Consider a history where, in stage 1, the sellers post $\alpha_s \in C$, the buyer posts $\alpha_b \in C$ with $0 < \alpha_b < \alpha_s$, the buyer visits store 1 first at time zero and the probability that the buyer and the sellers are commitment types are z_b and $\hat{z}_s$, respectively. The following results characterise the equilibrium continuation strategies of the game G in stage 2.

Lemma B.1. *In equilibrium where $z_b \geq (\hat{z}_s/A)^{\lambda_b/\lambda_s}$ holds, the rational buyer makes a take it or leave it offer to the seller and goes directly to store 2. Rational seller 1 immediately accepts the buyer's demand and finishes the game at time zero with probability one. In case seller 1 does not concede to the buyer, the buyer infers that seller 1 is obstinate, and so he never comes back to this store again. The concession game with the second seller may continue until the time $T_2^e = -\log(z_b)/\lambda_b$ with the following strategies: $F_b^2(t) = 1 - e^{-\lambda_b t}$ and $F_2(t) = 1 - \hat{z}_s z_b^{-\lambda_s/\lambda_b} e^{-\lambda_s t}$.*

Proof. Suppose that $z_b \geq (\hat{z}_s/A)^{\lambda_b/\lambda_s}$. The rational buyer (weakly) prefers to go to store 2 over conceding seller 1. In equilibrium, rational seller 1 anticipates that the buyer will never concede to him, and hence accepts α_b at time zero without any delay. Therefore, if 1 is rational the game is over at time zero. Otherwise, the buyer leaves the first store at time zero and directly goes to 2. Therefore, the concession game in store 2 ends at time $T_2^e = \tau_b^2 = \min\{\tau_b^2, \tau_2\}$ for sure where $\tau_b^2 = \inf\{t \geq 0 | \ F_b^2(t) = 1 - z_b\} = -\frac{\log z_b}{\lambda_b}$ and $\tau_2 = \inf\{t \geq 0 | \ F_2(t) = 1 - \hat{z}_s\} = -\frac{\log \hat{z}_s}{\lambda_s}$ denote the times that the buyer's and seller 2's reputations reach 1, respectively. Given the equilibrium strategies by Proposition 3.1, the rest follows. $\square$

Lemma B.2. *In equilibrium where $z_b \leq (\hat{z}_s^2/A)^{\lambda_b/\lambda_s}$ holds, the buyer leaves store 1 at time $T_1^d = -\log(\hat{z}_s)/\lambda_s$ for sure, if the game has not yet ended, and goes directly to store 2. The concession game with seller 2 may continue until the time $T_2^e = -\log(\hat{z}_s/A)/\lambda_s$. The players' concession game strategies are $F_b^1(t) = 1 - z_b(A/\hat{z}_s^2)^{\lambda_b/\lambda_s} e^{-\lambda_b t}$ and $F_1(t) = 1 - e^{-\lambda_s t}$ in store 1, and $F_b^2(t) = 1 - e^{-\lambda_b t}$ and $F_2(t) = 1 - Ae^{-\lambda_s t}$ in store 2.*

Proof. The proof is given in the main text, Appendix A. $\square$

Lemma B.3. *In equilibrium where $(\hat{z}_s^2/A)^{\lambda_b/\lambda_s} < z_b < (\hat{z}_s/A)^{\lambda_b/\lambda_s}$ holds, the buyer leaves store 1 at time $T_1^d = -\log(z_b)/\lambda_b + \log(\hat{z}_s/A)/\lambda_s$ for sure, if the game has not yet ended, and goes directly to store 2. The concession game with seller 2 may continue until the time $T_2^e = -\log(\hat{z}_s/A)/\lambda_s$. The players' concession game strategies are $F_b^1(t) = 1 - e^{-\lambda_b t}$ and $F_1(t) = 1 - (\hat{z}_s^2/A)(z_b)^{-\lambda_s/\lambda_b} e^{-\lambda_s t}$ in store 1, and $F_b^2(t) = 1 - e^{-\lambda_b t}$ and $F_2(t) = 1 - Ae^{-\lambda_s t}$ in store 2.*

Proof. Consider an equilibrium where $(\hat{z}_s^2/A)^{\lambda_b/\lambda_s} < z_b < z_b^* = (\hat{z}_s/A)^{\lambda_b/\lambda_s}$ holds. Again, the rational buyer leaves 1 when his reputation reaches z_b^*, implying that the equality $c_b^1 e^{-\lambda_b T_1^d} = \frac{z_b}{z_b^*}$ must hold. If $c_b^1 = 1$, then $T_1^d = -\frac{\log z_b}{\lambda_b} + \frac{\log \hat{z}_s/A}{\lambda_s}$, and it is smaller than $-\frac{\log \hat{z}_s}{\lambda_s}$ as $(\hat{z}_s^2/A)^{\lambda_b/\lambda_s} < z_b$. Similar to Lemma B.2, the game ends in store 2 at time $T_2^e = \frac{\log z_b^*}{\lambda_b}$. Given the values of T_1^d and T_2^e, Proposition 3.1 implies the concession game strategies. $\qquad\square$

Now, let σ_ϵ denote a sequential equilibrium of the discrete-time competitive-bargaining game $G(g_\epsilon)$ and σ_i be the rational buyer's equilibrium strategy for store selection at time zero. Given σ_i, the random outcome corresponding to σ_ϵ is a random object $\theta_\epsilon(\sigma_i)$ which denotes any realization of an agreed division as well as a time and store at which agreement is reached. Let $\theta(\sigma_i)$ denote the unique equilibrium distribution of the second stage of the game G that is characterised in Lemmata B.1 — B.3.

Proposition B. *As ϵ converges to 0, $\theta_\epsilon(\sigma_i)$ converges in distribution to $\theta(\sigma_i)$.*

I first present a series of results which I will later use in the proof of Proposition B.

Lemma 1. *As ϵ converges to zero, in any sequential equilibrium of the discrete-time competitive-bargaining game $G(g_\epsilon)$ in stage two after any history h_t such that the buyer is in store i and unknown to be rational while seller i is known to be rational, the payoff to the rational buyer is no less than $1 - \alpha_b - \epsilon$ and the payoff to seller i is no more than $\alpha_b + \epsilon$ (payoffs are evaluated at time t).*

The proof of this result is the same as the proof of Theorem 8.4 in Myerson (1991) and Lemma 1 in Abreu and Gul (2000) and very similar to the proof of Lemma 4: I show that the payoff to the rational buyer if he continues to stay in store i and mimics the obstinate type converges to $1 - \alpha_b$ as ϵ converges to zero. Given this, we can conclude that in any sequential equilibrium, the rational buyer chooses not to reveal his type and he stays in store i unless his expected payoff of doing the opposite exceeds $1 - \alpha_b$. I first show that the game ends (by seller i's acceptance of the buyer's offer α_b) with probability 1 in finite time, given history h_t, if the rational buyer continues to stay in store i and mimics the obstinate type. Finally, I show that as players make offers frequent enough ($\epsilon \to 0$), the game ends immediately with (almost) no delay. Therefore, I skip the proof.

With a similar spirit, Lemma 4 claims that as ϵ converges to 0, at any sequential equilibrium of the game $G(g_\epsilon)$ after the history h_t such that the buyer is in store i and known to be rational while both sellers are not known to be rational, the buyer makes immediate agreement with seller i, and the payoff to rational seller i (which depends on the details of the bargaining protocol g_ϵ) cannot be lower than α_s in the limit.

Before presenting the proof of Lemma 4, I prove two Lemmata that I use extensively later:

Lemma 2. *Let $\epsilon \to 0$ and h_t be a history such that the buyer is in store i, known to be rational, seller i is unknown to be rational and seller $j \in \{1, 2\}$, $j \neq i$ is known to be obstinate. Then, for any sequential equilibrium of the game $G(g_\epsilon)$ in stage two after the history h_t, the payoff to the buyer is no more than $1 - \alpha_s + \epsilon$ and the payoff to rational seller i is no less than $\alpha_s - \epsilon$ (payoffs are evaluated at time t).*

Proof. Given that seller j is the obstinate type, the buyer's continuation payoff in store j is at most $1 - \alpha_s$. Therefore, the buyer has no incentive to leave store i to get a price better than α_s. Given this, seller i does not reveal his type unless he gets a payoff higher than α_s by doing the opposite. Hence, the payoff to the buyer is no more than $1 - \alpha_s$ as ϵ converges to zero. $\square$

Lemma 3. *Let ϵ converge to 0 and let h_t be a history such that the buyer is in store i and known to be rational while both sellers are unknown to be rational. Then in any sequential equilibrium of the game $G(g_\epsilon)$ in stage two after the history h_t it cannot be the case that rational seller i finishes the game at time t at some price $x < \alpha_s - \epsilon$ with probability one.*

Proof. Suppose for a contradiction that rational seller i makes a deal with the buyer at some price $x < \alpha_s$ at time t with probability 1. Given that this is an equilibrium strategy, both seller j and the buyer assign probability 1 to the event that seller i is the obstinate type if the seller does not accept the buyer's offer. But then, according to Lemma 2, the buyer accepts the price α_s and finishes the game immediately at time t^* where $t < t^* \leq t + \epsilon$.

However, for arbitrarily small ϵ, rational seller i would prefer to deviate from his equilibrium strategy and wait until time t^* by mimicking the obstinate type so that he can get the payoff of α_s which is higher than x.[1] Hence, in equilibrium after the history h_t rational seller i delays the game with a positive probability. $\square$

Lemma 4. *As ϵ converges to zero, in any sequential equilibrium of the discrete-time competitive-bargaining game $G(g_\epsilon)$ in stage two after any history h_t such that the buyer is in store i and known to be rational while both sellers are not known to be rational, the payoff to the buyer is no more than $1 - \alpha_s + \epsilon$ and the payoff to rational seller i is no less than $\alpha_s - \epsilon$ (payoffs are evaluated at time t).*

Proof. Without loss of generality, suppose that the buyer is in store 1 at time t after the history h_t. I will show that as rational seller 1 continues to mimic the obstinate type, the payoff to the buyer converges to $1 - \alpha_s$ and the payoff to rational seller 1 converges to

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[1] Receiving α_s at time t^* is equivalent to receiving $\alpha_s e^{-r_s(t^*-t)}$ at time t, which is arbitrarily close to α_s as ϵ converges to 0.

α_s, as ϵ converges to zero. For the remainder of this proof, assume that rational seller 1 continues to mimic the obstinate type, while the buyer and rational seller 2 execute their equilibrium strategies.

For each i, let $\hat{z}_i(h_t)$ denote the probability that seller i is the obstinate type at time t after the history h_t. By Bayes' rule, $\hat{z}_i(h_t)$ is either zero or higher than z_s. By our assumption, however, we must have $\hat{z}_i(h_t) \geq z_s$.

If the buyer continues to stay in store 1 for long enough according to his equilibrium strategy while rational seller 1 continues to act irrationally, we know by Lemma 1 that the payoff to the buyer converges to $1 - \alpha_s$ as ϵ converges to zero, and this proves the claim. It is, however, possible that in equilibrium the buyer does not stay in store 1 long enough if seller 1 continues to mimic the obstinate type. This implies that the buyer leaves store 1 at some time $t' \geq t$. Note that the second seller's reputation at time t' is still $\hat{z}_2(h_t)$.

The buyer's decision of leaving store 1 at time t' implies that $\hat{z}_2(h_t) \leq \frac{\delta + \alpha_s - 1}{\delta \alpha_s} = \hat{\rho} < 1$ This is true because, if the buyer goes to store 2 and seeks an agreement with seller 2, the highest payoff he could achieve is $\delta[1 - \hat{z}_2(h_t) + (1 - \alpha_s)\hat{z}_2(h_t)]$. But leaving store 1 and going to store 2 at time t' is optimal for the buyer only if $1 - \alpha_s \leq \delta[1 - \hat{z}_2(h_t) + (1 - \alpha_s)\hat{z}_2(h_t)]$ which implies the desired result.

According to his strategy, if the buyer continues to stay in store 2 long enough, conditional on rational seller 2 mimicking the obstinate type, we know again by Lemma 1 that the payoff to the buyer converges to $1 - \alpha_s$ as ϵ converges to zero. This implies that $1 - \alpha_s$ is the highest payoff the buyer can attain in store 2. If this is the case, however, the buyer does not leave store 1 at time t', which contradicts our supposition. Therefore, it must be the case that the buyer leaves store 2 as well, conditional on seller 2 insisting on his demand, at some time t'' where $t'' > t'$.

According to his equilibrium strategy, rational seller 2 may be playing a strategy that ends the game while the buyer is in store 2. However, according to Lemma 3, we know that rational seller 2 will not play a strategy that will end the game with a price less than α_s (in the limit) with probability one. If rational seller 2 is playing a strategy which ends the game with a price higher than α_s, then he buyer does not leave store 1 at time t', which contradicts our supposition. Therefore, it must be the case that rational seller 2 is playing a strategy that extends the game, i.e. rational seller 2 will mimic the obstinate type with a positive probability, until time t''.

Conditional on the buyer arriving at store 1 once more, the same arguments show that the buyer shall leave store 1 once again as rational seller 1 continues to mimic the obstinate type (because otherwise, the payoff to the buyer will be at most $1 - \alpha_s$ and this contradicts our supposition that the buyer leaves store 2 when rational seller 2 continues

to mimic the obstinate type).

Therefore, conditional on both sellers extending the game and the buyer leaving store 1 twice, we have $\hat{z}_2(h_t) \leq \hat{\rho}^2$, so that extending the game by going back and forth between the sellers (twice) is more profitable for the buyer than seeking an immediate agreement with inflexible seller 1. Similarly, for the game that lasts until the kth departure of the buyer, it must be true that $\hat{z}_2(h_t) \leq \hat{\rho}^k$. Choosing k such that $\hat{\rho}^k < z_s$ establishes contradiction since, as argued earlier, $\hat{z}_2(h_t) \geq z_s$.

Therefore, as rational seller 1 continues to mimic the obstinate type, rational seller 2 will continue to play a strategy which extends the game with positive probability (immediate consequence of Lemma 3). The buyer, however, will travel back and forth between the sellers only for some finite time in order to get a deal better than α_s. This implies that the buyer will end up at some store $i \in \{1, 2\}$ at some finite time $\bar{t}$. That is, the buyer does not leave store i after time $\bar{t}$ while rational seller i continues to mimic the obstinate type.

This implies that the buyer's continuation payoff in store i is at most $1 - \alpha_s$, evaluated at time $\bar{t}$. This leads to a contradiction because, given that the buyer's continuation payoff in his final destination is less than $1 - \alpha_s$, the buyer should not have left store j when seller j continues to be inflexible. Hence, repeating this argument backward, we can conclude that the buyer should not delay the game, but instead seek an immediate agreement with seller 1 at time t. $\qquad\square$

Proof of Proposition B. This proof is adapted from the proof of Proposition 4 in Abreu and Gul (2000). Let $G(g_{\epsilon_n})$ be a sequence of discrete-time competitive-bargaining games and σ_n (drop the term ϵ to ease the notation) be the corresponding sequence of sequential equilibria. Then, for each seller i and $T \in H^i$ define $F_n^{i,T} : [T, T'] \to [0, 1]$, where $T' \leq \infty$ depends on the buyer's equilibrium strategy on timing and store selection, and $F_n^{i,T}(t)$ is the cumulative probability that seller i takes an action not consistent with the obstinate type in the interval $[T, t]$, conditional on the buyer and the other seller have ben inflexible until time t. Similarly, define $F_n^{b,i,T} : [T, T'] \to [0, 1]$ where $F_n^{b,i,T}(t)$ is the cumulative probability that the buyer takes an action not consistent with the obstinate type in the interval $[T, t]$, conditional on the buyer is in store i in this time interval and both sellers have been inflexible until time t.

To prove the Proposition, arbitrarily choose some $\bar{n} \geq 0$, an equilibrium strategy $\sigma_{\bar{n}}$, $i \in \{1, 2\}$ and a history $h_T \in H^i$. Then, I show that as $n \geq \bar{n}$,

Step (1) Every subsequence of $F_n^{i,T}$ and $F_n^{b,i,T}$ have a convergent subsequence: Similar to Steps 1 and 2 in the proof of Proposition 4 in Abreu and Gul (2000), define G_n^i

such that

$$G_n^{i,T}(t) = \frac{F_n^{i,T}(t)}{F_n^{i,T}(T')} \quad \text{whenever} \quad F_n^{i,T}(T') \neq 0$$

for all $t \leq T'$ where $T' \leq \infty$ depends on the buyer's equilibrium timing decision and $F_n^{i,T}(T') = 1 - \hat{z}_s(T)/\hat{z}_s(T')$. Note that $\{F_n^{i,T}(T')\}$ is a bounded real sequence, which is bounded below by 0 and above by $1 - z_s$ for all n. The same arguments hold for the buyer. Moreover, by Helly's selection Theorem (See Billingsley (1986)), the sequence $G_n^{i,T}$ has a subsequence $G_{n_k}^{i,T}$ which converges to a right continuous, non-decreasing function G_i^T at every continuity point of G_i^T. Let $F_{n_k}^{i,T} = F_{n_k}^{i,T}(T')G_{n_k}^{i,T}$. Since the real sequence $F_{n_k}^{i,T}(T')$ is bounded below 0 and bounded above $1 - z_s$ for any n, there must exist a subsequence $F_{n_{k_j}}^{i,T}(T')$ which converges to some real number $F_i^T(T')$. Therefore, $F_{n_{k_j}}^{i,T} = F_{n_{k_j}}^{i,T}(T')G_{n_{k_j}}^{i,T}$ implies that $F_i^T = F_i^T(T')G_i^T$. Applying the same arguments to the buyer's strategy and renumbering the sequence n_{k_j} will yield the desired result.

Step (2) the limit points of $(F_n^{i,T}, F_n^{b,i,T})$ do not have common points of discontinuity in the domain $[T, T']$; The proofs of this claim utilizes the exact methods used in the proof of steps 3-6 of Proposition in Abreu and Gul (2000). Therefore, I do not represent the proof here to prevent duplication.

Step (3) if $(F_n^{i,T}, F_n^{b,i,T})$ converges to $(F_i^T, F_b^{i,T})$ and if the limit functions do not have common points of discontinuity then $(F_i^T, F_b^{i,T})$ is a part of equilibrium strategies of the continuous-time competitive-bargaining game G.

The following arguments prove Step 3 and complete the proof of Proposition 9.1. Recall that σ_n is the equilibrium strategy of the game $G(g_{\epsilon_n})$. For any $t > 0$ and $\hat{\epsilon} > 0$ define a strategy $\tilde{\sigma}_n^i$ to be a strategy of seller i within the interval $[T, T']$ as follows: Seller i behaves according to σ_n^i until time t_n where t_n is the last time the buyer makes an offer prior to $t + \bar{\epsilon}$ (for some $\bar{\epsilon} > 0$) and at time t_n seller i accepts the buyer's offer α_b. Let U_n^i denote the utility function of rational seller i in the game $G(g_{\epsilon_n})$. Then there exist finite integers N_1, N_2, N_3 and $\bar{\epsilon} > 0$ sufficiently close to 0, such that $t + \bar{\epsilon}$ is a continuity point of $F_b^{i,T}$ and

$$U^i(t, F_b^{i,T}) - U^i(t + \bar{\epsilon}, F_b^{i,T}) < \hat{\epsilon}, \tag{1}$$

$$U^i(t + \bar{\epsilon}, F_b^{i,T}) - U^i(t_n, F_n^{b,i,T}) < \hat{\epsilon} \quad \forall n \geq N_1, \tag{2}$$

$$U^i(t_n, F_n^{b,i,T}) - U^i(\tilde{\sigma}_n^i, \sigma_n^b) < \hat{\epsilon} \quad \forall n \geq N_2, \tag{3}$$

$$U_n^i(\tilde{\sigma}_n^i, \sigma_n^b) - U_n^i(\sigma_n^i, \sigma_n^b) \leq 0 \quad \forall n, \tag{4}$$

$$U_n^i(\sigma_n^i, \sigma_n^b) - U_n^i(F_n^{i,T}, F_n^{b,i,T}) < \hat{\epsilon} \quad \forall n \geq N_2 \tag{5}$$

$$U^i(F_n^{i,T}, F_n^{b,i,T}) - U^i(F_i^T, F_b^{i,T}) < \hat{\epsilon} \quad \forall n \geq N_3. \tag{6}$$

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Equation (1) follows immediately from the definition of U^i. That is, $U^i(., F_b^{i,T})$ is continuous at continuity points of $F_b^{i,T}$. If t is not a continuity point of $F_b^{i,T}$, for $\bar{\epsilon}$ small enough the left-hand side of (1) is strictly negative (similar logic to the proof of step 3 of Abreu and Gul (2000): if the buyer makes a mass acceptance at time t, seller i would prefer conceding at time $t + \bar{\epsilon}$ over conceding at time t). Since $t + \bar{\epsilon}$ is a continuity point of $F_b^{i,T}$, (2) follows from Step 6 of Abreu and Gul (2000). Equation (3) follows from the definition of $\tilde{\sigma}_n^i$ and Lemma 4. Equation(4) is the consequence of the fact that (σ_n^i, σ_n^b) is equilibrium. Equation (5) is an application of Lemma 1; seller i can never get more than α_b after revealing his rationality. Moreover, in equilibrium, since his opponent makes offers frequently, he can reveal himself to be rational in a manner that guarantees α_b. Equation (6) follows from Steps 3-6 of Abreu and Gul (2000).

Choosing $n \geq \max\{N_1, N_2, N_3\}$ and adding Equations (1)-(6) will yield

$$U^i(t, F_b^{i,T}) - U^i(F_i^T, F_b^{i,T}) < 5\hat{\epsilon}$$

Since this inequality is true for any $\hat{\epsilon} > 0$, it must be the case that

$$U^i(t, F_b^{i,T}) - U^i(F_i^T, F_b^{i,T}) \leq 0.$$

Hence, F_i^T is a best response to $F_b^{i,T}$. Symmetric arguments imply that $(F_i^T, F_b^{i,T})$ is a Nash equilibrium of the concession game within the interval $[T, T']$. Note that if rational seller i is the first to reveal his type, he can guarantee α_b by accepting the buyer's offer. This would yield the buyer a payoff of $1 - \alpha_b$. If rational seller i reveals his type in some other way, then by Lemma 1 he is still, in the limit, guaranteed α_b. This happens only if agreement is reached immediately at these terms. Analogous arguments are valid for the buyer. Therefore, convergence in expected payoffs implies convergence in distribution within the interval $[T, T']$.

After an arbitrary history h_T and continuation strategy $\sigma_{\bar{n}}$, I proved the convergence of concession game strategies in any interval $[T, T']$. So, for any given history, the concession game strategies of the discrete-time competitive-bargaining game $G(g_{\epsilon_n})$, i.e. $\mathscr{F}_n = (\mathscr{F}_n^b, \mathscr{F}_n^1, \mathscr{F}_n^2)$ (the function which maps histories to the set of right-continuous distribution functions), converge to the the concession game strategies of the continuous-time competitive-bargaining game G, $\mathscr{F} = (\mathscr{F}^b, \mathscr{F}^1, \mathscr{F}^2)$, history by history (i.e., interval by interval) in the product topology.

Given that $\mathscr{F}_n$ converges to $\mathscr{F}$ history by history, similar arguments in the proof of Proposition 3.1 suffice to show that for sufficiently large n, the buyer visits each store at most once according to the equilibrium strategy of the game $G(g_{\epsilon_n})$. As a result, fixing the rational buyer's store selection at time zero, convergence in distribution in all subgames implies that the buyer's timing and location decisions together with the distribution

functions, $\mathscr{F}_n$, converge to the unique equilibrium of the competitive-bargaining game G that is characterized by Lemmata B.1 — B.3.