Soc Choice Welfare (2008) 30:89–101
DOI 10.1007/s00355-007-0223-6

ORIGINAL PAPER

Strategy-proof resolute social choice correspondences

Selçuk Özyurt · M. Remzi Sanver

Received: 26 August 2004 / Accepted: 24 January 2007 / Published online: 20 March 2007
© Springer-Verlag 2007

Abstract We qualify a social choice correspondence as *resolute* when its set valued outcomes are interpreted as mutually compatible alternatives which are altogether chosen. We refer to such sets as "committees" and analyze the manipulability of resolute social choice correspondences which pick fixed size committees. When the domain of preferences over committees is unrestricted, the Gibbard–Satterthwaite theorem—naturally—applies. We show that in case we wish to "reasonably" relate preferences over committees to preferences over committee members, there is no domain restriction which allows escaping Gibbard–Satterthwaite type of impossibilities. We also consider a more general model where the range of the social choice rule is determined by imposing a lower and an upper bound on the cardinalities of the committees. The results are again of the Gibbard–Satterthwaite taste, though under more restrictive extension axioms.

1 Introduction

We know from Gibbard (1973) and Satterthwaite (1975) that every strategy-proof social choice function with a range containing at least three alternatives must be dictatorial. The fact that social choice functions pick a single alternative at each preference profile is critical in establishing this result. On the other

S. Özyurt
Department of Economics, New York University, New York, NY, USA

M. Remzi Sanver (✉)
Department of Economics, İstanbul Bilgi University, Inonu Cad. No. 28 Kustepe, 80310 Istanbul, Turkey
e-mail: sanver@bilgi.edu.tr

hand, there is a vast literature exploring the strategy-proofness of social choice correspondences.[1] For such an analysis to be carried, individual preferences over sets are necessary information. A typical approach is to deduce this information from individual preferences over alternatives through certain extension axioms which assign to every ordering over alternatives the list of acceptable orderings over sets.

The plausibility of these axioms depends on how one interprets a set. One standard interpretation is to see a set as a first refinement of the initial set of alternatives, from which finally a single outcome will be chosen. This is the conception of social choice correspondences as non-resolute social choice rules. The literature under this interpretation dates back to Pattanaik (1973), Barberà (1977), Kelly (1977), Gärdenfors (1976, 1978), Feldman (1979a,b) and is more recently followed by Duggan and Schwartz (2000), Ching and Zhou (2002), Benoit (2002), Barberà et al. (2001), Özyurt and Sanver (2005). All these papers analyze strategy-proofness of social choice correspondences defined over domains restricted through extension axioms appropriate to the given interpretation of a set. The basic result is negative: The Gibbard–Satterthwaite theorem is very robust and for many reasonable domain restrictions, strategy-proofness is equivalent to various versions of dictatoriality.

Another interpretation of a non-singleton set is a list of mutually compatible alternatives which are altogether chosen, as it may be the case for a department of mathematics recruiting more than one assistant professor. We refer to such sets as “committees”. We restrict the possible outcomes through the committee size and consider social choice rules which assign a committee to every preference profile over committees. We refer to these social choice rules as *resolute social choice correspondences*.

It is clear that if we allow all possible orderings of committees as individual preferences, then the Gibbard–Satterthwaite theorem directly applies to resolute social choice correspondences. For, in this case the social choice problem is equivalent to a standard one defined over the full domain of alternatives, where every committee is an “alternative”. On the other hand, individual preferences over committees may be restricted. In particular, connecting preferences over committees to preferences over the basic set of alternatives (which are the possible committee members) restricts the domain of orderings over committees. In fact, every logical relationship between preferences over committees and preferences over the basic set of alternatives implies a particular restriction of the domain of orderings over committees. This leads to ask whether it is possible to escape Gibbard–Satterthwaite type of results under reasonable axioms to extend orderings over a set to its power set.

We bring a negative answer to this question through a previous result of Aswal et al. (2003) who establish that any unanimous and strategy-proof social choice function defined over a “linked” domain has to be dictatorial. We show that in the framework of resolute social choice correspondences, very weak

[1] By a social choice correspondence, we mean a social choice rule that assigns a non-empty set of alternatives to every preference profile.

axioms to connect preferences over committees to preferences over the basic set of alternatives lead to "linked" domains, thus ending up in Gibbard–Satterthwaite type of impossibilities.

Section 2 gives the preliminaries. Section 3 states the main impossibility result: Given m alternatives, fix some $k < m$ and consider a unanimous social choice rule f whose range is committees of size k. The domain of f contains preference profiles over these committees. Individual preferences in the domain are obtained through extension axioms that relate preferences over alternatives to preferences over sets. We show that virtually all extension axioms lead to the equivalence of strategy-proofness and dictatoriality. Section 4 considers a more general model where the range of the social choice rule is determined by imposing a lower and an upper bound on the cardinalities of the committees. The results are again of the Gibbard–Satterthwaite taste, though under more restrictive extension axioms. Section 5 makes some concluding remarks.

2 Preliminaries

Taking any two integers n and m with $n \geq 2$ and $m \geq 3$, we consider a society $\mathbf{N} = \{1, \ldots, n\}$ confronting a set of alternatives $\mathbf{A}$ with $\#\mathbf{A} = m$. Fixing any integer k with $1 \leq k < m$, we write $\mathbf{A_k} = \{X \subseteq \mathbf{A} : \#X = k\}$ for the set of all k element subsets of $\mathbf{A}$, to which we refer as a "committee". We denote a typical committee by $c \in \mathbf{A_k}$.

We let Π stand for the set of all complete, transitive and antisymmetric binary relations over $\mathbf{A}$. Every $\rho \in \Pi$ represents an individual preference on the elements of $\mathbf{A}$ in the following manner: For any $a, b \in \mathbf{A}, a\rho b$ means "a is at least as good as b".[2]

Similarly, we let $\Re$ stands for the set of all complete and transitive orderings over $\mathbf{A_k}$. Every $\mathrm{R} \in \Re$ represents an individual preference on the elements of $\mathbf{A_k}$ in the following manner: For any $c, c' \in \mathbf{A_k}, cRc'$ means "c is at least as good as c'". We denote P for the strict counterpart of R.[3] In case the preference ordering over $\mathbf{A_k}$ is specified to belong to a particular agent $i \in \mathbf{N}$, we write it as R_i, with its respective strict counterpart P_i. A typical preference profile over $\mathbf{A_k}$ is denoted by $\underline{R} = (R_1, \ldots, R_n) \in \Re^{\mathbf{N}}$.

Given any non-empty $D \subseteq \Re$, we define a *social choice rule* as a mapping $f : D^{\mathbf{N}} \to \mathbf{A_k}$. So we conceive a social choice rule as a resolute social choice correspondence, as it assigns to every preference profile $\underline{R} \in D^{\mathbf{N}}$, some committee $f(\underline{R}) \in \mathbf{A_k}$. Note that we only consider social choice rules whose domains are Cartesian products of some $D \subseteq \Re$, in which case we say that the social choice rule is defined over the domain D. A social choice rule $f : D^{\mathbf{N}} \to \mathbf{A_k}$ is *manipulable* at $\underline{R} \in D^{\mathbf{N}}$ by some $i \in \mathbf{N}$ if and only if there exists $\underline{R}' \in D^{\mathbf{N}}$ with

[2] As ρ is antisymmetric, for any distinct $a, b \in \mathbf{A}$ we have a $\rho\ b \to$ not $b\ \rho\ a$. In other words, for distinct alternatives, $a\ \rho\ b$ means "a is preferred to b".

[3] For any $c, c' \in \mathbf{A_k}$, we write $c\ P\ c'$ if and only if $c\ R\ c'$ holds but $c'\ R\ c$ does not, i.e., c is preferred to c'.

$R_j = R'_j$ for all $j \in \mathbf{N}\backslash\{i\}$ such that $f(\underline{R}')\ P_i\ f(\underline{R})$. We say that f is *strategy-proof* if and only if there exists no $\underline{R} \in D^{\mathbf{N}}$ at which f is manipulable by some $i \in \mathbf{N}$. A social choice rule $f : D^{\mathbf{N}} \to \mathbf{A_k}$ is *dictatorial* if and only if there exists $\mathrm{d} \in \mathbf{N}$ such that for all $\underline{R} \in D$ we have $f(\underline{R})R_{\mathrm{d}}\, c$ for all $c \in \mathbf{A_k}$.

We know by the Gibbard–Satterthwaite theorem that when $D = \Re$, a social choice rule $f : \Re^{\mathbf{N}} \to \mathbf{A_k}$ with a range containing at least three elements is strategy-proof if and only if f is dictatorial. In fact, it is not possible to escape this equivalence through reasonable domain restrictions, as we show in the next section.

3 The main impossibility result

We start by quoting two definitions from Aswal et al. (2003):

Definition 3.1 Two committees $c, c' \in \mathbf{A_k}$, are said to be *connected* on a domain $D \subseteq \Re$ if and only if there exists $R, R' \in D$ such that $\operatorname{argmax}_{\mathbf{A_k}} R = c$, $\operatorname{argmax}_{\mathbf{A_k}\backslash\{c\}} R = c'$, $\operatorname{argmax}_{\mathbf{A_k}} R' = c'$ and $\operatorname{argmax}_{\mathbf{A_k}\backslash\{c\}'} R' = c$.

So we say that committees c and c' are connected on the domain D if there exists a preference ordering in D where c is ranked first, c' second and another ordering where c' is ranked first, c second. We denote this as $c \sim c'$.[4]

Before quoting the next definition, let $s = \#\mathbf{A_k} = m!\backslash(m-k)!k!$

Definition 3.2 A domain $\mathrm{D} \subseteq \Re$ is said to be *linked* if and only if one can order the elements of $\mathbf{A_k} = \{c_1,\ c_2, \ldots, c_s\}$ such that

(i) $c_1 \sim c_2$
(ii) c_i is connected to at least two elements of the set $\{c_1,\ c_2, \ldots, c_{i-1}\}$ $(i = 3, \ldots, s)$.

It follows from Theorem 3.1 in Aswal et al. (2003) that unanimous[5] and strategy-proof social choice rules defined over linked domains are dictatorial. We refer to this result as Fact 3.1 and state it formally below:

Fact 3.1 *Consider any linked domain $D \subseteq \Re$. A unanimous social choice rule $f : D^{\mathbf{N}} \to \mathbf{A_k}$ is strategy-proof if and only if f is dictatorial.*

We wish to note that our statement of Fact 3.1 is stronger than the statement of Theorem 3.1 in Aswal et al. (2003), as they consider antisymmetric orderings only. In other words, their Theorem 3.1 is Fact 3.1 restricted to domains D which only contain strict orderings. On the other hand, the proof of their Theorem 3.1 establishes the more general result announced by our Fact 3.1.

We now turn to domain restrictions. An *extension map* is a rule α which assigns to every $\rho \in \Pi$, a non-empty set $\alpha(\rho) \subseteq \Re$ of admissible orderings over

[4] Note that the relation $\sim$ is symmetric, i.e., $c \sim c'$ implies $c' \sim c$.

[5] A social choice rule $f : D^{\mathbf{N}} \to \mathbf{A_k}$ is *unanimous* if and only if given any $c \in \mathbf{A_k}$ and any $R \in D$ with cP_ic' for all $i \in \mathbf{N}$ and for all $c' \in \mathbf{A_k}$, we have $f(R) = c$.

$\mathbf{A_k}$. A domain $D \subseteq \Re$ is said to be obtained through an extension map α if and only if $D = \bigcup_{\rho \in \Pi} \alpha(\rho)$.

An extension map α is *reasonable* if and only if given any $\rho \in \Pi$, there exists $R \in \alpha(\rho)$ such that given $c_1, c_2 \in \mathbf{A_k}$ which differ only by one element and satisfy

(i) $x\rho y$ for all $x \in c_1$ and for all $y \in \mathbf{A} \backslash c_1$.
(ii) $c_2 \backslash c_1 = \text{argmax}_{\mathbf{A} \backslash c_1} \rho$ and $c_1 \backslash c_2 = \text{argmin}_{c_1} \rho$

we have $c_1 P c$ for all $c \in \mathbf{A_k} \backslash \{c_1\}$ and $c_2 P c$ for all $c \in \mathbf{A_k} \backslash \{c_1, c_2\}$.

Note that given an ordering ρ over **A,** c_1 is the committee formed by the k best elements of **A** and c_2 is obtained by replacing the worst element in c_1 by the best element outside of c_1. An extension map is reasonable if it allows individuals to admit c_1 as the best and c_2 as the second best committee. This is a fairly weak condition. In fact, negating reasonability is too demanding: It means that individuals are not allowed to consider the committee formed by their first k best members as their favorite one or their second best committee cannot be the one consisting of the first $k-1$ best and $k+1$th ranked members. As we will explore throughout the paper, virtually all axioms of the literature on extending an order over a set to its power set induce reasonable extension maps.

Our main result is an impossibility theorem which states that every unanimous and strategy-proof social choice rule defined over a domain obtained through a reasonable extension map must be dictatorial.

Theorem 3.1 *Let* $\#\mathbf{A} \geq 3$. *Take any* $D \subseteq \Re$ *which is obtained through a reasonable extension map. A unanimous social choice rule* f*:* $D^{\mathbf{N}} \to \mathbf{A_k}$ *is strategy-proof if and only if* f *is dictatorial.*

Proof Let **A** and D be as in the statement of the theorem. The "if" part is obvious. To prove the "only if" part, we will show that D is a linked domain, hence proving the theorem through Fact 3.1.

We start by introducing a lexicographic extension map: The lexicographic extension of any $\rho \in \Pi$ is an ordering $\lambda(\rho)$ over $\mathbf{A_k}$ which is defined as follows: Take any two distinct $X, Y \in \mathbf{A_k}$. Let, without loss of generality, $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ such that $x_j \; \rho \; x_{j+1}$ and $y_j \; \rho \; y_{j+1}$ for all $j \in \{1, \ldots, k-1\}$. We have $X \; \lambda(\rho) \; Y$ if and only if $x_h \; \rho \; y_h$ for the smallest $h \in \{1, \ldots, k\}$ such that $x_h \neq y_h$. Note that, at each $\rho \in \Pi, \lambda(\rho)$ gives a unique complete, transitive and antisymmetric ordering over $\mathbf{A_k}$.

Now fix some total order θ over **A**. Assume, without loss of generality that $a_i \; \theta \; a_{i+1}$ for all $i \in \{1, \ldots, m-1\}$. Order the elements of $\mathbf{A_k} = \{c_1, c_2, \ldots, c_s\}$ according to the lexicographic ordering $\lambda(\theta)$. Let, without loss of generality, $c_i \; \lambda(\theta) \; c_{i+1}$ for all $i \in \{1, \ldots, s-1\}$. We now make three claims, which altogether prove the theorem:

Claim 1 $\#(c_1 \backslash c_2) = 1$.

Claim 2 For all $c_j \in \mathbf{A_k}$ with $j > 2$, there exist distinct $c_h, c_i \in \mathbf{A_k}$ with $h < j$ and $i < j$ such that $\#(c_h \backslash c_j) = \#(c_i \backslash c_j) = 1$.

Claim 3 For any $c, c' \in \mathbf{A_k}$, if $\#(c \backslash c') = 1$ then $c \sim c'$ on the domain D.

Note that, if these claims are true, then $c_1 \sim c_2$ and c_i is connected to at least two elements of the set $\{c_1, c_2, \ldots, c_{i-1}\}$ for all $i \in \{3, \ldots, s\}$, which shows that D is linked. So we complete the proof by showing our claims.

To show Claim 1, it suffices to note that under $\lambda(\theta)$, c_1 consists of the first k best elements of $\mathbf{A}$ according to θ, i.e., $c_1 = \{a_1, \ldots, a_k\}$, while $c_2 = [c_1 \backslash \{a_k\}] \bigcup \{a_{k+1}\}$.

To show Claim 2, take any $c_j \in \mathbf{A_k}$ with $j > 2$. Let $x = \text{argmin}_{c_j}\theta$ be the lowest ranked alternative in c_j according to θ. Note that there exist distinct $a, b \in \mathbf{A} \backslash c_j$ such that $a\, \theta\, x$ and $b\, \theta\, x$, as otherwise we would have either $c_j = c_1 = \{a_1, \ldots, a_k\}$ or $c_j = c_2 = [c_1 \backslash \{a_k\}] \bigcup \{a_{k+1}\}$, contradicting $j > 2$. Now consider the committee $c_i = [c_j \backslash \{x\}] \cup \{a\}$. Clearly $\#\ (c_i \backslash c_j) = 1$. Moreover, $i < j, i.e., c_i\ \lambda(\theta)\ c_j$. Similarly, given the committee $c_h = [c_j \backslash \{x\}] \cup b$, we have $\#\ (c_h \backslash c_j) = 1$, as well as $h < j$, i.e., $c_h\ \lambda(\theta)\ c_j$. To show Claim 2, it suffices to check that c_h and c_j are distinct, which directly arises from the fact that a and b are distinct.

To show Claim 3, take any two distinct committees $c, c' \in \mathbf{A_k}$ with $\#(c \backslash c') = 1$. Let, without loss of generality, $c = \{x_1, \ldots, x_k\}$ and $c' \backslash c = \{x_{k+1}\}$ and $c \backslash c' = \{x_k\}$. Now pick $\rho \in \Pi$ with $x_i\, \rho\, x_{i+1}$ for all $i \in \{1, \ldots, k\}$ while $x\, \rho\, y$ for all $x \in c \cup c'$ and for all $y \in \mathbf{A} \backslash (c \cup c')$. Pick also $\rho' \in \Pi$ with $x_i\ \rho'\ x_{i+1}$ for all $i \in \{1, \ldots, k-2\}, x_{k-1}\ \rho'\ x_{k+1}$, $x_{k+1}\ \rho'\ x_k$, and $x\ \rho'\ y$ for all $x \in c \cup c'$ and for all $y \in \mathbf{A} \backslash (c \cup c')$. As D is obtained through a reasonable extension map α, there exists $R \in \alpha(\rho) \subset D$ as well as $R' \in \alpha(\rho') \subset D$ with argmax $_{\mathbf{A_k}} R = c$, argmax $_{\mathbf{A_k} \backslash \{c\}} R = c'$, argmax $_{\mathbf{A_k}} R' = c'$ and argmax $_{\mathbf{A_k} \backslash \{c'\}} R' = c$. Hence, $c \sim c'$ on the domain D which shows Claim 3 and thus completes the proof. □

Theorem 3.1 covers many interesting domains, such as those where members of the committees do not impose externalities on each other, i.e., agents have separable preferences over committees. This property is particularly used in the many-to-one matching literature where it is called "responsiveness".[6] We refer to it as **RES** and define as follows:

RES: For any $X \subseteq \mathbf{A}$ and any $x, y \in \mathbf{A} \backslash X$ we have

$$(X \cup \{x\})\ R\ (X \cup \{y\}) \Leftrightarrow x\ \rho\ y$$

For every $\rho \in \Pi$, define the extension map $\alpha^{\text{RES}}(\rho) = \{R \in \Re : \text{R satisfies } \mathbf{RES}\}$. We write $\Re^{\text{RES}} = \bigcup_{\rho \in \Pi} \alpha^{\text{RES}}(\rho)$. It is straightforward to see that α^{RES} is a reasonable extension map, thus $\Re^{\text{RES}}$ is covered by Theorem 3.1. We state this formally in the following corollary:

Corollary 3.1 *Let $\#\mathbf{A} \geq 3$. A unanimous social choice rule f: $[\Re^{\text{RES}}]^{\mathbf{N}} \to \mathbf{A_k}$ is strategy-proof if and only if f is dictatorial.*

---

6 For a detailed treatment of the responsiveness axiom, one can see Roth and Sotomayor (1990). Remark that responsiveness is a modified version of the monotonicity axiom of Kannai and Peleg (1984).

As another example, we consider the "dominance" axiom of Kelly (1977) to which we refer as **DOM** and define as follows:

DOM: For any distinct X, Y $\in 2^{\mathbf{A}}\backslash\{\emptyset\}$ such that x ρ y $\forall$ x $\in$X and $\forall$ y $\in$ Y, we have X P Y

For every $\rho \in \Pi$, define the extension map $\alpha^{\mathrm{DOM}}(\rho) = \{R \in \Re : R$ satisfies **DOM**$\}$ We write $\Re^{\mathrm{DOM}} = \bigcup_{\rho\in\Pi} \alpha^{\mathrm{DOM}}(\rho)$. It is straightforward to see that α^{DOM} is a reasonable extension map, thus $\Re^{\mathrm{DOM}}$ is covered by Theorem 3.1. We state this formally in the following corollary:

Corollary 3.2 *Let #$\mathbf{A} \geq 3$. A unanimous social choice rule f*: $[\Re^{\mathrm{DOM}}]^{\mathbf{N}} \to \mathbf{A_k}$ *is strategy-proof if and only if f is dictatorial.*

Finally, we wish to note that Theorem 3.1 may cover very narrow domains. Consider the lexicographic extension map λ defined in the proof of Theorem 3.1.[7] Write $\Re^{\lambda} = \bigcup_{\rho\in\Pi}\{\lambda(\rho)\}$.[8] Clearly, the lexicographic extension map is reasonable, hence leading to dictatorial social choice rules, which we state in the following corollary.

Corollary 3.3 *Let #$\mathbf{A} \geq 3$. A unanimous social choice rule f*: $[\Re^{\lambda}]^{\mathbf{N}} \to \mathbf{A_k}$ *is strategy-proof if and only if f is dictatorial.*

Corollary 3.3 is independently shown by Campbell and Kelly (2002) who characterize strategy-proof social choice correspondences when preferences over sets are obtained through the lexicographic extension map λ. Although they analyze non-resolute social choice correspondences, they also consider social choice rules which pick (non-empty and non-singleton) sets of fixed cardinality—which we call resolute social choice correspondences. Campbell and Kelly (2002) establish the dictatoriality of strategy-proof and unanimous social choice rules defined over the domain $\Re^{\lambda}$—which is Corollary 3.3 to our Theorem 3.1.[9]

4 A Generalization

The impossibility expressed by Theorem 3.1 can be carried to a more general framework where possible outcomes are determined by imposing a lower and an upper bound on the cardinalities of the committees. We start by considering the case where the lower and upper bound differ by one. Fix some k with $1 \leq k < m-1$. Let $\mathbf{A_k} = \{X \subseteq \mathbf{A} : \#X = k\}$ and $\mathbf{A_{k+1}} = \{X \subseteq \mathbf{A} : \#X = k+1\}$. We stick to the notation used in the previous sections with the sole difference that $\Re$ stands for the set of all complete and transitive orderings over $\mathbf{A_k} \bigcup \mathbf{A_{k+1}}$.

[7] This extension map is a particular case of the lexicographic rank-ordered rule which Bossert (1995) characterizes.

[8] Note the narrowness of $\Re^{\lambda}$ which contains precisely $m!$ orderings over the set $\mathbf{A_k}$ containing $m!\,/\,[(m-k)!k!]$ elements.

[9] In fact, they use a weaker surjectivity condition instead of unanimity. On the other hand, surjectivity can replace unanimity in our Theorem 3.1 as well.

Now, given any non-empty $D \subseteq \Re$, we conceive a *social choice rule* as a mapping $f : D^{\mathbf{N}} \to \mathbf{A_k} \bigcup \mathbf{A_{k+1}}$.

The definition of reasonable extension maps introduced in Sect. 3 has a natural adaptation to this framework. We say that an extension map α is *separately reasonable* if and only if for all $j \in \{k, k+1\}$ and for all $\rho \in \Pi$, there exists $R \in \alpha(\rho)$ such that given $c_1, c_2 \in \mathbf{A}_j$ which differ only by one element and satisfy

(i) $x \,\rho\, y$ for all $x \in c_1$ and for all $y \in \mathbf{A} \backslash c_1$.
(ii) $c_2 \backslash c_1 = \mathrm{argmax}_{\mathbf{A} \backslash c_1} \rho$ and $c_1 \backslash c_2 = \mathrm{argmin}_{c_1} \rho$

we have $c_1\, Pc$ for all $c \in \mathbf{A_k} \bigcup \mathbf{A_{k+1}} \backslash \{c_1\}$ and $c_2 Pc$ for all $c \in \mathbf{A_k} \bigcup \mathbf{A_{k+1}} \backslash \{c_1, c_2\}$.

Note that separate reasonability, as the name implies, applies the reasonability condition introduced in Sect. 3 for committees of size k and $k+1$ separately. However, Theorem 3.1 cannot be carried to this framework by replacing reasonability with separate reasonability. To see this, consider the following Example 4.1 where domains $D_1, D_2 \subseteq \Re$ are defined as $D_1 = \{R \in \Re : c\, P\, d$ for all $c \in \mathbf{A}_k$ and $d \in \mathbf{A}_{k+1}\}$ and $D_2 = \{R \in \Re : d\, P\, c$ for all $c \in \mathbf{A_k}$ and $d \in \mathbf{A_{k+1}}\}$. So D_1 contains all orderings over $\mathbf{A_k} \bigcup \mathbf{A}_{k+1}$ where every committee of size k is preferred to every committee of size $k+1$. Similarly, D_2 is the set of orderings over $\mathbf{A_k} \cup \mathbf{A}_{k+1}$ where every committee of size $k+1$ is preferred to every committee of size k. Let $D = D_1 \bigcup D_2$. Clearly, D can be obtained by a separately reasonable extension map.[10] However, given a two person–society $\mathbf{N} = \{1, 2\}$, a social choice rule $f : D^{\mathbf{N}} \to \mathbf{A_k} \bigcup \mathbf{A_{k+1}}$ defined for each $\underline{R} = (R_1, R_2) \in D^{\mathbf{N}}$ by the following three exhaustive cases is unanimous, non-dictatorial and strategy–proof:

(i) If $R_1 \in D_1$ then $f(\underline{R}) R_1 X$ for all $X \in \mathbf{A_k} \cup \mathbf{A_{k+1}}$;
(ii) If $R_1, R_2 \in D_2$ then $f(\underline{R}) R_2 X$ for all $X \in \mathbf{A_k} \cup \mathbf{A_{k+1}}$;
(iii) If $R_1 \in D_2$ and $R_2 \in D_1$ then $f(\underline{R}) R_2 X$ for all $X \in \mathbf{A_{k+1}}$.

So whenever agent 1 announces a preference from D_1, he becomes the dictator. If agent 1 and agent 2 both announce a preference from D_2, then agent 2 is the dictator. Finally, if agent 1 announces a preference from D_2 and agent 2 announces a preference from D_1, then the $k+1$ element committee which is most preferred by agent 2 is chosen. As D contains orderings with indifferences, we fix at the outset some linear ordering over sets to break all possible ties that may occur. Checking the unanimity, non-dictatoriality and strategy-proofness of f is left as an exercise to the reader.

We now propose a variant of the reasonability condition which enables us to state an analogous of Theorem 3.1 in this framework. We explore the general case where the lower and upper bounds of the committees need not differ by one. So let k_1 with $1 \le k_1 < m-1$ be the lower bound and k_2 with $k_1 \le k_2 < m$ be the upper bound. Write $K = \{j : k_1 \le j \le k_2\}$ for the set of admissible committee sizes.[11] By an ordering of K, we mean a bijection κ between the set of positive integers $\{1, \ldots, \#K\}$ and K. So given any $i \in \{1, \ldots, \#K\}$, $\kappa(i)$ is the

[10] In fact, D contains all possible orderings of the committees of size k and all possible orderings of the committees of size $k+1$.

[11] So we always have $m \notin K$.

i'th ranked element of K. As usual, for any $k \in K$, we let $\kappa^{-1}(k) \in \{1, \ldots, \#K\}$ stand for the rank of k.

Let $\mathbf{A_k} = \{X \subseteq \mathbf{A} : \#X = k\}$ for any $k \in K$. Now $\Re$ stands for the set of all complete and transitive orderings over $\bigcup_{k \in K} \mathbf{A_k}$ and given non-empty $D \subseteq \Re$, a social choice rule is a mapping f: $D^{\mathbf{N}} \to \bigcup_{\mathrm{k} \in \mathrm{K}} \mathbf{A_k}$.

We say that an extension map $\underset{.}{\alpha}$ satisfies *condition* $\underset{.}{\Lambda}$ if and only if there exists an ordering κ of K such that for all $\rho \in \Pi$, the following two conditions Λ_1 and Λ_2 hold:

(Λ_1) There exists $R \in \alpha(\rho)$ such that given $c_1, c_2 \in \mathbf{A}_{\kappa(1)}$ which differ only by one element and satisfy

(i) $x \, \rho \, y$ for all $x \in c_1$ and for all $y \in \mathbf{A} \backslash c_1$.
(ii) $c_2 \backslash c_1 = \mathrm{argmax}_{\mathbf{A} \backslash c_1} \rho$ and $c_1 \backslash c_2 = \mathrm{argmin}_{c_1} \rho$

we have $c_1 \mathrm{P} c$ for all $c \in \bigcup_{\mathrm{k} \in K} \mathbf{A_k} \backslash \{c_1\}$ and $c_2 \mathrm{P} c$ for all $c \in \bigcup_{k \in K} \mathbf{A_k} \backslash \{c_1, c_2\}$.

(Λ_2) For all $k \in K \backslash \kappa(1)$, there exists $j \in K$ with $\kappa^{-1}(j) < \kappa^{-1}(k)$ such that given the committees $c^* \in \mathbf{A}_j$ and $d^* \in \mathbf{A_k}$ which respectively consist of the first j and k elements according to ρ,[12] there exists $R, R' \in \alpha(\rho)$ with $c^* \, P \, d^* \, P \, b$ for all $b \in \bigcup_{k \in K} \mathbf{A_k} \backslash \{c^*, d^*\}$ and $d^* \, P' \, c^* \, P' \, b$ for all $b \in \bigcup_{k \in K} \mathbf{A_k} \backslash \{d^*, c^*\}$.

Condition Λ has two facades. On one side, it is a variant of the reasonability condition: Both Λ_1 and Λ_2 require the existence of orderings where certain "good" committees are ranked first and second. On the other side, by Λ_2, it requires the existence of orderings where committees are not a priori compared according to their sizes. In other words, for certain committee sizes (say k and k'), it ensures the existence an ordering where a committee c of size k is best while a committee c' of size k' is second-best and the existence of another ordering where c' is best and c is second-best. Note that this property is not satisfied by the domain in Example 4.1 which consists of orderings where committees of size k are always preferred to committees of size $k+1$ or vice versa—hence the existence of unanimous, strategy-proof and non-dictatorial social choice rules. On the other hand, the satisfaction of Condition Λ allows us to state the following impossibility theorem:

Theorem 4.1 *Let $\#\mathbf{A} \geq 3$. Take any $D \subseteq \Re$ which is obtained through an extension map which satisfies condition Λ. A unanimous social choice rule f: $D^{\mathbf{N}} \to \bigcup_{k \in K} \mathbf{A_k}$ is strategy-proof if and only if f is dictatorial.*

Proof The "if" part being obvious, we will show the "only if" part. Let $\mathbf{A}$ and D be as in the statement of the theorem. Let κ be the ordering of K which ensures that the extension map which induces D satisfies condition Λ. We will show that D is a linked domain, hence proving the theorem through Fact 3.1.

Let T be a linear ordering of $\bigcup_{k \in K} \mathbf{A_k}$ such that given any $j, k \in K$ with $\kappa^{-1}(j) < \kappa^{-1}(k)$, any $c \in \mathbf{A_j}$ and any $d \in \mathbf{A_k}$, we have $c \, T \, d$. Moreover, T orders the elements of $\mathbf{A}_{\kappa(\mathbf{1})}$ according to the lexicographic ordering $\lambda(\theta)$ where θ is

[12] Thus, we have $x \, \rho \, y$ for all $x \in c^*$ and for all $y \in \mathbf{A} \backslash c^*$ as well as $x \, \rho \, y$ for all $x \in d^*$ and for all $y \in \mathbf{A} \backslash d^*$ while $c^* \in \mathbf{A}_j$ and $d^* \in \mathbf{A}_k$.

some total order over $\mathbf{A}$, i.e., given any $c, c' \in \mathbf{A}_{\kappa(1)}$, we have $c\ T\ c' \Leftrightarrow c\ \lambda(\theta)\ c'$. Writing s for the cardinality of $\mathbf{A}_{\kappa(1)} = \{c_1, c_2, \ldots, c_s\}$, let, without loss of generality, $c_i\ \lambda(\theta)\ c_{i+1}$ for all i $\in \{1, \ldots, s-1\}$. By condition Λ_1 and using the arguments in the proof of Theorem 3.1, we have $c_1 \sim c_2$ and c_i is connected to at least two elements of the set $\{c_1, c_2, \ldots, c_{i-1}\}$ for all i $\in \{3, \ldots, s\}$.

Now we will show that any element of $\mathbf{A}_{\kappa(2)}$ is connected to at least two elements of $\mathbf{A}_{\kappa(1)}$. Take some d $\in \mathbf{A}_{\kappa(2)}$. Pick some $\rho \in \Pi$ with $x\ \rho\ y$ for all $x \in d$ and for all $y \in \mathbf{A}\backslash d$. Let $c \in \mathbf{A}_{\kappa(1)}$ be the committee consisting of the first $\kappa(1)$ elements according to ρ. By condition Λ_2 which is satisfied for $k = \kappa(2)$, there exists $R, R' \in \alpha(\rho)$ with $c\ P\ d$ P b for all $b \in \bigcup_{k \in K} \mathbf{A_k}\backslash\{c, d\}$ and $d\ P'\ c\ P'\ b$ for all $b \in \bigcup_{k \in K} \mathbf{A_k}\backslash\{d, c\}$. Thus, $d \sim c$. Now take some $\rho' \in \Pi\backslash\{\rho\}$ with $x\ \rho'\ y$ for all $x \in d$ and for all $y \in \mathbf{A}\backslash d$ while the committee c' which consists of the first $\kappa(1)$ elements according to ρ' differs from c.[13] Again by condition Λ_2 which is satisfied for k = $\kappa(2)$, we have d $\sim$ c'. Thus, any d $\in \mathbf{A}_{\kappa(2)}$ is connected to at least two elements in $\mathbf{A}_{\kappa(1)}$.

By similar arguments, one can take any d $\in \mathbf{A}_{\kappa(3)}$ and show that it is connected to at least two elements in $\mathbf{A}_{\kappa(1)} \bigcup \mathbf{A}_{\kappa(2)}$. In fact the same arguments show that any d $\in \mathbf{A}_{\kappa(i)}$ is connected to at least two elements in $\mathbf{A}_{\kappa(1)} \bigcup \mathbf{A}_{\kappa(2)} \bigcup \ldots \bigcup \mathbf{A}_{\kappa(i-1)}$, showing that D is linked. □

We now give examples of domains covered by Theorem 4.1. First, reconsider the responsiveness condition introduced in Sect. 3. Again for every $\rho \in \Pi$, let $\alpha^{\mathrm{RES}}(\rho) = \{R \in \Re :$ R satisfies **RES**$\}$ and write $\Re^{\mathrm{RES}} = \bigcup_{\rho \in \Pi} \alpha^{\mathrm{RES}}(\rho)$.[14] Take any ordering κ of K. We already checked in Sect. 3 that α^{RES} satisfies condition Λ_1. Moreover **RES** does not bring any restriction on how sets of different cardinalities are ordered. As a result, α^{RES} trivially satisfies condition Λ_2. Thus $\Re^{\mathrm{RES}}$ is covered by Theorem 4.1. We state this formally in the following corollary:

Corollary 4.1 *Let* $\#\mathbf{A} \geq 3$. *A unanimous social choice rule f*: $[\Re^{\mathrm{RES}}]^{\mathbf{N}} \rightarrow \bigcup_{\mathrm{k} \in \mathrm{K}} \mathbf{A_k}$ *is strategy-proof if and only if f is dictatorial.*

As another example, we reconsider the dominance axiom introduced in Sect. 3. Again for every $\rho \in \Pi$, let $\alpha^{\mathrm{DOM}}(\rho) = \{R \in \Re : R$ satisfies **DOM**$\}$ and write $\Re^{\mathrm{DOM}} = \bigcup_{\rho \in \Pi} \alpha^{\mathrm{DOM}}(\rho)$. Take any ordering κ of K. We already checked in Sect. 3 that α^{DOM} satisfies condition Λ_1. Moreover, **DOM** does not bring any restriction on how sets which are sub/supersets of each other are ordered—except for singleton sets. As a result, when the range of the social choice rule excludes singleton sets, α^{DOM} trivially satisfies condition Λ_2.[15] Thus $\Re^{\mathrm{DOM}}$ is covered by Theorem 4.1, as stated in the following corollary:

13 The fact that $m \notin K$ ensures the existence of ρ'.

14 Recall that $\Re$ now stands for the set of all complete and transitive orderings over $\bigcup_{\mathrm{k} \in \mathrm{K}} \mathbf{A_k}$.

15 Remark that when $1 \in K$, α^{DOM} exemplifies a failure of condition Λ_2. As another example of an extension map failing condition Λ_2, we can cite the one obtained through the Gärdenfors (1976) principle which requires $X\ P\ Y$ for any $X, Y \in 2^{\mathbf{A}}\backslash\{\emptyset\}$ with $X \subset Y$ such that $x\ \rho\ y\ \forall x \in X$ and $\forall y \in Y\backslash X$.

Corollary 4.2 *Let* $\#\mathbf{A} \geq 3$ *and* $1 \notin K$. *A unanimous social choice rule* $f : [\Re^{\mathrm{DOM}}]^{\mathbf{N}} \to \bigcup_{k\in K} \mathbf{A_k}$ *is strategy-proof if and only if* f *is dictatorial.*

As a third example, we consider preferences over sets that are additively separable. For each $\rho \in \Pi$, let $U\rho$ be the set of real-valued "utility" functions (defined over $\mathbf{A}$) which represent ρ. Let $\alpha^{\mathrm{AS}}(\rho) = \{R \in \Re : \exists u \in U\rho$ such that $X\ R\ Y \Leftrightarrow \sum_{x\in X} u(x) > \sum_{y\in Y} u(y)$ for all $X, Y \in \bigcup_{k\in K} \mathbf{A_k}\}$ and $\Re^{\mathrm{AS}} = \bigcup_{\rho\in\Pi} \alpha^{\mathrm{AS}}(\rho)$. Take any ordering κ of K. The satisfaction of condition Λ_1 is straightforward to check. As alternatives can be assigned negative "utilities", condition Λ_2 is also satisfied.[16] Hence the domain $\Re^{\mathrm{AS}}$ is covered by Theorem 4.1, as we state below:

Corollary 4.3 *Let* $\#\mathbf{A} \geq 3$. *A unanimous social choice rule* $f : [\Re^{\mathrm{AS}}]^{\mathbf{N}} \to \bigcup_{k\in K} \mathbf{A_k}$ *is strategy-proof if and only if* f *is dictatorial.*

As a final example, we consider a domain obtained through the lexicographic extension map λ defined in the proof of Theorem 3.1. For each $\rho \in \Pi$, write $\alpha^{\lambda}(\rho) = \{R \in \Re :$ given any $k \in K$ and any $X, Y \in \mathbf{A_k}$ we have $X\ R\ Y \Leftrightarrow X\ \lambda(\rho)\ Y\}$ and let $\Re^{\lambda} = \bigcup_{\rho\in\Pi} \alpha^{\lambda}(\rho)$. Take any ordering κ of K. We already checked in Sect. 3 that the lexicographic extension map satisfies condition Λ_1. Moreover, given the way $\alpha^{\lambda}(\rho)$ is defined, it does not bring any restriction on how sets of different cardinalities are ordered—hence the trivial satisfaction of condition Λ_2. Thus $\Re^{\lambda}$ is covered by Theorem 4.1, as stated in the following corollary:

Corollary 4.4 *Let* $\#\mathbf{A} \geq 3$. *A unanimous social choice rule* f: $[\Re^{\lambda}]^{\mathbf{N}} \to \bigcup_{k\in K} \mathbf{A_k}$ *is strategy-proof if and only if* f *is dictatorial.*

5 Concluding remarks

We conclude by giving a brief account of our contribution. By making use of an impossibility result by Aswal et al. (2003), we explore the strategy-proofness of resolute social choice correspondences. We start by considering the case where outcomes are committees of a unique size. Clearly, when all possible orderings over committees are allowed, the Gibbard–Satterthwaite theorem directly applies. Moreover, every restricted domain obtained through a reasonable extension map exhibits a Gibbard–Satterthwaite type of impossibility. This is quite a strong result. For, being reasonable is a very mild condition: It does not rule out any ordering as inadmissible but only requires a particular natural ordering to be among the admissible ones. In fact, virtually all the extension axioms and conditions which appear in the literature satisfy the reasonability

[16] Suppose we replace additive separability by average utility, i.e., we let $\alpha^{\mathrm{AU}}(\rho) = \{R \in \Re : \exists u \in U\rho$ such that $X\ R\ Y \Leftrightarrow (1/\#X)\sum_{x\in X} u(x) > (1/\#Y)\sum_{y\in Y} u(y)$ for all $X, Y \in \bigcup_{k\in K} \mathbf{A_k}\}$. The domain $\Re^{\mathrm{AU}} = \bigcup_{\rho\in\Pi} \alpha^{\mathrm{AU}}(\rho)$ is not covered by Theorem 4.1, as the extension map α^{AU} fails to satisfy condition Λ_2.

condition.[17] Hence, we can confidently conclude that there exists no domain restriction which allows escaping Gibbard–Satterthwaite type of impossibilities, in case we wish to relate preferences over committees to preferences over committee members in a sensible manner.

This impossibility announced by Theorem 3.1 can be carried to a more general framework where the range of the social choice rule is determined by putting a lower and an upper bound on the size of the committees. The Gibbard–Satterthwaite type of impossibilities we establish are quite general. They cover virtually all cases regarding the lower and upper bounds imposed on the sizes of the committees.[18] On the other hand, they prevail under extension maps which satisfy condition Λ which is more restrictive that being reasonable. In fact, axioms which conceive sets as non-resolute outcomes typically require a singleton set to be the best outcome—hence being inconsistent with condition Λ. So the impossibility announced in Sect. 4 purely belongs to a world where sets are resolute outcomes while, as illustrated by Corollaries 4.1, 4.2, 4.3 and 4.4, it applies to a fairly large class of extension maps.

A particular application of Theorem 4.1 is to the framework of Barberà et al. (1991) who consider social choice rules which pick a (possibly empty) set of alternatives at each preference profile over sets of alternatives. Here, individual orderings include the empty-set and they are assumed to be separable. Given that the set of admissible outcomes is unrestricted, non-manipulable social choice rules are characterized in terms of a class which Barberà et al. (1991) call *voting by committees*.[19] Aswal et al. (2003) show that restricting the set of feasible outcomes through lower and upper bounds imposed on the cardinality of admissible committees ends up in the equivalence of strategy-proofness and dictatoriality. This equivalence is "almost" implied by our Theorem 4.1, as separable preferences satisfy condition Λ.[20]

Acknowledgments We thank Maurice Salles, Arunava Sen and two anonymous referees.

References

Aswal N, Chatterji S, Sen A (2003) Dictatorial domains. Econ Theory 22(1):45–62

Barberà S (1977) The manipulation of social choice mechanisms that do not leave too much to chance. Econometrica 45:1573–1588

Barberà S, Sonnenschein H, Zhou L (1991) Voting by committees. Econometrica 59(3):595–609

[17] Many of these axioms are defined in a world where sets are conceived as non-resolute outcomes. Nevertheless, most of them incorporate an idea which can be translated to the framework of resolute social choice correspondences. For example, "dominance" in the sense of Kelly (1977); "(additive) separability" in the sense of Barberà et al. (1991); "expected utility consistency" in the sense of Barberà et al. (2001) can all be naturally adapted to our framework where they satisfy reasonability.

[18] Except for the trivial case of committee size m where all candidates are chosen.

[19] It should be remarked that their usage of the term "committee" has a different meaning than ours.

[20] The use of the qualification "almost" is due to the fact that we do not allow committees of cardinality zero, i.e., the empty set.

Barberà S, Dutta B, Sen A (2001) Strategy-proof social choice correspondences. J Econ Theory 101:374–394
Benoit JP (2002) Strategic manipulation in voting games when lotteries and ties are permitted. J Econ Theory 102:421–436
Bossert W (1995) Preference extension rules for ranking sets of alternatives with a fixed cardinality. Theory Decis 39:301–317
Campbell DE, Kelly JS (2002) A Leximin characterization of strategy-proof and non-resolute social choice procedures. Econ Theory 20:809–829
Ching S, Zhou L (2002) Multi-valued strategy-proof social choice rules. Soc Choice Welf 19:569–580
Duggan J, Schwartz T (2000) Strategic manipulability without resoluteness or shared beliefs: Gibbard–Satterthwaite generalized. Soc Choice Welf 17:85–93
Feldman A (1979a) Nonmanipulable multi-valued social decision functions. Public Choice 34:177–188
Feldman A (1979b) Manipulation and the Pareto Rule. J Econ Theory 21:473–482
Gärdenfors P (1976) Manipulation of social choice functions. J Econ Theory 13:217–228
Gärdenfors P (1978) On definitions of manipulation of social choice functions. In: Laffont JJ (ed) Aggregation and revelation of preferences. North Holland, Amsterdam
Gibbard A (1973) Manipulation of voting schemes: a general result. Econometrica 41:587–601
Kannai Y, Peleg B (1984) A note on the extension of an order on a set to the power set. J Econ Theory 32:172–175
Kelly JS (1977) Strategy-proofness and social choice functions without single-valuedness. Econometrica 45:439–446
Özyurt S, Sanver MR (2005) Almost all social choice correspondences are subject to the Gibbard–Satterthwaite theorem. Unpublished working paper
Pattanaik PK (1973) On the stability of sincere voting situations. J Econ Theory 6:558–574
Roth A, Sotomayor MAO (1990) Two-sided matching: a study in game theoretic modelling and analysis. Cambridge University Press, London
Satterthwaite MA (1975) Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. J Econ Theory 10:187–217