{p(x|S)} = \frac{\sum_{y \in S \setminus x} s_y}{s_x} \quad \Rightarrow \quad p(x|S) = \frac{s_x}{\sum_{y \in S} s_y},$$

as desired. Since x was arbitrary, this concludes the proof of the claim. □

For any S and $x \in S$, let $A(x|S)$ denote the set of alternatives in S for which x is the maximal element in S:

$$A(x|S) = \{y \in S | \, x = \arg\max(\succ_y, S)\}.$$

Hence, the representation we want to prove can alternatively be stated as

$$p(x|S) = \frac{\sum_{a \in A(x|S)} s_a}{\sum_{b \in S} s_b}.$$

The last claim shows that Axioms 1-4 and 6-9 are sufficient for the representation.

CLAIM 7: *Suppose* $p(\cdot|\cdot)$ *satisfies Axioms 1-4 and 6-9, and let* $\{\succ_x\}_{x \in X}$ *and*

$\{s_x\}_{x\in X}$ be defined as in the previous claims. Then, for any S and $x \in S$,

$$p(x|S) = \frac{\sum_{a\in A(x|S)} s_a}{\sum_{y\in S} s_b}.$$

PROOF:

Note that we have already proven the result for S with $|S| \leq 3$. We will extend the result for all S by induction. Suppose the characterization holds for all S with $|S| \leq n$, where $n \geq 3$, and let $S \ni x$ with $|S| = n+1$ be given. If $p(x|S) = 1$ for some $x \in S$, then the same argument used in Claim 5 can still be used to show the characterization. Hence, we can assume that $p(x|S) = 1$ for no $x \in S$. In addition, if $p(x|S) > 0$ for all $x \in S$, then Claim 6 guarantees that the representation holds. Hence, suppose there exists $z \in S$ such that $p(z|S) = 0$. There are two cases to consider.

Case 1: $p(z|S) = 0$ and $p(x|S) > 0$ for all $x \in S \setminus z$. By definition of $\succ_z$, we should have that $z \in A(x|S)$ for some $x \in S\setminus z$, and the representation holds for z. In addition, every $x \in S \setminus z$ is $\succ_x$-maximal in S. Hence, assume $A(x|S) = \{x, z\}$ for some $x \in S \setminus z$ and $A(y|S) = \{y\}$ for $y \in S \setminus \{x, z\}$. Since $|S| \geq 4$, there exist at least two alternatives $y, y' \in S \setminus x$ such that $p(y|S)p(y'|S) > 0$. By Axioms 3 and 4, $p(y|S) \leq p(y|S\setminus t)$ for all $t \in S\setminus y$ and $p(y'|S) \leq p(y'|S\setminus t)$ for all $t \in S\setminus y'$. By Axiom 7, for any $t \in S \setminus \{x, y, z\}$,

$$\frac{p(t|S)}{p(x|S)} = \frac{p(t|S\setminus y)}{p(x|S\setminus y)}.$$

In addition, by Axiom 7,

$$\frac{p(y|S)}{p(x|S)} = \frac{p(y|S\setminus y')}{p(x|S\setminus y')}.$$

By induction argument, we have

$$\frac{p(t|S\setminus y)}{p(x|S\setminus y)} = \frac{s_t}{s_x + s_z} \quad \text{and} \quad \frac{p(y|S\setminus y')}{p(x|S\setminus y')} = \frac{s_y}{s_x + s_z}.$$

Combining the previous two lines, we get

$$\frac{1 - p(x|S)}{p(x|S)} = \frac{\sum_{t\in S\setminus\{x,z\}} s_t}{s_x + s_z},$$

which implies

$$p(x|S) = \frac{s_x + s_z}{\sum_{y\in S} s_y} \quad \text{and} \quad p(t|S) = \frac{s_t}{\sum_{y\in S} s_y} \text{ for } t \in S \setminus \{x, z\},$$

as desired.

Case 2: $p(z|S) = p(t|S) = 0$ for $z \neq t \in S$. There are two subcases to consider.

First, suppose $z, t \in A(x|S)$ for some $x \in S$. Since we assumed that $p(x|S) \neq 1$, Axiom 2 implies $p(x|S) > p(x|S \setminus z)$ and $p(x|S) > p(x|S \setminus t)$. By Axiom 9, for any $y \in S \setminus x$ with $p(y|S) > 0$, we have

$$\frac{p(x|S)}{p(y|S)} = \frac{p(x|S \setminus z)}{p(y|S \setminus z)} + \frac{p(x|S \setminus t)}{p(y|S \setminus t)} - \frac{p(x|S \setminus \{z,t\})}{p(y|S \setminus \{z,t\})}.$$

Since $A(y|S) = A(y|S \setminus z) = A(y|S \setminus t) = A(y|S \setminus \{z,t\})$, by induction argument, we get

$$\frac{p(x|S)}{p(y|S)} = \frac{\sum_{a \in A(x|S)\setminus z} s_a}{\sum_{b \in A(y|S)} s_b} + \frac{\sum_{a \in A(x|S)\setminus t} s_a}{\sum_{b \in A(y|S)} s_b} - \frac{\sum_{a \in A(x|S)\setminus \{z,t\}} s_a}{\sum_{b \in A(y|S)} s_b} = \frac{\sum_{a \in A(x|S)} s_a}{\sum_{b \in A(y|S)} s_b}.$$

Since this is true for any $y \in S \setminus x$ with $p(y,S) > 0$, we get

$$p(x|S) = \frac{\sum_{a \in A(x|S)} s_a}{\sum_{b \in S} s_b} \quad \text{and} \quad p(y|S) = \frac{\sum_{a \in A(y|S)} s_a}{\sum_{b \in S} s_b}.$$

Next, suppose $z \in A(x|S)$ and $t \in A(y'|S)$ for some $y' \neq x$. Hence, we should have $p(x|S) > p(x|S \setminus z)$ and $p(x|S) \leq p(x|S \setminus t)$. By Axiom 8, for any $y \neq x$ such that $p(y|S) > 0$, we have

$$\frac{p(x|S)}{p(y|S)} = \frac{p(x|S \setminus z)}{p(y|S \setminus z)} \frac{p(x|S \setminus t)}{p(y|S \setminus t)} \frac{p(y|S \setminus \{z,t\})}{p(x|S \setminus \{z,t\})}.$$

Note that for $y \neq x, y'$, we have $A(y|S) = A(y|S \setminus z) = A(y|S \setminus t) = A(y|S \setminus \{z,t\})$. Similarly, we have $A(x|S)\setminus z = A(x|S\setminus t)\setminus z = A(x|S\setminus z) = A(x|S\setminus\{z,t\})$. Hence, for $y \neq x, y'$, the induction hypothesis implies

$$\frac{p(x|S)}{p(y|S)} = \frac{\sum_{a \in A(x|S)\setminus z} s_a}{\sum_{b \in A(y|S)} s_b} \frac{\sum_{a \in A(x|S)} s_a}{\sum_{b \in A(y|S)} s_b} \frac{\sum_{b \in A(y|S)} s_b}{\sum_{a \in A(x|S)\setminus z} s_a} = \frac{\sum_{a \in A(x|S)} s_a}{\sum_{b \in A(y|S)} s_b}.$$

In addition, since $A(y'|S) \setminus t = A(y'|S \setminus z) \setminus t = A(y'|S \setminus t) = A(y'|S \setminus \{z,t\})$,

$$\frac{p(x|S)}{p(y'|S)} = \frac{\sum_{a \in A(x|S)\setminus z} s_a}{\sum_{b \in A(y'|S)} s_b} \frac{\sum_{a \in A(x|S)} s_a}{\sum_{b \in A(y'|S)\setminus t} s_b} \frac{\sum_{b \in A(y'|S)\setminus t} s_b}{\sum_{a \in A(x|S)\setminus z} s_a} = \frac{\sum_{a \in A(x|S)} s_a}{\sum_{b \in A(y'|S)} s_b}.$$

Combining the previous two lines, we get

$$p(x|S) = \frac{\sum_{a \in A(x|S)} s_a}{\sum_{b \in S} s_b} \quad \text{and} \quad p(y|S) = \frac{\sum_{a \in A(y|S)} s_a}{\sum_{b \in S} s_b},$$

as desired. □

This concludes the proof of the theorem. □

A3. Bertrand competition with differentiated products

In this section, we derive the symmetric equilibrium prices and profits in the Bertrand competition model presented in Section VI and prove all the results related to this model. Recall that there are two products, x_1 produced by Firm 1 and x_2 produced by Firm 2. Let s_i be the salience parameter, a_i the loyalty parameter, and p_i the price of product x_i. The costs are assumed to be zero. The demand function for Firm i is given by

$$D_i^{RAR}(p_1, p_2) = \frac{s_i}{s_i + s_j} D_i(p_i - a_i, p_j) + \frac{s_j}{s_i + s_j} D_i(p_i, p_j - a_j),$$

where

$$D_i(p_1, p_2) = \int_{p_i}^{1} \int_0^{\min\{v_i - p_i + p_j, 1\}} dv_j dv_i.$$

Multiplying the demand function with price, we get the profit function of Firm i. It is easy to check that fixing p_j, Firm i's profit function is continuously differentiable and strictly concave in p_i so that there is a unique best response to any p_j. Assuming $p_i - a_i \geq 0$ and $p_i \in (p_j - a_j, p_j + a_j)$ for $i, j \in \{1, 2\}$, we can write the profit function of Firm i as

$$\Pi_i(p_1, p_2) = \frac{s_i p_i (1 - p_j^2 + 2p_j - 2p_i + 2a_i)}{2(s_i + s_j)} + \frac{s_j p_i (1 - p_i)(1 - p_i + 2p_j - 2a_j)}{2(s_i + s_j)}.$$

The first order condition is then given by

$$s_i(1 - p_j^2 + 2p_j - 4p_i + 2a_i) + s_j(1 - 2p_i)(1 - p_i + 2p_j - 2a_j) - s_j(p_i - p_i^2) = 0.$$

To simplify the analysis, we assume that $a_1 = a_2 = a$ and $s_1 = s_2 = s$. Under these assumptions, we have a symmetric equilibrium $p_i = p_j = p$. Using this in the first order condition, we get

$$-2p^2 - (4 - 4a)p + 2 = 0 \quad \Rightarrow \quad p^* = a - 1 + \sqrt{a^2 - 2a + 2}.$$

Notice that $p^* - a = -1 + \sqrt{a^2 - 2a + 2} \geq 0$ and $p_i \in (p_j - a, p_j + a)$ so that our assumptions are satisfied. In addition, Firm i's equilibrium profits are given by

$$\Pi_i(p^*, p^*) = \frac{(2 - a)(a - 1 + \sqrt{a^2 - 2a + 2})^2}{2}.$$

We next prove the results stated in Proposition 6. Our proof uses the implicit function theorem to analyze the implications of changes in the salience and loyalty parameters on equilibrium prices and profits.

PROOF OF PROPOSITION 6:

We can simplify the first order condition as

$$s_i(1 - p_j^2 + 2p_j - 4p_i + 2a_i) + s_j(1 + 2(p_j - a_j) - 4p_i(1 + p_j - a_j) + 3p_i^2) = 0.$$

If $a_1 = a_2 = a$, $s_1 = s_2 = s$, we found that there is a symmetric equilibrium $p_1 = p_2 = p^* = a - 1 + \sqrt{a^2 - 2a + 2}$. Starting from the symmetric equilibrium, we first look at a unilateral change in s_i. We can calculate the effects of a change in s_i on equilibrium prices and equilibrium profits as:

$$\frac{\partial p_i}{\partial s_i} = -\frac{\partial p_j}{\partial s_i} = \frac{a}{s(6 - a + \sqrt{a^2 - 2a + 2})} > 0,$$

$$\frac{\partial \Pi_i}{\partial s_i} = -\frac{\partial \Pi_j}{\partial s_i} = \frac{a(3 - a - \sqrt{a^2 - 2a + 2})(a + \sqrt{a^2 - 2a + 2} - 1)}{4s} > 0.$$

For the second item, we follow a similar methodology to obtain the effects of a change in a_i on equilibrium prices as:

$$\frac{\partial p_i}{\partial a_i} = \frac{23\sqrt{a^2 - 2a + 2} + 6a(4 - a - \sqrt{a^2 - 2a + 2}) - 19}{(34 - 10a)\sqrt{a^2 - 2a + 2}} > 0,$$

$$\frac{\partial p_j}{\partial a_i} = \frac{1 - a}{20 - 12a - 8\sqrt{a^2 - 2a + 2}} + \frac{\sqrt{a^2 - 2a + 2} - 2(1 - a)}{4\sqrt{a^2 - 2a + 2}} > 0.$$

A comparison of the two expressions shows that $\frac{\partial p_i}{\partial a_i} > \frac{\partial p_j}{\partial a_i}$. We can also obtain the effects of a change in a_i on equilibrium profits as:

$$\frac{\partial \Pi_i}{\partial a_i} = \frac{a - 1 + \sqrt{a^2 - 2a + 2}}{2} > 0,$$

$$\frac{\partial \Pi_j}{\partial a_i} = -\frac{(2 - a - \sqrt{a^2 - 2a + 2})(a - 1 + \sqrt{a^2 - 2a + 2})}{2} < 0.$$

Since $(2 - a - \sqrt{a^2 - 2a + 2}) \in (0, 1)$, the increase in Firm i's profits is higher in absolute value than the decrease in Firm j's profits.

In item three, we analyze how a change in the common a affects the impact of a change in s_i on equilibrium prices:

$$\frac{\partial^2 p_i}{\partial s_i \partial a} = -\frac{\partial^2 p_j}{\partial s_i \partial a} = \frac{2 - a + 6\sqrt{a^2 - 2a + 2}}{s\sqrt{a^2 - 2a + 2}\left(6 - a + \sqrt{a^2 - 2a + 2}\right)^2} > 0.$$

Lastly, we also look how a change in the common s affects the impact of a change in a_i on equilibrium prices:

$$\frac{\partial^2 p_i}{\partial a_i \partial s} = -\frac{\partial^2 p_j}{\partial a_i \partial s} = 0.$$

This concludes the proof of the Proposition. □