Strategy-proof and Efficient Mediation: An Ordinal Mechanism Design Approach*

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Abstract

Mediation is a dispute resolution method that has gained increasing popularity over the last few decades and given rise to a multi-billion-dollar industry. This paper develops an "ordinal" market/mechanism design approach, where the mediator seeks a resolution over two issues in which negotiators have diametrically opposed ordinal preferences. Each negotiator has private information about her own ranking of the outside option, i.e., the point beyond which the negotiator would rather take the case to a conventional court proceeding. A necessary and sufficient condition for the existence of strategy-proof and efficient mechanisms is the availability of "logrolling bundles" that form a special (semi)lattice structure and allow negotiators to make compromises on different issues. We characterize the full class of strategy-proof, efficient, and individually rational mediation rules. A central member of this class, the constrained shortlisting rule, stands out as the unique strategy-proof, efficient, and individually rational mechanism that minimizes rank variance.

Keywords: Mediation; logrolling; outside options; strategy-proofness. (JEL C78, D47, D74, D78, D82)

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“ *Mediation has rapidly become, with precious little fanfare, the ocean we swim in and the air we breathe. It would now be hard to imagine a world where it wasn't.*” Jim Melamed[1]

1. Introduction

The bestseller *Getting to Yes* by Roger Fisher and William Ury, is arguably one of the most famous—if not the most famous—works on the topic of negotiation. The authors identify conflict as a growth industry, and the last few decades have proved them right. Courts in all US states currently offer some form of ADR (alternative dispute resolution) for the cases filed in state courts. Seventeen states require mandatory mediation: 6% of civil cases in Northern California courts in 2011, and 35.6% of civil and 21.6% of divorce cases in New York state courts in 2016 were mediated.[2] The total value of mediated cases in the UK is estimated to be £10.5bn. in 2011, excluding mega-cases and family and community disputes.[3] Empirical studies of mediation and program evaluations suggest a 60-90% success rate, a 90-95% satisfaction rate by the disputants, and a higher rate of compliance with mediator recommendations relative to court-imposed orders. Employment, patent/copyright, construction, and family conflicts are some of the most common types of mediated disputes. In addition to face-to-face mediation practices, online dispute resolution has also gained increasing popularity over the last decade. The dispute resolution centers of ebay, PayPal, Uber, and Amazon tackle more than a billion disputes a year. Many online dispute resolution websites use automated protocols to help parties resolve their conflicts.

Unlike litigation and arbitration, mediation does not aim at finding the truth, but rather seeks mutual satisfaction through a nonbinding recommendation from which the disputants are free to walk away. In mediation, a neutral third party orchestrates a collaborative solution by facilitating communication/negotiation and promoting the exploration of mutually acceptable outcomes. The emphasis is not on who is right or wrong, but rather on establishing a workable solution that meets the participants' needs. Many proponents advocate mediation as a means to broaden the range of issues being negotiated from legal matters and remedies to other concerns and resolutions that might be important to the parties.[4] Disputants also prefer mediation over its alternatives because

[1] Founder and CEO of Mediate.com and recipient of American Bar Association Institutional Problem Solver Award.

[2] Sources: National Center for State Courts (https://www.ncsc.org) and dispute resolution center of State of New York (https://www.nysdra.org).

[3] The Seventh Mediation Audit, Centre for effective dispute resolution.

[4] For example, Rule 6 of the ADR Local Rules of the United States District Court for the Northern District of California asserts that “[a] hallmark of mediation is its capacity to expand traditional settlement discussion and broaden resolution options, often by exploring litigant needs and interests that may be formally independent of the legal issues of the controversy.”

it is more cost effective.[5] Airline companies and hospitals, for example, prefer mediation because mediation sessions are private and confidential. It is impossible to discuss a legally "irrelevant" issue in litigation/arbitration, and some disputes are not just about money or being right.[6] In mediation, however, parties can discuss and negotiate issues that are not directly linked to the case. The infinite flexibility of being able to bring any issue to the table can be used to transform a competitive, "zero-sum" negotiation problem into a "multi-issue" negotiation problem that enlarges the set of acceptable outcomes.

Notwithstanding the practical conveniences it affords, the mediation process is often considered to be less formal and less transparent than binding adjudication processes. The traditional view among legal theorists and researchers is that the competitive presentation of evidence in the formal adversarial system "counteracts decision maker bias and produces fairer and more accurate decisions than less formal systems."[7] Many supporters of this view argue that the low visibility and lack of formal rules and structure in mediation, facilitated settlement, and other relatively informal processes reduce the rights of less powerful participants.[8] This view is not entirely without support from the field. In a seminal work, LaFree and Rack (1996) provide empirical evidence from the small claims court mediation program in Bernalillo County in Albuquerque, New Mexico, and conclude that ethnicity and gender could be more important determinants in mediation than adjudication. In particular, they report that white males receive significantly more favorable outcomes in mediation than minority females.[9]

In this paper, which takes a market design approach, we argue that it may be possible to make meditation a more structured and rigorous process without compromising the mediator's primary role of facilitating communication and exploring common ground between the disputants. To this end, we search for neutral, efficient, and incentive compatible recommendation mechanisms that mediators can use for systematic and consistent decision making in practice. Market design has been fruitful in many applications, most notably in auctions and matching. The goal of this paper is to offer the first market design setting to analyze dispute resolution via mediation that is simple enough to be practically relevant while maintaining the informational richness and complexities faced

[5]According to Hadfield (2000), it costs a minimum of \$100,000 to litigate a straightforward business claim in the US, whereas a mediation session varies from few hours to a day and even the most reputable mediators charge around \$10,000 - \$15,000 for a day. In addition, disputants do not have to pay any fees for experts, witnesses, document preparation, investigation, or paralegal services, which would easily make the costs pile up.

[6]A plaintiff, for example, could be suing her employer for discrimination, and reinstatement or a good reference might be more important for her than compensation.

[7]Damaska (1975).

[8]See, for example, Nader (1969), Abel (1982), and Norton (1989).

[9]In a similar vein, many others emphasize the factors that can cause disputant dissatisfaction that are under the direct control of mediators. As a remedy, Tyler and Huo (2002) advocate the use of fair procedures that are described as those in which decisions are viewed as *neutral, objective, and consistent.*

in actual disputes. Our modeling differs markedly from the traditional mechanism design approach to bargaining that builds on the seminal work of Myerson and Satterthwaite (1983), which postulates that traders have private valuations drawn from prespecified distributions and commonly known utility functions. This type of "cardinal approach" has been the subject of the famous Wilson critique, for its lack of "detail-freeness" and potential inability to provide "robust incentives" to participants in the sense of Bergemann and Morris (2005). While stressing the powerful insights that mechanism design offers in bargaining problems, Ausubel et al. (2002) voice a similar concern:

> "... *Despite these virtues, mechanism design has two weaknesses. First, the mechanisms depend in complex ways on the traders' beliefs and utility functions, which are assumed to be common knowledge. Second, it allows too much commitment. In practice, bargainers use simple trading rules—such as a sequence of offers and counteroffers—that do not depend on beliefs or utility functions.*" [10]

We aim to avoid this type of critique via an "ordinal approach" whereby the designer truthfully elicits ordinal preference information in dominant strategies.[11] Two common justifications for the ordinal approach are the limited rationality of the agents participating in the mechanism and the genuine simplicity of implementing such mechanisms.[12] This approach has been championed by its remarkable success in applications of matching and assignment such as medical residency, college admissions, school choice, and kidney exchange, where a plethora of strategy-proof and efficient mechanisms have been proposed, extensively studied, and some even adopted in practice.

Our model assumes that two negotiators are in a dispute and aim to reach a resolution through a mediator. There is a main issue, issue X, consisting of a finite number of alternatives, which is relevant to both parties' welfare.[13] The negotiators have diametrically opposed preferences over the alternatives in the sense that if one negotiator

[10] Handbook of Game Theory, Chapter 50: Bargaining with Incomplete Information.

[11] Our view should not be taken as one of universal endorsement of ordinal mechanisms over their cardinal counterparts, but rather as advocating their use in a specific context where the former can offer the practical convenience from a market design point of view. In a general mechanism design setting, Carroll (2018) shows that one loses generality by restricting to ordinal mechanisms. He also provides a foundation for ordinal mechanisms by showing that a planner can implement her goals robustly to uncertainty about cardinal preferences only if she uses an ordinal mechanism. Another limitation of ordinal mechanisms is that they cannot elicit preference intensities. See also Jehiel et al. (2006), Carroll (2012), and Pycia (2014).

[12] There is a large body of experimental evidence that finds that the representation of preferences by VNM utility functions may be inadequate; see, for example, Kagel and Roth (2016). This literature argues that the formulation of rational preferences over lotteries is a complex process that most agents prefer not to engage in if they can avoid it.

[13] We later relax the finiteness assumption.

prefers one alternative over another, then the other negotiator has the exact opposite ranking of the two alternatives.[14] However, not all alternatives are acceptable for any given negotiator. When offered an unacceptable alternative for her, a negotiator rejects the mediator's proposal and pursues alternative means of resolution, e.g., litigation. We capture such circumstances by assuming an outside option whose ranking is each negotiator's private information. The mediator's objective is to truthfully elicit negotiators' private information about how they rank their respective outside options and propose an efficient and mutually acceptable, i.e., individually rational, outcome.

We first show that if there is a single issue, i.e., no other issues than issue X, then there is no strategy-proof, efficient, and individually rational mechanism. A long-standing and commonly recognized theme in the practice of successful negotiation has been the two sides' ability to jointly discuss multiple issues where parties would be asked to consider a compromise in one issue for a more favorable treatment in another. However, the previous impossibility cannot be avoided by simply allowing for multiple issues if each issue has an independent, similarly defined outside option, i.e., if for each issue each negotiator has an outside option whose ranking is her private information. This motivates us to consider a setting that treats different issues asymmetrically: Consider a second issue, issue Y, where the outside option is commonly known to be the least-preferred outcome for both negotiators. This asymmetric treatment of the outside options for the two issues can be motivated by various employment, family, construction, or patent/copyright infringement disputes. Litigation is naturally the default option when the issue is compensation or division of property. If parties expect litigation to be a very long and costly process, then any division of surplus would be efficient relative to litigation. In that regard, compensation could be considered to be issue Y in many such disputes.[15]

In the two-issue mediation problem,[16] the mediator recommends a bundle (x, y) of outcomes from $X \times Y$. A mediation rule is a systematic way of choosing an outcome for any reported pair of types of the two negotiators. Since the mediator asks the negotiators to report their least preferred acceptable alternatives in issue X (recall that there is no uncertainty regarding negotiators' preferences over alternatives in issue Y), one needs to

[14] We justify such modeling by showing that under a mild efficiency requirement, any situation where negotiators' preferences are not diametrically opposed can be equivalently represented as one where they are (Proposition 1).

[15] In a bilateral negotiation between a worker and an employer, for example, issue Y could represent wage and location options could represent issue X. In other contexts where Y again captures compensation, agreements over change orders, extra work requirements, the scope of work (in construction disputes), or child custody or terms of visitation (in family disputes) could be viewed as the main issue, i.e., issue X, where each negotiator is uncertain about her opponent's set of acceptable alternatives. However, it is worth noting that although money is an important issue in disputes, it is rarely the only issue (Malhotra and Bazerman, 2008).

[16] Our main model easily extends to the case of more than two issues. See Section 7.

invoke extension mappings to obtain the set of all possible underlying preferences over bundles.[17] The main question we ask is whether there is an impartial and dominant strategy incentive compatible, i.e., strategy-proof, way of soliciting true preferences so that mediation outcomes are always efficient and individually rational. By invoking a standard revelation principle with veto rights, any mediation game can be represented as a direct revelation mechanism where negotiators announce their least-acceptable alternatives in a first stage. This announcement stage is then followed by a ratification stage where negotiators are voluntarily asked to accept or veto the mediator's recommendation deliberated through the mediation rule. A mediation rule can simply be represented by an $m \times m$ matrix, where m is the number of alternatives in either issue, and rows (respectively, columns) correspond to different types of negotiator 1 (respectively, negotiator 2). A mediation game has a dominant strategy equilibrium in which recommendations are never vetoed if and only if the corresponding mediation rule is strategy-proof and (ex post) individually rational.

A sufficient condition for obtaining a positive answer to our main question is the so-called *logrolling (quid pro quo)* condition on negotiators' preferences. This assumption imposes a form of substitutability between issues X and Y. More specifically, logrolling requires preferences to be rich enough that a negotiator is able to make concessions in issue X for a more preferred alternative in issue Y, e.g., for a given pair of alternatives x and x' in X where x is preferred over x', there exists a corresponding pair of alternatives y and y' in Y such that when bundled together, (x', y') is preferred over (x, y). This condition rules out lexicographic treatment of the two issues and is compatible with many common utility functions such as the CES and the quasi-linear utility.

Our main result is a complete characterization of the class of strategy-proof, efficient, and individually rational mediation rules. These rules operate through an exogenously specified precedence order over a set of special bundles, which we call the *logrolling bundles*, and choose the highest-precedence logrolling bundle among those that are mutually acceptable to both negotiators (Theorems 1 and 2). As the precedence order varies, the characterized class of rules span what we refer to as the *family of adjacent rules*. A visual characterization of this family establishes that a rule f belongs to the family if and only if the lower half of the matrix representing f can be partitioned into rectangular regions, where each rectangle corresponds to the set of problems for which the same logrolling bundle is recommended (Theorem 3). The logrolling bundles form a simple lattice structure with respect to either negotiator's preferences: given a set of mutually acceptable

[17] Alternatively, it is conceivable that the mediator elicits full preferences over bundles of alternatives. This approach, which we do not pursue, however, has two drawbacks: First, the number of bundles to rank increases quadratically with the number of alternatives in each issue, which in turn makes asking for full-fledged rankings over bundles highly impractical for our problem. Second, an impossibility similar to that in single-issue mediation would arise in this case.

alternatives, for each negotiator there is always an optimal-logrolling bundle that she prefers over all other acceptable logrolling bundles; this bundle is the pessimal-logrolling bundle for the opposite negotiator. As a consequence, the family of adjacent rules nest interesting special members. When the precedence order coincides with the preference ranking of a given negotiator over the logrolling bundles, we obtain the corresponding negotiator-optimal rule.

A negotiator-optimal rule represents situations when a mediator may be categorically biased toward one side of the dispute. In keeping with our main objective of finding impartial mediation rules, we search for members of this class of rules that satisfy sensible fairness criteria. To this end, we define the "rank variance" of an outcome as the sum of the square of each negotiator's ranking of each alternative in each issue. It turns out there is a unique member of the family of strategy-proof, efficient, and individually rational mediation rules that minimizes rank variance across all mediation problems (Theorem 4). This is the so-called *constrained shortlisting* rule, which recommends the median logrolling bundle when it is mutually acceptable, and when it is not, the mutually acceptable logrolling bundle closest to it. An equivalent way for the mediator to implement this rule is as follows: The mediator solicits from one negotiator her favorite acceptable logrolling bundle. He then offers a menu of options to the other negotiator, which includes this bundle together with the median bundle if it is acceptable to the first negotiator and an efficient disagreement bundle.

Logrolling assumes that it should be possible to reverse the ranking of every pair of alternatives in issue X when bundled with a corresponding pair in issue Y. While sufficient, the full power of this condition is not necessary for obtaining a possibility result. We show that a weaker form of logrolling that requires only certain pairs of alternatives in issue X to have a compensating pair of alternatives in issue Y is both necessary and sufficient to obtain a strategy-proof, efficient, and individually rational rule (Theorem 5). Alternatively, we offer an order-theoretic necessary and sufficient condition for the existence of a possibility result. Although the full logrolling condition implies a simple lattice structure, we find that to acquire a positive result, it is both necessary and sufficient for the logrolling bundles to form a specific type of semilattice structure. More precisely, a strategy-proof, efficient, and individually rational rule exists on a preference domain if and only if any linked subset of the set of logrolling bundles forms a join-semilattice when equipped with a partial order based on a concatenation of negotiators' preferences (Theorem 6).

An important extension of our model concerns situations where each issue consists of a continuum of alternatives. For a continuous analogue of our main model, we show that a mediation rule is strategy-proof, efficient, and individually rational if and only if it maximizes a strict and quasi upper-semicontinuous precedence relation over the set of logrolling bundles (Theorem 7).

Related Literature

Both our paper and our modeling approach span and connect literature on bargaining, two-sided assignment/matching, fair division, and political economy.

Foremost, mediation has been studied as part of the traditional bargaining literature with incomplete information, which is primarily based on the cardinal approach discussed above. A central question is whether private information prevents the bargainers from reaping all possible gains from trade. The mechanism design approach to this problem was pioneered by the classic paper by Myerson and Satterthwaite (1983) [henceforth MS], which shows that for a model with transferable utility, there is no ex post efficient, individually rational, and Bayesian incentive compatible mechanism when there is uncertainty about whether gains are possible. The MS impossibility crucially depends on types being independent.[18] In our setup, since each negotiator privately knows the realization of her outside option—which is also assumed to be independent of the other negotiator's realization—the MS model can be seen to correspond roughly to a case where outside options are treated symmetrically. This helps explain the impossibility of attaining strategy-proof, efficient, and individually rational mediation with symmetric outside options (see the detailed discussion in Supplementary Appendix about how our model compares with the MS setting).[19]

On the context of mediation, specifically, there are very few papers. For a model featuring a continuum of types, Bester and Warneryd (2006) show that asymmetric information about relative strengths as an outside option in a conflict may render agreement impossible even if there is no uncertainty about the agreement being efficient. In their model, conflict shrinks the pie and agreement on a peaceful settlement is always ex post efficient. Following Bester and Warneryd (2006), Hörner et al. (2015) compare the optimal mechanisms with two types of negotiators under arbitration, mediation, and unmediated communication. They show that there is no ex post efficient and Bayesian incentive compatible mechanism: the optimal mechanism is necessarily inefficient. Börgers and Postl (2009) consider an arbitration problem with three alternatives (and no outside options) between two agents who have diametrically opposed ordinal preferences. Their

[18]Subsequently, it was shown that efficient trade may be possible when types are correlated (e.g., Gresik 1991 and McAfee and Reny 1992).

[19]Compte and Jehiel (2007) consider bargaining problems where outside options are private but correlated and parties have a veto right similarly to our mediation game. They show that inefficiencies are inevitable whatever the exact form of correlation, which resonates with the negative result in our benchmark model of single-issue mediation. This contrasts with the famous full rent extraction result of Crémer and McLean (1988) mainly because of the assumption that agents can quit the mechanism. We also adopt this perspective in our modeling and allow agents to veto a recommendation to exercise their outside options.

cardinal setting assumes that agents' utilities of the middle-ranked alternative are i.i.d. and privately observed. They show that there is no rule that truthfully elicits utilities and implements efficient outcomes.

Obtaining a possibility result in our model hinges crucially on the availability of (at least) a second issue. Linking multiple decisions/issues to overcome welfare and incentive constraints has been a useful tool in many economic applications such as bundling of goods by a monopolist (e.g., McAfee et al. 1989), agency problems (e.g., Maskin and Tirole 1990), and logrolling in voting (e.g., Wilson 1969). A common insight in these approaches is based on applying a law of large numbers theorem to ensure that truthtelling incentives are restored in a sufficiently large market. In this vein, Jackson and Sonnenschein (2007) show that by linking different issues in many situations, including the bilateral bargaining setting of MS, it is possible to achieve outcomes that are approximately efficient in an approximately incentive compatible way as the number of issues goes to infinity. In contrast with these approaches, we establish efficiency in dominant strategies with only two issues in an application where the number of potential issues is inherently limited.

Our "ordinal approach" to mediation is inspired primarily by the success of market design in several matching and assignment problems such as school choice, organ exchange, course allocation, landing slot assignments, and cadet-branch matching.[20] Our setup shares important conceptual and mathematical parallels with two-sided matching and assignment models,[21] when the negotiators are viewed to be on opposite sides. Conceptually, logrolling bundles are analogous to stable matchings in the sense that the negotiators have opposite preferences over them: if one negotiator would rather have the mediator recommend an alternative logrolling bundle, the other negotiator would object to it.[22] Mathematically, the set of logrolling bundles coupled with either side's preferences forms a simple lattice (or a semilattice in the extended model) much like the structure of the set of stable matchings in a two-sided market. Notably, a certain type of semilattice structure proves both necessary and sufficient for attaining strategy-proofness in our model. As a consequence, one connection with matching models that surfaces is that we find the class of strategy-proof, efficient, and individually rational rules to contain the negotiator-optimal rules. These rules, though logically unrelated, are akin to the proposing-side optimal deferred acceptance mechanisms in two-sided matching and the buyer/seller optimal core assignments in the Shapley-Shubik assignment game.

Although ordinal mechanisms are known to achieve better incentive properties than

[20]See, for example, Gale and Shapley (1962), Shapley and Shubik (1971), Crés and Moulin (2001), and recent applications of ordinal assignment mechanisms such as Bogomolnaia and Moulin (2001), Abdülkadiroglu and Sönmez (2003), Budish (2011), Switzer and Sönmez (2013), and Ergin et al. (2017).

[21]See Roth and Sotomayor (1990) for a survey of two-sided matching and assignment problems.

[22]The same can also be said for individual alternatives bundled together in a logrolling bundle: whenever one negotiator would favor a different alternative in either issue than the corresponding alternative in the logrolling bundle, the other would oppose to the change.

their cardinal contenders in matching problems,[23] strategy-proofness (for all participants) is not guaranteed in general. In two-sided one-to-one and many-to-one matching problems, for example, a stable mechanisms can be strategy-proof only for one side of the market (see e.g., Roth and Sotomayor 1990).[24]

Absent outside options, our setting can also be viewed as a type of multi-unit assignment problem, e.g., course allocation, with only two agents, each of whom needs to be assigned two objects, one from each of two sets A and B, where an alternative in issue X (respectively Y) represents a specific pair of objects from set A (respectively B) that must be assigned simultaneously.[25] The multi-unit assignment setting, however, provides little reason to remain optimistic for positive results. The literature contains a series of papers that show impossibility results. The main result of this literature is that the only strategy-proof and efficient rules are serial dictatorships; e.g., see Papai (2001), Klaus and Miyagawa (2002), and Ehlers and Klaus (2003).[26] Clearly, dictatorship rules have little appeal in a dispute resolution situation. Worse still, dictatorships violate individual rationality in our model,[27] i.e., such recommendations will be vetoed in equilibrium.

A dispute resolution problem can also be interpreted as a type of fair division problem involving indivisible items. The focal rule within the characterized family of adjacent rules, the constrained shortlisting rule, allows one negotiator to effectively reduce the set of possible outcomes to a short list, from which the other negotiator makes her favorite selection. In that sense, the constrained shortlisting rule is reminiscent of the well-known

[23]For example, the most prominent cardinal mechanism in the context of unit-assignment problems (possibly allowing for stochastic assignments), the competitive equilibrium from equal incomes solution (Hylland and Zeckhauser 1979), is not strategy-proof. This difficulty of achieving strategy-proofness is generally attributed to the tension with efficiency since cardinal mechanisms achieve stronger welfare properties (e.g., maximization of utilitarian welfare) than ordinal mechanisms. Zhou (1990) shows that no cardinal mechanism is strategy-proof, efficient, and symmetric, whereas ordinal mechanisms, e.g., random priority, are well known to attain the three properties.

[24]Similar to the literature on linking decisions, a common method of circumventing these impossibilities is to resort to large market arguments by allowing for the number of participants and resources to grow. Once again, such methods are obviously inapplicable in the context of mediation.

[25]Suppose set A contains three objects in the order of decreasing desirability, a, b, and c, where a and c are in unit supply and b has two copies. Then issue X can be viewed as consisting of the following object pairings $X = \{(a, c), (b, b), (c, a)\}$. That is, if one agent gets a, the other must get c, and b cannot be assigned together with any other object.

[26]Results continue to be negative with private endowments (Konishi, Quint, and Wako, 2001) and even with stochastic mechanisms (Kojima, 2009). In the course allocation context, two notable contributions that identify nondictatorship mechanisms are Sönmez and Ünver (2010) and Budish (2011). The former paper argues for eliciting bids from students together with ordinal preferences over courses and then using a Gale-Shapley mechanism where bids are interpreted as course priorities. However, the mechanism is strategy-proof only if the bids are treated as exogenously given. The latter paper proposes an approximately efficient mechanism that is incentive compatible in a large market.

[27]A constrained dictatorship where one negotiator maximizes her welfare among the set of mutually acceptable outcomes would satisfy individual rationality, but such a rule is easily manipulable.

biblical rule of divide-and-choose, which has been extensively studied in fair cake-cutting problems. Two advantages of the constrained shortlisting relative to divide-and-choose is that it is strategy-proof and its outcome is independent of the ordering of the negotiators. More generally, the fair division literature almost exclusively focuses on fairness and efficiency issues due to inherent incompatibilities with strategy-proofness similar to those in the multi-unit assignment context; see, e.g., Brams and Taylor (1996).

In our setting, in contrast to a matching or a fair division model, mediation is an entirely voluntary process. As such, the mediator has no enforcement power and the negotiators are free to walk away to exercise their private outside options. Such lack of commitment to the mediator's recommendation causes negotiators to create negative externalities on each other. When one negotiator chooses to exercise her outside option by vetoing the proposal, the other negotiator is automatically compelled to also exercise her outside option.

Our model also resembles a voting setting where a number of voters have single-peaked preferences over the political spectrum and a voting scheme aggregates individual preferences (Black 1948). In our model, when restricted to each issue, preferences can also be seen as single-peaked, with each negotiator preferring the opposite extremes of the spectrum. This type of voting domains allows to overcome the Gibbard-Satterthwaite impossibility, and the famous median voter theorem states that the majority-rule voting system that selects the Condorcet winner, i.e., the outcome most preferred by the median voter, is strategy-proof; see Moulin (1980) for a classic generalization of this result. The constrained shortlisting rule can be viewed as similar to a Condorcet winner in the sense that it recommends the median logrolling bundle when the median is mutually acceptable for both negotiators and the closest logrolling bundle to it when it is not. Nevertheless, this connection is superficial as our model differs in several ways from a voting framework. In these voting models, there are several voters whose bliss point (peak value) is their private information, whereas in our model there are two agents (the negotiators) whose peaks in each issue are publicly known. What is private information here consists in the two negotiators' outside options, which have no analogues in a voting model. Consequently, there is no clear way to adopt such voting schemes in our setup, as they would violate individual rationality. Moreover, the above analogy between the two types of models applies only when each issue is considered separately, since negotiators' underlying joint preferences over bundles in our two-issue model are not necessarily single-peaked.[28]

[28]An alternative view could be based on a multi-issue voting setting. However, in multidimensional voting models where people vote on several issues, a main conclusion is that strategy-proofness effectively requires each dimension to be treated separately in the sense that each dimension should admit its own generalized median voter schemes, see, e.g., Barberà et al. (1991) and Barberà et al. (1997). Our strategy-proofness result, by contrast, depends critically on having more than one dimension and relies

Finally, with the hope of arriving at possibility results, there is a tradition of searching for strategy-proof mechanisms in restricted economic environments that make it possible to escape Arrow-Gibbard-Satterthwaite impossibilities. Well-known examples include VCG mechanisms (Vickrey 1961, Groves 1973, and Clarke 1971) for public goods and private assignment with transfers;[29] the uniform rule (Sprumont 1991) for the distribution of a divisible private good under single-peaked preferences; generalized median-voters (Moulin 1980); proportional-budget exchange rules (Barberà and Jackson 1995) that allow for trading from a finite number of prespecified proportions (budget sets), deferred acceptance (Gale and Shapley 1962) and top trading cycles (Shapley and Scarf 1974, Abdülkadiroglu and Sönmez 2003); and hierarchical exchange and brokerage (Pápai 2001, Pycia and Ünver 2017). We also add to this literature by introducing and characterizing an entirely new class of strategy-proof and efficient rules.

The rest of the paper is organized as follows. Section 2 introduces the problem with a simple example. Section 3 describes the main model. Section 4 provides various characterizations of strategy-proof, efficient, and individually rational rules. Section 5 gives two sets of necessary and sufficient conditions for the existence of a strategy-proof mediation rule. Section 6 extends the main model to the case when there is a continuum of alternatives. Section 7 provides a discussion of our modeling assumptions and other extensions. Section 8 concludes. The Supplementary Appendix contains illustrations of logrolling and the proof of the revelation principle. All proofs are relegated to the Appendix.

2. The Environment

We begin by describing the environment with a simple example and a brief discussion about why the assumption of diametrically opposed preferences is without loss of generality.

A simple example: Single-issue mediation

Negotiators 1 and 2 are in dispute over a single issue that is important to both of them. Let x_1 and x_2 denote the available alternatives for (solutions to) the dispute. Each negotiators is also entitled to a private outside option, o, e.g., the negotiator's private

heavily on leveraging the exchangeability between the two issues together with an asymmetric treatment of outside options.

[29] One may draw a conceptual parallel with the family of adjacent rules and the VCG mechanisms in a combinatorial multi-item setting with transferable utility. In the VCG model, cardinal preferences over items are private information and the preferences over money is common knowledge. This is much like negotiators' preferences over issue X (where complete preference rankings including the outside option are private information) versus issue Y (where complete preference rankings including the outside option are publicly known). Note, however, that VCG mechanisms are cardinal; assignments and transfers depend on reported utilities. Also see Supplementary Appendix.

belief about the outcome of an alternative adjudication process, in case one or both of them reject the mediator's recommendation. Therefore, the set $X = \{x_1, x_2, o\}$ denotes the set of all possible outcomes of the dispute.

It is common knowledge that the negotiators have diametrically opposed preferences over the alternatives x_1 and x_2, i.e., negotiator 1 (strictly) prefers alternative x_1 to x_2 and negotiator 2 prefers x_2 to x_1. The ranking of the outside option, however, is the negotiators' private information. Therefore, each negotiator has two types:[30]

$\theta_1^{x_1}$	$\theta_1^{x_2}$
x_1	x_1
o	x_2
x_2	o

$\theta_2^{x_2}$	$\theta_2^{x_1}$
x_2	x_2
o	x_1
x_1	o

Consider the mediation process, denoted by f, as a mechanism with veto rights that maps the negotiators' private information to an outcome in X. Then, it would be represented by the following matrix:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$
$\theta_1^{x_1}$	$f_{1,1}$	$f_{1,2}$
$\theta_1^{x_2}$	$f_{2,1}$	$f_{2,2}$

where $f_{\ell,j} \in X$ for all $\ell, j \in \{1, 2\}$.

We can assign $f_{1,2} = o$, without loss of generality, because there is no mutually acceptable alternative when the negotiators' types are $\theta_1^{x_1}$ and $\theta_2^{x_2}$, and thus, the outside option o is effectively the only result in all voluntary mediation processes. If the outcomes of the mediation process are (Pareto) efficient, then $f_{1,1}$ should be x_1 or x_2. Moreover, if the process produces individually rational outcomes, then we must have $f_{1,1} = x_1$. Likewise, an efficient and individually rational mediation process suggests $f_{2,2} = x_2$ and $f_{2,1} \in \{x_1, x_2\}$.

However, none of these processes is immune to strategic manipulation. To see this point, suppose that $f_{2,1} = x_1$. In this case, type $\theta_2^{x_1}$ of negotiator 2 would deviate and declare his type as $\theta_2^{x_2}$ to obtain x_2, contradicting strategy-proofness. Symmetrically, if $f_{2,1} \neq x_1$, then type $\theta_1^{x_2}$ of negotiator 1 would deviate and declare her type as $\theta_1^{x_1}$ to obtain x_1, again contradicting strategy-proofness.

Note that this impossibility prevails even when the mediation mechanism is allowed to be stochastic.[31] It is straightforward to extend this example to the case with more than two alternatives, and so extrapolate that there exists no strategy-proof, efficient, and individually rational single-issue mediation process.

[30] We assume, without loss of generality, that there is at least one acceptable alternative for each negotiator.

[31] In that case, the only difference in the argument would be that $f_{2,1}$ would choose a lottery over x_1 and x_2. However, the above deviations would still remain profitable.

Modeling conflicting preferences

When describing a dispute, using diametrically opposed preferences over alternatives is intuitive because it resembles the standard bargaining problem, which is modeled as a zero-sum game. It is also unavoidable when the number of available alternatives is just two. However, it is conceivable that many other situations, where preferences are not necessarily diametrically opposed, could also depict a dispute where there are more than two alternatives. Consider, for example, a case where the set of available alternatives (other than the outside option) is $A = \{x_1, x_2, x_3, x_4, x_5\}$ and the negotiators' preferences are as follows:

θ_1	θ_2
x_1	x_3
x_2	x_5
x_3	x_4
x_4	x_2
x_5	x_1

These preferences are not diametrically opposed, but they are certainly conflicting—to some degree—as the agents cannot agree on their best alternative. Notice, however, that alternatives x_4 and x_5 are (Pareto) dominated by x_3, and so, if selecting an efficient outcome by the mediation protocol is desired, then the presence of these two alternatives is irrelevant for the problem. Thus, this particular dispute problem can be transformed into a simplified and "outcome equivalent" version where the only available alternatives are x_1, x_2, and x_3 and the negotiators' preferences over these three are diametrically opposed. We can generalize this observation for any (discrete) set of alternatives and for any preference profile, where negotiators cannot agree on their first best alternative.

Let A be nonempty set of available alternatives and Θ be the set of all complete, transitive, and antisymmetric preference relations on A. Define $max(\theta)$ as the maximal element of the preference ordering $\theta \in \Theta$, namely if $x^\star = max(\theta)$, then $x^\star \ \theta \ x$ for all $x \in A \setminus \{x^\star\}$. Therefore, a ***two-person, single-issue dispute*** (dispute in short) problem is a list $D = (\theta_1, \theta_2, A)$ where $\theta_i \in \Theta$ for $i = 1, 2$ and $max(\theta_1) \neq max(\theta_2)$.

For any nonempty subset $\widetilde{A} \subseteq A$, let $\theta|_{\widetilde{A}}$ denote the restriction of the preference ordering $\theta \in \Theta$ on $\widetilde{A}$. Therefore, define $\widetilde{D} = (\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{A})$ to be a dispute reduced from $D = (\theta_1, \theta_2, A)$ whenever $\widetilde{A} \subseteq A$ and $\widetilde{\theta}_i = \theta_i|_{\widetilde{A}}$ for $i = 1, 2$.

Proposition 1. *By eliminating all the Pareto inefficient alternatives, any dispute D can be reduced to an equivalent dispute $\widetilde{D}$ where the negotiators' preferences are diametrically opposed.*

All proofs are deferred to the Appendix. A similar result, which we omit for brevity, holds for two-person, multi-issue disputes whenever preferences over bundles satisfy monotonicity.[32]

[32]See the next section for the formal definition of monotonicity.

3. The Main Model: Multi-Issue Mediation

There are two negotiators, $\mathrm{I} = \{1, 2\}$, in a dispute who aim to reach a resolution through a mediator. Without loss of generality, there are two **issues** that are important to the negotiators' welfare.[33] Let the sets $X = \{x_1, ..., x_m, o_X\}$ and $Y = \{y_1, ..., y_m, o_Y\}$, where $m \geq 2$, respectively denote the finite sets of potential **outcomes** for the main issue and the second issue. We assume, for expositional simplicity, that the cardinality of the sets of alternatives in the two issues is the same.[34] The sets $X \setminus \{o_X\}$ and $Y \setminus \{o_Y\}$ are the sets of available **alternatives**. Each negotiator is entitled to an **outside option** (disagreement point) for each issue, o_X and o_Y, in case either negotiator refuses to accept an alternative that is available for that issue. We refer to negotiators as "she" and to the mediator as "he".

Preferences over Outcomes: The negotiators' preferences over outcomes for each individual issue satisfy the following three conditions:

1. The negotiators' preferences over alternatives (not including the outside option) for each individual issue are diametrically opposed and public information.

2. Each negotiator's ranking of her outside option o_X (relative to the other alternatives) in issue X is her private information.

3. It is public information that both negotiators rank the outside option o_Y in issue Y as their worst outcome.

More formally, for any issue $Z \in \{X, Y\}$, where $Z = \{z_1, ..., z_m, o_Z\}$, let Θ_i^Z denote the set of all complete, transitive, and antisymmetric preference relations of negotiator $i \in \mathrm{I}$ over issue Z and θ_i^Z denote an ordinary element of the set Θ_i^Z. It is publicly known that $z_k\ \theta_1^Z\ z_{k+1}$ and $z_{k+1}\ \theta_2^Z\ z_k$ for all $k = 1, ..., m-1$. Namely, the negotiators' preferences over the alternatives for each issue are diametrically opposed (the first condition). The ranking of the outside option in issue X, o_X, is the negotiators' private information (the second condition). Finally, it is common knowledge that $y\ \theta_i^Y\ o_Y$ for all i and $y \in Y \setminus \{o_Y\}$ (the third condition). Therefore, the set of acceptable alternatives for issue X is privately known by the negotiators, and it is unknown to them whether there is a mutually acceptable alternative for that issue. However, any alternative in issue Y is acceptable to both negotiators and efficient. Note that there is a unique preference ordering in Θ_i^Y and $m+1$ possible orderings in Θ_i^X. Without loss of generality, we ignore

[33]The extension to the case with more than two issues is discussed in Section 7.

[34] This assumption is without loss of generality. All that is needed for our results to go through is that the number of alternatives in issue Y must be greater than or equal to the number of alternatives in issue X. A more detailed discussion of how to extend the analysis to the case where $\#Y > \#X$ is deferred to Section 7.

those types that declare all alternatives in X unacceptable. Let $\Theta_i = \Theta_i^X$ denote the set of all **types** for negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ the set of all type profiles. For the rest of the paper we use θ_i instead of θ_i^X to indicate negotiator i's preferences over the outcomes in issue X. However, when we need to distinguish i's preferences over issues X and Y, then we use θ_i^X and θ_i^Y, respectively.

The asymmetric treatment of the outside options in the above formulation is motivated by practical and theoretical considerations. For example, the quality of the reference letter that a former employer would be willing to write (in an employment dispute), or the terms of child custody or visitation (in a family dispute), could be considered the main issue. In this type of issues, typically every alternative is not acceptable for a negotiator. Moreover, a negotiator's ranking of the outside option is not clear to all the parties. Thus, such situations would correpond to issue X. In various employment, family, construction, or patent/copyright infringement disputes, litigation is the standard form of resolution in cases of disagreement when the issue is one of monetary compensation or division of property. Often in such disputes litigation, i.e., the outside option, is a very long, administratively costly, and highly inefficient process relative to other potential divisions (alternatives), as discussed in the Introduction. As such, these types of issues would correspond to issue Y in our framework. Nonetheless, it is possible to find situations where the ranking of the outside option in both issues is the negotiators' private information. For that reason, the symmetric treatment of the outside option in both issues is formally investigated in Section 7. Unfortunately, it turns out that it is not possible to achieve strategy-proof, efficient, and individually rational mediation in this case, which also leads us to focus on the current setting.

Preferences over Bundles: A **bundle** (x, y) is a vector of outcomes, one for each issue, and the set $X \times Y$ denotes the set of all bundles. Let $\Re$ denote the set of all complete and transitive binary relations over the bundles. Relation R is a standard element of the set $\Re$, and for any two bundles $b, b' \in X \times Y$, $b \; R \; b'$ means "b is at least as good as b'." Let P denote the strict counterpart of R.[35] We assume that the mediator asks each negotiator to report her type, i.e., her least-acceptable alternative in X, rather than her full-fledged preferences over all bundles. Negotiators' underlying preferences over bundles are then assumed to be compatible with the reported types and to satisfy certain conditions, which we shortly define. While it is not without loss of generality, we offer two justifications for this assumption. First and foremost, asking negotiators to report full preferences over $(m+1)^2$ bundles is arguably impractical and cumbersome, which conflicts with the ease and convenience expected from the informal mediation process. Second, it can be shown that such a modeling would again lead to an impossibility similar

[35]That is, $b \; P \; b'$ if and only if $b \; R \; b'$ but not $b' \; R \; b$.

to that in the case of single-issue mediation. To obtain the set of possible preferences compatible with the reported types, we invoke an extension map that satisfies certain regularity axioms. Using extension maps to deduce complete preferences is a common tool in social choice theory pioneered by Barberà (1977) and Kelly (1977) as a way to explore the strategy-proofness of social choice correspondences.[36] An extension map is a rule Λ that assigns to every negotiator i and type $\theta_i \in \Theta_i$ a nonempty set $\Lambda(\theta_i) \subseteq \Re$ of admissible orderings over bundles.

For any negotiator i and type $\theta_i \in \Theta_i$, let $A(\theta_i) = \{x \in X | \; x \; \theta_i \; o_X\}$ denote the set of **acceptable** alternatives in issue X. For any type profile $(\theta_1, \theta_2) \in \Theta$, let the set $A(\theta_1, \theta_2) = \{x \in X | \; x \; \theta_i \; o_X \text{ for all } i \in I\}$ denote the set of **mutually acceptable** alternatives in issue X. In case we need to specify a type's acceptable alternatives, we use $\theta_i^x \in \Theta_i$: it denotes the preference relation (type) of negotiator i in which alternative $x \in X$ is the least acceptable alternative. Namely, for any $x' \in X \setminus \{o_X\}$, $x \; \theta_i^x \; x' \implies o_X \; \theta_i^x \; x'$.

Definition 1. *The* ***extension map*** Λ *is* ***regular*** *if the following hold for all* i, $\theta_i \in \Theta_i$ *and all* $R_i \in \Lambda(\theta_i)$*:*

i. [Monotonicity] For any $x, x' \in X$ *and* $y, y' \in Y$ *with* $(x, y) \neq (x', y')$,

$$(x, y) \; P_i \; (x', y') \text{ whenever } \left[x \; \theta_i^X \; x' \text{ or } x = x'\right] \text{ and } \left[y \; \theta_i^Y \; y' \text{ or } y = y'\right].$$

ii. [Deal-breakers] For any $y, y' \in Y$ *and* $x, x' \in X$ *with* $x \; \theta_i \; o_X \; \theta_i \; x'$ *and* $y \neq o_X$,

$$(x, y) \; R_i \; (o_X, y') \; R_i \; (x', y).$$

Monotonicity is a standard requirement. The second condition says that a bundle with an acceptable alternative in the main issue is always preferred over a bundle with the outside option, which in turn is always preferred over a bundle with an unacceptable alternative, regardless of the alternatives chosen for the second issue. In particular, unacceptable alternatives in issue X are "deal-breakers" for the negotiators. Namely, a bundle including an unacceptable alternative is never acceptable regardless of the alternative it chooses for the second issue. Put differently, for an unacceptable alternative x in issue X, there is no sufficiently attractive alternative y in issue Y that can make the bundle combining the two alternatives acceptable. For example, in an employment negotiation between a candidate and a company with multiple offices in different cities/countries, the candidate would have strict locational preferences that make some alternatives unacceptable regardless of the wage offered by the employer. Similarly, in family disputes, certain

[36]For such an analysis to be carried out, individual preferences over sets are required. A typical approach is to infer this information from individual preferences over alternatives through certain extension axioms which assign to every ordering over alternatives a list of acceptable orderings over sets. See also Duggan and Schwartz (2000) and Barberà et al. (2001).

terms of visitation may be unacceptable independent of the outcome of the division of family property. We make this assumption for mathematical tractability, but otherwise, it is not needed for a possibility result. (See Section 7 for further discussion, where we provide a strategy-proof, efficient, and individually rational mediation rule in a domain of preferences that violate this assumption.)

Direct Mechanisms with Veto Rights: Mediation would potentially be a very complicated, multistage game between the negotiators and the mediator. The mediation protocol, whatever the details may be, produces proposals for agreement that are always subject to unanimous approval by the negotiators. That is, before finalizing the protocol, each negotiator has the right to veto the proposal and exercise her outside option.

A version of the revelation principle, which we prove in the Supplementary Appendix, guarantees that we can stipulate the following type of a direct mechanism without loss of generality when representing mediation. The direct mechanism consists of two stages: an *announcement* stage and a *ratification* stage; and it is characterized by a mediation rule $f : \Theta \to X \times Y$. After being informed of her type, each negotiator i privately reports her type, $\widehat{\theta}_i$, to the mediator, who then proposes $f(\widehat{\theta}_1, \widehat{\theta}_2) \in X \times Y$. In the ratification stage, each party decides whether to accept or veto the proposed bundle. If both negotiators accept the proposed bundle, then it becomes the final outcome. If either or both negotiators veto the proposal, each party gets the outside option for both issues, i.e., (o_X, o_Y).[37] Such two-stage mechanisms will be called *direct mechanisms with veto rights*. Next we define the main properties we impose on mediation rules.

Definition 2. *The mediation rule f is **strategy-proof** if for all i and all $\theta_i \in \Theta_i$, $f(\theta_i, \theta_{-i})\ R_i\ f(\theta'_i, \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$, $\theta'_i \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$.*

Definition 3. *The mediation rule f is **individually rational** if for all i and all $(\theta_i, \theta_{-i}) \in \Theta$, $f(\theta_i, \theta_{-i})\ R_i\ (o_X, o_Y)$ for all $R_i \in \Lambda(\theta_i)$.*

Definition 4. *The mediation rule f is **efficient** if there exists no $(\theta_i, \theta_{-i}) \in \Theta$ and $(x', y') \in X \times Y$ such that $(x', y')\ R_i\ f(\theta_i, \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$ and all $i \in I$, and for at least one $i \in I$, $(x', y')\ P_i\ f(\theta_i, \theta_{-i})$ for some $R_i \in \Lambda(\theta_i)$.*

Strategy-proofness requires truthful revelation of one's type to be her dominant strategy whatever her underlying preferences may be regardless of the type the opposite negotiator reports. It is worth noting that, coupled with the extension map Λ, this is a stronger incentive requirement than a standard strategy-proofness property that would be based on a full preference report of negotiators. As will be discussed following Theorem

[37] It does not matter whether voting in the ratification stage is simultaneous or sequential. However, it is critical that the parties vote on the proposed bundle as a whole rather than voting on each component separately. As we further discuss in Section 7, the latter alternative leads to impossibility.

1, this will in turn require us to impose additional restrictions on the underlying preferences to attain possibility results. Individual rationality guarantees an outcome at least as good as what each negotiator would receive were she to walk away from mediation and thus ensures that the mediator's proposals are never vetoed no matter what the underlying preferences are. Efficiency says that it should not be possible to find an alternative proposal that would make both parties better off at all possible preferences and one party strictly better off at at least one preference profile. Contrary to the improved strength of strategy-proofness, the dependence of the efficiency definition on the extension map Λ implies a weaker form of efficiency than would be the case under a standard requirement. In fact, it is easy to verify that an outcome is efficient (individually rational) if and only if it is issue-wise efficient (individually rational), i.e., an efficient bundle chooses a mutually acceptable alternative for issue X when such an alternative exists (and never chooses a mutually unacceptable alternative for issue X otherwise) and never chooses the outside option for issue Y; an individually rational bundle never chooses an alternative from X that is unacceptable to at least one negotiator.

We seek direct mechanisms with veto rights in which truthful reporting of types at the announcement stage is a *dominant strategy equilibrium* and the mediator's proposals are never vetoed in equilibrium. It immediately follows from the definitions that such an equilibrium exists if and only if the mediation rule f is strategy-proof and individually rational.[38]

4. Main Results: Strategy-Proof Mediation

It is convenient to represent a mediation rule f by an $m \times m$ matrix $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$, where $f_{\ell,j} = f(\theta_1^{x_\ell}, \theta_2^{x_j})$ and $M = \{1, ..., m\}$. The rows of this matrix correspond to possible types of negotiator 1 and the columns to possible types of negotiator 2.

$$f = \begin{array}{c|c|c|c|} & \theta_2^{x_1} & \cdots & \theta_2^{x_m} \\ \hline \theta_1^{x_1} & f_{1,1} & \cdots & f_{1,m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \theta_1^{x_m} & f_{m,1} & \cdots & f_{m,m} \\ \hline \end{array}$$

More specifically, in the matrix representation above, row (respectively, column) ℓ indicates the type of negotiator 1 (respectively, 2) that finds all alternatives $\{x_k | k \leq \ell\}$ (respectively, $\{x_k | k \geq \ell\}$) acceptable. For any reported pair of types $(\theta_1^{x_\ell}, \theta_2^{x_j})$, rule f chooses an outcome $f_{\ell,j} \in X \times Y$. It follows that there is a unique mutually acceptable alternative in issue X for the type pairs that correspond to an entry on the diagonal of the matrix, i.e., $\{f_{\ell,\ell} \mid \ell \in M\}$. Furthermore, there is no mutually acceptable alternative in

[38]See the Supplementary Appendix.

issue X for the type pairs that correspond to an entry in the upper half of the matrix. We start with a partial characterization of the set of strategy-proof, efficient, and individually rational mediation rules.

Theorem 1. *Suppose that the extension map Λ is regular and f is a strategy-proof, efficient, and individually rational mediation rule. Then there exists a unique one-to-one function $t : X \to Y$ such that for all i, $\theta_i \in \Theta_i$ and all $x, x' \in A(\theta_i)$, we have $x\ \theta_i\ x' \implies t(x')\ \theta_i^Y\ t(x)$. Furthermore, the following hold for all $\ell, j \in M$:*

(i) If $\ell < j$, then $f_{\ell,j} = (o_X, y)$ for some $y \in Y \setminus \{o_Y\}$.

(ii) If $\ell = j$, then $f_{\ell,j} = (x_\ell, t(x_\ell))$.

(iii) (Adjacency) If $\ell > j$, then $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$, where $\mathbf{B} \equiv \{(x_k, t(x_k)) \in X \times Y \big| k = 1, ..., m\}$, and there exists a strict (i.e., complete, transitive, and anti-symmetric) precedence order $\rhd$ on $\mathbf{B}$ such that

$$f_{\ell,j} = \begin{cases} f_{\ell-1,j}, & \text{if } f_{\ell-1,j} \rhd f_{\ell,j+1} \\ f_{\ell,j+1}, & \text{oth.} \end{cases}$$

Part (i) of Theorem 1 says that when there is no mutually acceptable alternative in issue X, the mediation rule always chooses a designated bundle at which the outside option in X is coupled with some efficient alternative in Y. In this case, the mediation rule provides only a partial resolution to the dispute because of the severity of the diasagrement on issue X.

Parts (ii) and (iii) reveal that any mediation rule satisfying the three properties, i.e., strategy-proofness, efficiency, and individual rationality, must always make selections from a special set of bundles when the set of mutually acceptable alternatives in issue X is nonempty. At these bundles, for each alternative in X, there is a corresponding distinct alternative in Y with which it must be paired. Interestingly, these bundles have the property that a more preferred alternative from issue X is paired with a ranking-wise equally less-preferred alternative from issue Y. Specifically, at such a bundle, if a negotiator is receiving her first-most favorite alternative from X, she must be receiving her least favorite alternative from Y; if she is receiving her second-most favorite alternative from X, she must be receiving her second-least favorite alternative from Y and so on. By transitivity, $x\ \theta_i\ x' \implies t(x')\ \theta_i^Y\ t(x)$ implies that alternative x_k is paired with alternative y_{m-k+1}. We interpret these bundles as representing possible "compromises" between the two issues. As such, we henceforth call a bundle $(x_k, y_{m-k+1}) \in X \times Y$ a **logrolling bundle**. The set $\mathbf{B}$ consists of all the logrolling bundles.

Part (ii) says that the set of logrolling bundles constitute the "backbone" of every strategy-proof, efficient, and individually rational rule. That is, the diagonal of any such rule must always be comprised of the logrolling bundles. Part (iii) characterizes all possible assignment formations in the lower half of the diagonal. In particular, the logrolling bundles on the diagonal "propagate" in the southwestern direction following an adjacency requirement. Any entry in the lower half of the diagonal must be assigned a bundle that coincides with the outcome in the entry either immediately above it or immediately to its right. Given the structure of the main diagonal in part (ii), using the adjacency condition $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\}$ in part (iii), one can obtain the distribution of the bundles for the diagonal entries immediately to the left of the main diagonal, which can then be used to determine the distribution of the diagonal entries immediately to the left of those, and so on, until the corner entry $f_{m,1}$ is finally determined. Another important implication of the three properties is that the mediator has discretion over the choice of a **precedence order** $\triangleright$ on the set of logrolling bundles, which he must use to make "consistent" selections as to which of the two (diagonally) adjacent bundles takes precedence when fullfilling the adjacency condition. The higher precedence bundle is always the "winner" between any two diagonally adjacent logrolling bundles.

For the rest of the paper, we call a mediation rule f an **adjacent rule** if it satisfies parts $(i)-(iii)$ of Theorem 1 and denote it by $f^{\triangleright}$ whenever we need to specify the binary relation $\triangleright$ that is used to construct f. Different precedence orders can lead to different mediation rules. Theorem 3 provides a complementary visual characterization of the set of all possible adjacent rules. Before giving a sketch of the proof of Theorem 1, we provide an example of these rules.

Example 1 (An adjacent rule): Let each issue consist of five alternatives, i.e., $m = 5$. The set of logrolling bundles in this case is $\mathbf{B} = \big\{(x_1, y_5), (x_2, y_4), (x_3, y_3), (x_4, y_2), (x_5, y_1)\big\}$. Let us construct the adjacent rule $f^{\triangleright}$ associated with the precedence order $\triangleright$ where

$$(x_5, y_1) \;\triangleright\; (x_1, y_5) \;\triangleright\; (x_4, y_2) \;\triangleright\; (x_2, y_4) \;\triangleright\; (x_3, y_3).$$

Whenever the negotiators have no mutually acceptable alternative in issue X, i.e., part (i), let the rule pick the bundle (o_X, y_3). The main diagonal, i.e., part (ii), is filled with the members of the set of logrolling bundles, $\mathbf{B}$, e.g., we have $f_{1,1} = (x_1, y_5)$ in the first diagonal entry, $f_{2,2} = (x_2, y_4)$ in the second diagonal entry, etc. For the rest of the matrix, iterating the adjacency condition together with the precedence order fully determines the distribution of the logrolling bundles. Since the logrolling bundle (x_5, y_1) has higher precedence than all other bundles in $\mathbf{B}$, it "beats" all of its diagonally adjacent neighbors at any possible binary comparison. Hence, bundle (x_5, y_1) claims all the entries to its southwest, which amounts to the set of all entries on the bottom row to the left of $f_{5,5}$. The second-highest precedence bundle is (x_1, y_5), and it similarly claims all the unfilled entries to its southwest. Thus, starting from the entry $f_{1,1}$ on the main diagonal, all the remaining empty entries on the first column fill up with (x_1, y_5). Iterating this process for all the logrolling bundles in the precedence order yields the following matrix:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_2}$	(x_1, y_5)	(x_2, y_4)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_3}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_4}$	(x_1, y_5)	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(o_X, y_3)
$\theta_1^{x_5}$	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)

Figure 1: *A standard member of the adjacent rules family*

Sketch of the proof of Theorem 1: We now provide a sketch of some of the main ideas behind the proof of Theorem 1 using a simple example. Suppose that $m = 3$ and f is a strategy-proof, efficient, and individually rational mediation rule over a domain of preferences that satisfy the monotonicity and deal-breakers properties. We first argue that there must exist three distinct alternatives $y^{x_1}, y^{x_2}, y^{x_3}$ in issue Y such that negotiator 1 prefers y^{x_3} to y^{x_2} and y^{x_2} to y^{x_1}. Therefore, given that $Y \backslash \{o_Y\} = \{y_1, y_2, y_3\}$, we must have $y^{x_3} = y_1$, $y^{x_2} = y_2$, and $y^{x_1} = y_3$, and so, $t(x_k) = y_{4-k}$ for $k = 1, 2, 3$. Then, we argue that mediation rule f always offers one of the bundles from set $\mathbf{B} = \{(x_1, y_3), (x_2, y_2), (x_3, y_1)\}$ whenever negotiators have at least one mutually acceptable alternative in issue X.[39]

A strategy-proof, efficient, and individually rational mediation rule f is illustrated in the following matrix:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$
$\theta_1^{x_1}$	(x_1, y^{x_1})	(o_X, y^o)	(o_X, y^o)
$\theta_1^{x_2}$	$f_{2,1}$	(x_2, y^{x_2})	(o_X, y^o)
$\theta_1^{x_3}$	$f_{3,1}$	$f_{3,2}$	(x_3, y^{x_3})

In the upper half of the matrix, i.e., $\ell < j$, there is no mutually acceptable alternative in issue X. Thus, individual rationality and the regularity of preferences require that f must suggest a bundle with o_X, and efficiency requires that the second alternative must be different from o_Y. Finally, strategy-proofness and monotonicity imply that f must suggest the same bundle (o_X, y^o) at all $\ell, j \in M$ with $\ell < j$, for otherwise a negotiator can mimic a type where she receives a more preferred alternative from Y.

[39]Note that this example (and Theorem 1) can be extended to the case where $\#Y > 3$. In that case, it would be such that $\mathbf{B} = \{(x_1, y^{x_1}), (x_2, y^{x_2}), (x_3, y^{x_3})\}$ for some triplet of alternatives in $Y \setminus \{o_Y\}$ satisfying the property that $y^{x_3}\ \theta_1^Y\ y^{x_2}\ \theta_1^Y\ y^{x_1}$. See Section 7 for more on this.

For determining the values of the entries over the main diagonal, note that the only mutually acceptable alternative in issue X is x_1 when negotiator 1 is of type $\theta_1^{x_1}$ and negotiator 2 is of type $\theta_2^{x_1}$. Individual rationality, deal-breakers, and monotonicity imply that the entry $f_{1,1}$ is either (x_1, y) or (o_X, y) for some $y \in Y$. Efficiency and monotonicity require $y \neq o_Y$. The same reasoning applies to the rest of the diagonal entries, and so let (x_1, y^{x_1}), (x_2, y^{x_2}), and (x_3, y^{x_3}) denote the three bundles on the diagonal.

Next, we argue that $y^{x_3}\ \theta_1^Y\ y^{x_2}\ \theta_1^Y\ y^{x_1}$ must hold. Suppose that $f_{2,1} = (x, y)$ for some $x \in X$ and $y \in Y$. The set of mutually acceptable alternatives in issue X is $\{x_1, x_2\}$ when the negotiators' types are $\theta_1^{x_2}$ and $\theta_2^{x_1}$, and thus, individual rationality, efficiency, deal-breakers, and monotonicity imply that $x \in \{x_1, x_2\}$. If $x = x_1$, then we must have $y = y^{x_1}$. Suppose, for a contradiction, that $y \neq y^{x_1}$. Then either y is better or worse than y^{x_1} for negotiator 1. If it is better (worse), then type $\theta_1^{x_1}$ ($\theta_1^{x_2}$) prefers to deviate to $\theta_1^{x_2}$ ($\theta_1^{x_1}$) since monotonicity implies that (x_1, y) is better (worse) than (x_1, y^{x_1}), contradicting strategy-proofness. Once we establish that $(x, y) = (x_1, y^{x_1})$, it is easy to see that negotiator 2 prefers y^{x_1} to y^{x_2} by strategy-proofness and monotonicity, for otherwise type $\theta_2^{x_1}$ would deviate to $\theta_2^{x_2}$. Therefore, we must have $y^{x_1}\ \theta_2^Y\ y^{x_2}$, or equivalently, $y^{x_2}\ \theta_1^Y\ y^{x_1}$ since negotiators' preferences over alternatives are diametrically opposed. On the other hand, if $x = x_2$, then a symmetric argument for negotiator 2 implies that we must have $y = y^{x_2}$ and negotiator 1 prefers y^{x_2} to y^{x_1} by strategy-proofness and monotonicity. Thus, for any feasible values of x, we must have $y^{x_2}\ \theta_1^Y\ y^{x_1}$. This proves that (1) $f_{2,1} \in \{(x_1, y^{x_1}), (x_2, y^{x_2})\}$ and (2) $y^{x_2}\ \theta_1^Y\ y^{x_1}$. Symmetric arguments imply that $f_{3,2} \in \{(x_2, y^{x_2}), (x_3, y^{x_3})\}$ and $y^{x_3}\ \theta_1^X\ y^{x_2}$. Hence, transitivity implies $y^{x_3}\ \theta_1^Y\ y^{x_2}\ \theta_1^X\ y^{x_1}$ as we claimed.

Since f is efficient and individually rational, $f_{3,1}$ must be a bundle with x_1, x_2, or x_3. Similar to previous arguments, if it includes x_1 or x_3, then the second alternative must be y^{x_1} or y^{x_3}, respectively. However, if it is a bundle with x_2, then the alternative in issue Y must be different from y^{x_1} and y^{x_3} by strategy-proofness and monotonicity. Therefore, we have $f_{3,1} = (x_2, y)$ such that negotiator 1 ranks y above y^{x_1} and below y^{x_3}. Therefore, f must always offer a bundle from the set $\mathbf{B}$ when negotiators have at least one mutually acceptable alternative. Similar to earlier arguments, we can conclude that $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\}$ for all $\ell, j \in M$ with $\ell > j$.

Finally, it can be shown that strategy-proofness requires that when the mediation rule selects one of two diagonally adjacent bundles for the entry between them, it cannot reverse that choice at another instance where the two bundles are again diagonally adjacent. More specifically, strategy-proofness effectively leads us to identify a transitive and antisymmetric precedence order on the set of logrolling bundles. We construct the order $\rhd$ by performing pairwise comparisons for all the entries $f_{\ell,j}, f_{\ell-1,j}, f_{\ell,j+1}$. This construction generates a partial order $\rhd$ on $\mathbf{B}$, which can then be completed in an arbitrary fashion to obtain a strict precedence order.[40]

Full Characterization and Logrolling (Quid Pro Quo)

Theorem 1 characterizes the necessary conditions a strategy-proof, efficient, and individually rational mediation rule must satisfy. It is easy to verify that any adjacent rule satisfying the three parts of Theorem 1 is also efficient and individually rational.[41] Recall that our preference domain allows for any preferences meeting the monotonicity and the deal-breakers properties. In particular, we have so far made no assumptions whatsoever as to the relative importance of the two issues for the two negotiators. Consider, for example, a scenario where alternatives in issue Y have little appeal for the negotiators compared to those in issue X, e.g., preferences are lexicographic over the two issues with each negotiator mainly caring about her outcome from issue X. Then there is little reason to suspect that the impossibility in the single-issue case will be overturned in the two-issue world. In such a scenario, ignoring any possible alternatives she may be assigned from issue Y, a negotiator can easily manipulate an adjacent rule by using a truncation strategy much like in the example given in Section 2. Consequently, obtaining strategy-proofness inevitably necessitates more restrictions on the underlying preferences beyond monotonicity.

An adjacent rule always chooses a logrolling bundle when mediation is mutually beneficial for resolving the conflict in issue X, where such a bundle complements a more desirable alternative from one issue with a less desirable alternative from the other issue. Since the class of adjacent rules contains the only candidates that can achieve the three properties by Theorem 1, ensuring that an adjacent rule is strategy-proof automatically entails imposing a discipline on preferences regarding how negotiators rank the logrolling bundles. It then follows that the preference domain should permit the possibility that the negotiators are willing to make compromises in issue X for a more favorable treatment in issue Y. Put differently, it should be possible to find some alternatives in issue Y that are sufficiently attractive for at least one of the negotiators to reverse her ranking of some alternatives in issue X when they are bundled together. We start with a strong version of this requirement that will be sufficient to ensure the strategy-proofness of *all* adjacent

[40]Consequently, different strict precedence orders may lead to the same mediation rule. In this sense, it is possible to replace the strict precedence order in statement of Theorem 1 with a corresponding partial order. This is exactly what we do in the sequel when strengthening Theorems 1 and 2; e.g., see Theorem 4. We have chosen the current presentation as it provides considerable expositional simplicity.

[41]It suffices to check these properties issue by issue. When there is no mutually acceptable alternative in X, the outside option o_X is the only efficient and individually rational option. When the set of mutually acceptable alternatives in X is nonempty, any alternative in this set is efficient and individually rational. Finally, any alternative in $Y \setminus \{o_Y\}$ is always efficient and individually rational. These observations together imply that the bundles selected by an adjacent rule for the upper and lower halves of the matrix are always efficient and individually rational.

rules. We will later identify a weaker version that will be both sufficient and necessary for obtaining a strategy-proof, efficient, and individually rational mediation rule.

Definition 5. *The* ***extension map*** Λ *satisfies* ***logrolling (quid pro quo)*** *if there exists a function* $t : X \to Y$ *such that for all* i*,* $\theta_i \in \Theta_i$*,* $R_i \in \Lambda(\theta_i)$ *and all* $x, x' \in A(\theta_i)$ *with* $x \ \theta_i \ x'$*, we have* $\big(x', t(x')\big) \ R_i \ \big(x, t(x)\big)$*.*

Logrolling (quid pro quo) allows "trading" of alternatives. It requires two conditions. First, for any two acceptable alternatives x, x' in X where x is ranked above x' for type θ_i, there must exist two alternatives y, y' in Y, where y' is ranked above y, such that (x', y') is ranked at least as high as (x, y) at all admissible orderings (over bundles) $R_i \in \Lambda(\theta_i)$. Second, types must be "consistent." Namely, order-reversing mapping, t, is independent of types. Logrolling implies that the attractiveness of the alternatives in Y is sufficiently dispersed so that the negotiators are willing to trade any acceptable alternative in issue X with any other acceptable but less desirable alternative in X. This notion can be viewed as the nontransferable utility analogue of the *possibility of compensation* assumption in a transferable utility model; see, e.g., Thomson (2016). Logrolling imposes a certain type of substitutability between the two issues. As such, it rules out lexicographic preferences. In fact, many standard utility functions are compatible with this condition, as illustrated in Example 3 for quasi-linear utility and further elaborated in the Supplementary Appendix. Note that logrolling is a well-defined concept only if the number of alternatives in issue Y is no less than the number of alternatives in issue X.

Example 2 (logrolling): Suppose that each issue consists of three alternatives, i.e., $m = 3$. Since the alternatives in issues X and Y are equal in number, there is a unique one-to-one function t, where $t(x_k) = y_{4-k}$ for $k = 1, 2, 3$, which satisfies the requirements of Definition 5. Logrolling implies that the type $\theta_1^{x_3}$ of negotiator 1 who deems all three alternatives in issue X acceptable, i.e., $x_1 \ \theta_1^{x_3} \ x_2 \ \theta_1^{x_3} \ x_3 \ \theta_1^{x_3} \ o_X$, will rank (x_3, y_1) at least as high as the bundle (x_2, y_2) and rank (x_2, y_2) at least as high as the bundle (x_1, y_3) for all admissible orderings $R_1 \in \Lambda(\theta_1^{x_3})$. The consistency of the mapping t over the types implies, for example, that type $\theta_1^{x_2}$ of negotiator 1 who deems only x_1 and x_2 acceptable, i.e., $x_1 \ \theta_1^{x_2} \ x_2 \ \theta_1^{x_2} \ o_X \ \theta_1^{x_2} \ x_3$, will rank (x_2, y_2) at least as high as the bundle (x_1, y_3). Logrolling imposes no restriction on admissible orderings $R_1 \in \Lambda(\theta_1^{x_2})$ regarding how they rank the bundle (x_3, y_1) relative to the bundles (x_2, y_2) and (x_1, y_3).

Although our ordinal approach does not make any explicit assumptions about (or seek to elicit) negotiators' cardinal preferences, this certainly does not preclude the possibility that the negotiators are inherently endowed with such preferences. In that regard, it is interesting to ask what kind of underlying preferences would be compatible with the logrolling condition. The next example offers an illustration when the negotiators' utility functions are quasi-linear.

Example 3 (logrolling under quasi-linear utility): Suppose the negotiators are in a dispute over dividing 10 units of good x and 6 units of good y. It is commonly known that negotiator i's utility, for an acceptable amount of good x and y, is of the form $U_i(x, y) = x^{\alpha_i} + y$ where $\alpha_i \in (0, \frac{3}{5}]$ for $i = 1, 2$, but the exact value of α_i is not necessarily publicly known.[42] There are five possible ways of dividing each good such that a negotiator's share of good x is an alternative from the set $X = \{1, 3, 5, 7, 9\}$ and that of good y is an alternative from the set $Y = \{1, 2, 3, 4, 5\}$. Her least acceptable amount of good x is each negotiator's private information. Then the set of logrolling bundles is $\mathbf{B} = \{(1, 5), (3, 4), (5, 3), (7, 2), (9, 1)\}$, i.e., those bundles $(x, t(x))$ where $t(x) = \frac{11-x}{2}$, and the logrolling condition is satisfied, as can be easily verified.

Remark 1: Example 3 raises a natural question as to what conditions negotiators' underlying utility functions generally need to satisfy to ascertain that the logrolling condition is satisfied. In the Supplementary Appendix, where we consider a continuous analogue of our discrete model, we provide sufficiency conditions on negotiators' underlying preferences that ensure compatibility with the logrolling condition. Intuitively, these sufficiency conditions require the marginal rate of substitution between the two issues be no greater than the slope of the t function which governs the set of logrolling bundles. In other words, there is a certain degree of freedom in terms of choosing the set of logrolling bundles and the negotiators' utility functions. Specifically, for any given pair of differentiable and increasing utility functions, there is a corresponding set of decreasing functions, i.e., a set of all possible sets of logrolling bundles, that guarantees compatibility with the logrolling condition. Alternatively, for any given set of bundles that lie on a strictly decreasing curve, there is a corresponding set of differentiable and increasing utility functions that guarantees compatibility with the logrolling condition. See the Supplementary Appendix for formal statements.

We are now ready to provide a full characterization result.

Theorem 2. *Suppose that the regular extension map Λ satisfies logrolling (quid pro quo). The mediation rule f is strategy-proof, efficient, and individually rational if and only if it is an adjacent rule, i.e., $f = f^{\triangleright}$, for some strict precedence order $\triangleright$ on $\mathbf{B}$.*

Theorem 2 gives us a complete characterization of the set of strategy-proof, efficient, and individually rational mediation rules under logrolling. Any mediation rule satisfying the three properties must be an adjacent rule associated with some precedence order, and conversely, *any* adjacent rule possesses the three properties, i.e., any strict binary relation over $\mathbf{B}$ generates a strategy-proof, efficient, and individually rational mediation rule. In the next section we focus on the structure of the class of adjacent rules, provide simple

[42] When negotiator i gets bundle (x, y), negotiator j gets $(10 - x, 6 - y)$ which gives her a utility of $U_j(10 - x, 6 - y) = (10 - x)^{\alpha_j} + 6 - y$.

and intuitive interpretations of these rules, and identify interesting special members of this class of rules.

Sketch of the proof of Theorem 2: We provide only a sketch of the argument that an adjacent rule is strategy-proof, since all the other claims in Theorem 2 have already been established. The logrolling (quid pro quo) condition implies that the set of logrolling bundles forms a totally ordered set, or a simple distributive lattice, with respect to either negotiator's weak preferences. In particular, the bundle at the top corner of the diagonal, i.e., $f_{1,1} = (x_1, y_m)$ is negotiator 1's least-preferred (negotiator 2's most-preferred) logrolling bundle and the bundle at the bottom corner of the diagonal, i.e., $f_{m,m} = (x_m, y_1)$ is negotiator 1's most-preferred (negotiator 2's least-preferred) logrolling bundle whenever they are acceptable. The adjacency property in part (iii) of Theorem 1 implies that whenever a bundle $b \in \mathbf{B}$ lies above another bundle $b' \in \mathbf{B}$ along the main diagonal, i.e., b contains a lower-indexed alternative from X, it can never be the case that b' lies above b along any diagonal throughout the entire matrix. In other words, bundle b always lies to the left of bundle b' on any row that contains both bundles and lies above b' on any column that contains both bundles. Consider the outcome $f_{\ell,j} = f(\theta_1^{x_\ell}, \theta_2^{x_j})$ where $\ell \geq j$. Clearly, $f_{\ell,j}$ is a logrolling bundle. Consider, without loss of generality, negotiator 1. Suppose she were to report a less accepting type $\theta^{x'_\ell}$ with $\ell' < \ell$. If $\ell' \geq j$, then the new outcome $f_{\ell',j}$ is also a logrolling bundle. Moreover, by the previous implication of adjacency, bundle $f_{\ell',j}$ either coincides with or lies above bundle $f_{\ell,j}$ on the main diagonal, i.e., negotiator 1 is either as well off or worse off. If $\ell < j$, then the new outcome $f_{\ell',j}$ chooses the outside option in X and negotiator 1 is worse off by the deal-breakers property.

Suppose she were instead to report a more accepting type $\theta^{x'_\ell}$ with $\ell' > \ell$. Clearly, the new outcome $f_{\ell',j}$ is also a logrolling bundle. Recall that the adjacency property implies that the bundles on the main diagonal spread in the southwestern direction into the lower half of the matrix. This means that entry $f_{\ell,j}$ contains either bundle $f_{\ell,\ell}$ or a bundle above it on the main diagonal, i.e., no bundle on the main diagonal ever moves up to a higher row. Similarly, bundles below $f_{\ell,\ell}$ on the main diagonal also spread in the southwestern direction. This implies that any bundle below entry $f_{\ell,j}$ on column j either contains the same bundle as $f_{\ell,j}$ or a bundle that lies below $f_{\ell,\ell}$ on the main diagonal. But bundles below $f_{\ell,\ell}$ on the main diagonal contain unacceptable alternatives from issue X. Hence, reporting $\theta^{x'_\ell}$ makes her either as well off or worse off by the deal-breakers property.

Finally, if we started off from an entry $f_{\ell,j}$ with $\ell < j$, $f_{\ell,j}$ would choose a bundle with the outside option in X. If negotiator 1 were to report a more accepting type $\theta^{x_{\ell'}}$ with $\ell' > \ell$, then her outcome would either remain unchanged (when $j > \ell'$), or make her worse off by giving her an unacceptable alternative from X.

To provide further insight into the adjacent rules that are characterized in Theorem 2, and in particular, to gain a better understanding of the implications of part (iii) of Theorem 1, we offer a geometric analysis of these rules. The geometric analysis will lend itself to intuitive interpretations of these rules for practical use. In this subsection we fix a mediation rule $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$ that satisfies parts (i) and (ii) of Theorem 1. For any $k \in M$, let b_k denote the logrolling bundle $(x_k, y_{m-k+1}) \in \mathbf{B}$. We first introduce a couple of definitions to represent different rectangular and triangular regions of the matrix induced by rule f. In the following two definitions we slightly abuse notation and terminology in order to keep track of the entries contained in a rectangular/triangular region, e.g., we use $f_{\ell,j}$ to in fact refer to entry (ℓ, j) of the matrix rather than the specific bundle that rule f assigns to that entry.

Definition 6. *Consider the entry $f_{k,k} = b_k$ and an entry that lies (weakly) to its southwest, $f_{\ell,j}$ with $j \leq k \leq \ell$. The **rectangle** induced by b_k and $f_{\ell,j}$, denoted by $\Re^{b_k}_{\ell,j}$, is the set of all entries in the rectangular region of the matrix (inclusively) enveloped between rows k and ℓ and columns k and j. Namely, $\Re^{b_k}_{\ell,j} \equiv \bigcup_{\substack{j\leq s\leq k\\ k\leq t\leq \ell}} \{f_{t,s}\}$.*

Definition 7. *The **triangle** induced by an entry $f_{\ell,j}$, with $j \leq \ell$, denoted by $\triangle_{\ell,j}$, is the set of all entries in the triangular region of the matrix that is (inclusively) enveloped by the entry $f_{\ell,j}$, row ℓ, column j, and the main diagonal. Namely, $\triangle_{\ell,j} \equiv \bigcup_{j\leq t\leq \ell} \{f_{t,j}, f_{t,j+1}, ..., f_{t,t}\}$.*

Note that a rectangle/triangle is merely a collection of entries of the matrix induced by rule f, i.e., sets of pairs of indexes. Note also that an entry on the main diagonal is a special triangle (and also a special rectangle) that consists of a singleton entry. Furthermore, the entire main diagonal of the matrix and all the entries to its southwest constitute the largest possible triangle $\triangle_{m,1}$. Given a triangle $\triangle_{\ell,j}$, its entries that lie on the main diagonal are said to be on the *hypotenuse* of $\triangle_{\ell,j}$, and the set of these entries are denoted by $\mathbf{B}_{\ell j} \equiv \{f_{j,j}, ..., f_{\ell,\ell}\} = \{b_j, ..., b_\ell\}$. A partition of the lower half of the matrix is called a *rectangular (triangular) partition* if and only if it is the union of disjoint rectangles (triangles).[43] Given a set $B \subseteq \mathbf{B}$ and a strict precedence order $\rhd$ on $\mathbf{B}$, let $\boldsymbol{\max_{B}} \rhd \equiv \left\{b \in B \mid b \rhd a \text{ for all } a \in B\backslash\{b\}\right\}$ denote the unique bundle in B that has the highest precedence with respect to $\rhd$.

[43]Note that a rectangular partition consists of m disjoint rectangles. For example, $\{\Re^{b_k}_{k,1}\}_{k=1}^m$ and $\{\Re^{b_k}_{m,k}\}_{k=1}^m$ are two obvious rectangular partitions of $\triangle_{m,1}$. These two partitioning correspond respectively to what we will later refer to as the negotiator 1- and negotiator 2-optimal rules.

Theorem 3 (Visual Characterization). *Consider a mediation rule f satisfying parts (i) and (ii) of Theorem 1. The following statements are equivalent:*

(1) *f satisfies part (iii) of Theorem 1.*

(2) *$\triangle_{m,1}$ has a rectangular partition such that f assigns a unique bundle to each rectangle in this partition.*[44]

(3) *There exists a precedence order $\rhd$ on $\mathbf{B}$ such that $f_{\ell,j} = \max\limits_{\mathbf{B}_{\ell j}} \rhd$.*

Part (2) of Theorem 3 states that an adjacent rule f can be represented as the union of m disjoint rectangular regions. Each rectangle has a distinct corner entry on the main diagonal that contains the logrolling bundle that fills up the entire rectangle. Procedurally, these rectangles are obtained as follows. Given the precedence order, let the highest-precedence bundle on the hypotenuse of the largest triangle $\triangle_{m,1}$ fill up all the entries that are located to its southwest. This creates the first and largest rectangle $\Re$, and leads to a triangular partition of $\triangle_{m,1} \backslash \Re$. Next, pick any triangle from this partition and let the highest-precedence bundle on the hypotenuse of this triangle fill up all the entries that are located to its southwest. This leads to a second rectangle $\Re'$, as well as a unique triangular partition of $\triangle_{m,1} \backslash \{\Re, \Re'\}$. The process can be iterated in this fashion until the entire triangle $\triangle_{m,1}$ is partitioned into m disjoint rectangles in m steps. Conversely, any such geometric set, namely any rectangular partition of $\triangle_{m,1}$, can be used to construct a precedence order and a corresponding adjacent rule. Figure 2 provides an illustration of one such partitioning.

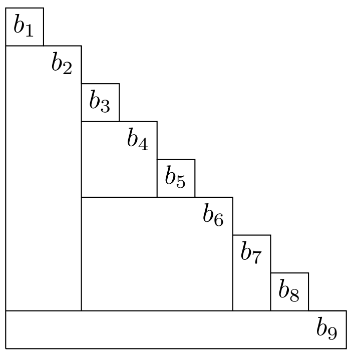


Figure 2: *An example of a rectangular partitioning with 9 alternatives*

Part (3) of Theorem 3 gives a shortcut formula for calculating the outcome of an adjacent rule f. Any entry $f_{\ell,j}$ with $\ell \geq j$ in the lower half of the matrix must always contain the logrolling bundle with the highest precedence among the bundles that are on the hypotenuse of the triangle generated by the entry $f_{\ell,j}$, i.e., $\triangle_{\ell,j}$. This eliminates the

[44] More formally, for any $\Re$ in the partition of $\triangle_{m,1}$ and any pair $a, b \in \Re$, $a = b$; but for any distinct pair $\Re, \Re'$ in the partition of $\triangle_{m,1}$, $a \in \Re$ and $b \in \Re'$ implies $a \neq b$.

need to calculate the immediately adjacent bundles of that entry as required by part (iii) of Theorem 1. Part (3) implies that the left-bottom corner entry $f_{m,1}$ of the matrix is always assigned the highest-precedence logrolling bundle, which is also the bundle that fills up the rectangle obtained at the first step of the partitioning procedure described above.

Theorem 3 can be used to obtain an alternative interpretation of an adjacent rule reminiscent of how a divide-and-choose type of rule from fair division works. In particular, an adjacent rule can be thought to operate as a "shortlisting rule": One negotiator offers a short list of bundles to the other negotiator and the other negotiator chooses her favorite bundle from this list. To see this, observe that when negotiator 1 reports her type as θ^{x_ℓ}, the rule must pick a bundle on row ℓ. Equivalently, this can be viewed as negotiator 1 forming a short list consisting of all the bundles on row ℓ. Suppose that the type of negotiator 2 is θ^{x_j}. When faced with the list of bundles negotiator 1 offers her, she indeed picks $f_{\ell,j}$ since it is in fact her favorite bundle on row ℓ by strategy-proofness. If the roles of the two negotiators in the procedure were reversed, the outcome would still be the same by a symmetric argument.[45] As an example, consider the adjacent rule depicted in Figure 2. Suppose negotiator 1 is of type x_3. Then we can think of her as proposing the short list $\{b_2, b_3, (o_X, y)\}$ to the other negotiator. The corresponding short lists for her types θ^{x_5} and θ^{x_7} are respectively $\{b_2, b_4, b_5, (o_X, y)\}$ and $\{b_2, b_6, b_7, (o_X, y)\}$.

Under this interpretation, an adjacent rule specifies the set of shortlisted bundles a negotiator can offer to the other party for each possible type she reports. When the proposing negotiator reports a more accepting type, then she can add new bundles to her previous list or remove some bundles from this list. However, Theorem 3 implies that a previously removed bundle can never be added back on to the list for a more accepting type. In our example, negotiator 1 adds bundles b_4 and b_5 to her list and removes b_3 when switching from θ^{x_3} to θ^{x_5} and adds b_6 and b_7 and removes b_4 and b_5 when switching from θ^{x_5} to θ^{x_7}. Note that once bundles b_3, b_4 or b_5 are removed, they are never added back in. Conversely, any collection of sets of shortlisted bundles (one set for each possible type of a negotiator) respecting this requirement can be shown to correspond to an adjacent rule.

Special members of the adjacent rules family

Three notable members of the adjacent rules family are worth pointing out. A **negotiator-optimal rule** represents a situation of extreme partiality to one side of the dispute and is constructed by using the strict counterpart of the preference of one negotiator over the logrolling bundles, $\mathbf{B}$, as the precedence order. For example, when

[45] In the context of fair division, however, the outcome of divide-and-choose may be order dependent. Divide-and-choose also violates strategy-proofness.

there are five alternatives in each issue, the negotiator 1-optimal rule takes

$$\rhd^1\!: (x_5, y_1) \;\rhd^1\; (x_4, y_2) \;\rhd^1\; (x_3, y_3) \;\rhd^1\; (x_2, y_4) \;\rhd^1\; (x_1, y_5)$$

whereas the negotiator 2-optimal rule takes

$$\rhd^2\!: (x_1, y_5) \;\rhd^2\; (x_2, y_4) \;\rhd^2\; (x_3, y_3) \;\rhd^2\; (x_4, y_2) \;\rhd^2\; (x_5, y_1).$$

Additionally, in case of disagreement, i.e., in case there is no mutually acceptable alternative in issue X, the corresponding designated bundle includes the favored negotiator's most-preferred alternative in issue Y. The two dual rules are shown below:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_1)	(o_X, y_1)	(o_X, y_1)	(o_X, y_1)
$\theta_1^{x_2}$	(x_2, y_4)	(x_2, y_4)	(o_X, y_1)	(o_X, y_1)	(o_X, y_1)
$\theta_1^{x_3}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(o_X, y_1)	(o_X, y_1)
$\theta_1^{x_4}$	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(o_X, y_1)
$\theta_1^{x_5}$	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)

Figure 3-a: *Negotiator 1-optimal rule*

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_5)	(o_X, y_5)	(o_X, y_5)	(o_X, y_5)
$\theta_1^{x_2}$	(x_1, y_5)	(x_2, y_4)	(o_X, y_5)	(o_X, y_5)	(o_X, y_5)
$\theta_1^{x_3}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(o_X, y_5)	(o_X, y_5)
$\theta_1^{x_4}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	(o_X, y_5)
$\theta_1^{x_5}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	(x_5, y_1)

Figure 3-b: *Negotiator 2-optimal rule*

A negotiator-optimal rule always chooses the corresponding negotiator's most-preferred bundle among the mutually acceptable logrolling bundles. The analogous shortlisting rule is rather simple: the favored negotiator's short list includes only two bundles, which are her favorite logrolling bundle and the designated disagreement outcome.[46] Clearly, these two polar members of the adjacent rules family are highly undesirable in practice. Nevertheless, they hint at the possibility of the mediator having the power to tilt the balance in a dispute despite using a rule that meets our desiderata. As discussed earlier, such a possibility has been acknowledged in the practical mediation literature, in which various field studies report and caution against biased treatment. Fortunately, there is a remarkable member of the adjacent rules family that treats negotiators symmetrically.

Given that the negotiators' preferences over the logrolling bundles are diametrically opposed, impartiality would require the mediator to focus on a central element of the set of logrolling bundles. It is then intuitive for him to recommend a *median* logrolling bundle, i.e., bundle (x_n, y_n) where n is the index of a median alternative,[47] when it is mutually acceptable, or seek a bundle as close to it as possible when it is not. Under the family of adjacent rules, this is achieved simply by assigning the highest precedence to a median

[46]Alternatively, the nonfavored negotiator's short list includes all logrolling bundles acceptable to her together with the designated disagreement outcome.

[47]If m is odd, there is a unique median alternative in each issue. If m is even, there are two median alternatives in each issue, in which case we assume that the mediator picks either of them.

logrolling bundle, and the next precedence to those bundles that are closest to the chosen median, and so on, and lowest precedence to the extremal logrolling bundles. Based on similar logic, when there is no mutually acceptable alternative in X, the designated bundle chosen by an impartial mediator should naturally include a median alternative in Y. This motivates the following type of rule, which we call a **constrained shortlisting (CS) rule**.

Definition 8. *Let $n \in \{\overline{n}, \underline{n}\}$ be the index of a median alternative, where $\overline{n} = \lceil \frac{m+1}{2} \rceil$ and $\underline{n} = \lfloor \frac{m+1}{2} \rfloor$. A rule is a constrained shortlisting rule, denoted $f^{CS} = [f_{\ell,j}]_{(\ell,j)\in M^2}$, if it is an adjacent rule that is associated with a precedence order $\triangleright$, where $b_n \triangleright b_{n-1} \triangleright \ldots \triangleright b_1$ and $b_n \triangleright b_{n+1} \triangleright \ldots \triangleright b_m$, and $f^{CS}_{\ell,j} = (o_X, y_n)$ whenever $\ell < j$.*

Note that there is a unique constrained shortlisting rule when the number of alternatives is odd. When the number of alternatives is even, however, a constrained shortlisting rule prescribes one of four possible types of outcomes.[48] Figure 4 illustrates the constrained shortlisting rule for the case of five alternatives.

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_2}$	(x_2, y_4)	(x_2, y_4)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_3}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_4}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(x_4, y_2)	(o_X, y_3)
$\theta_1^{x_5}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)	(x_4, y_2)	(x_5, y_1)

Figure 4: *Constrained shortlisting rule*

When the number of alternatives is odd, the CS rule is a symmetric member of the adjacent rules family.[49] In the lower half of the matrix, it acts as a negotiator-optimal rule whenever the median alternative in issue X is not mutually acceptable and recommends the median logrolling bundle whenever the set of mutually acceptable alternatives includes the median alternative. In other words, when both negotiators find at least half of the alternatives in X acceptable, the rule chooses the median logrolling bundle. On the other hand, when one negotiator finds at least half of the alternatives acceptable while the other finds less than half of the alternatives acceptable, the rule chooses the less accepting negotiator's favorite (acceptable) logrolling bundle.

[48]In this case, the rule depends on whether $b_{\overline{n}}$ or $b_{\underline{n}}$ has the highest precedence order and whether $y_{\overline{n}}$ or $y_{\underline{n}}$ is included in the designated disagreement bundle.
[49]When the number of alternatives is even, no adjacent rule is fully symmetric.

Assuming an odd number of alternatives,[50] the CS rule can intuitively be implemented as a shortlisting rule in the following way. The mediator solicits from one of the negotiators her favorite logrolling bundle, which effectively reveals the negotiator's type. Depending on the bundle the negotiator provides, the mediator proposes to the other negotiator one of two types of short lists to choose from. If the solicited negotiator finds less than half of the alternatives acceptable, then the short list includes only the designated disagreement bundle in addition to her favorite logrolling bundle. If the solicited negotiator finds at least half of the alternatives acceptable, then the short list includes the designated disagreement bundle, the median logrolling bundle, her favorite logrolling bundle, and all the logrolling bundles in between the latter two. It is clearly in the best interests of the opposite negotiator to choose truthfully from the proposed short list. Since CS is strategy-proof, the solicited negotiator also truthfully reveals her favorite logrolling bundle. It again does not matter for the outcome which negotiator the mediator approaches first.

In discrete resource allocation problems where agents are endowed with ordinal preference rankings, fairness properties (together with efficiency) have often proved difficult to attain in the absence of monetary transfers or a randomization device. It may nevertheless be worthwhile to investigate whether it is possible for a member of the adjacent rules family to achieve alternative fairness requirements beyond symmetry. We next formulate one such ordinal fairness notion as a normative requirement for our context.

Given the negotiators' fixed preferences over alternatives (not including the outside option), let $r_i(z) \in M$ denote negotiator i's ranking of an acceptable alternative $z \in Z \in \{X, Y\}$. We normalize the ranking of an outside option to be zero as we will restrict our attention to the family of adjacent rules.[51] Given an adjacent rule $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$, let $f_{\ell,j} = (f^X_{\ell j}, f^Y_{\ell j}) \in X \times Y$ denote the bundle it proposes when the negotiators' types are $\theta_1^{x_\ell}$ and $\theta_2^{x_j}$. The *rank variance of the bundle $f_{\ell,j}$* is defined as[52]

$$var(f_{\ell,j}) \equiv \sum_{i\in I} \left(r_i(f^X_{\ell j})\right)^2 + \left(r_i(f^Y_{\ell j})\right)^2 .$$

Then, the **rank variance** of a rule f is the total sum of the rank variances of all possible outcomes of f and defined as

$$Var(f) \equiv \sum_{\ell=1}^{m}\sum_{j=1}^{m} var(f_{\ell,j}).$$

[50]The interpretation a CS rule is analogous for the case of even number of alternatives.

[51]This normalization is without loss of generality for our analysis because all adjacent rules recommend the outside option only for the upper half of the matrix, in which case any adjacent rule selects the same outcome for issue X, i.e., the outside option o_X.

[52]The following formulation of rank variance assigns equal weights to both issues. One may also consider assigning different weights to different issues. Theorem 4 still holds in that case due to the symmetric structure of the logrolling bundles.

Intuitively, the larger the differences between the two negotiators' rankings of the alternatives in a given bundle, the higher the rank variance of that bundle. For example, while never recommended by an adjacent rule, the bundles (x_1, y_1) and (x_m, y_m) have the highest rank variance. Despite making one negotiator as well off as possible, they make the opposite negotiator as worse off as possible. In this sense, the larger the rank variance of a mediation rule, the more skewed it is toward extremal bundles. The next result shows that rank variance is minimized only via a CS rule within the class of adjacent rules.

Theorem 4. *A mediation rule minimizes rank variance within the class of strategy-proof, efficient, and individually rational rules if and only if it is a constrained shortlisting rule.*

The characterization of a CS rule in Theorem 4 essentially follows from the fact that the median logrolling bundle has the smallest rank variance among all logrolling bundles. Furthermore, the rank variance of a logrolling bundle grows as it gets further away from the median, e.g., the negotiator-optimal bundles have the highest rank variance. Put differently, the median bundle can be viewed as the "center of gravity" of the set of logrolling bundles. A CS rule assigns the highest precedence to the median logrolling bundle (and next precedence to those that are closest to it and so on) and thereby ensures that this bundle is the recommended outcome whenever it is mutually acceptable. For a continuous analogue of our model where the two issues represent goods x and y to be divided and the possible ways of dividing the two goods are symmetrically distributed where the median alternative corresponds to equal-split (as in Example 3), it is plausible to argue that the median logrolling bundle automatically achieves a strong form of fairness. Indeed, regardless of the negotiators' preferences, the median logrolling bundle corresponds to equal division of the two goods and thus always guarantees *envy-freeness*.

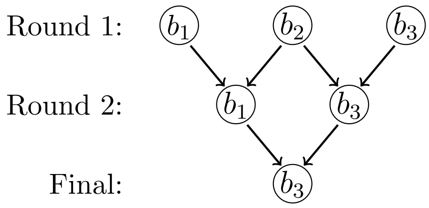
5. Necessary and Sufficient Conditions for Strategy-Proofness

Theorem 1 states that a strategy-proof, efficient, and individually rational rule must be an adjacent rule, and therefore must always choose a logrolling bundle when a mutually acceptable solution exists. The logrolling (quid pro quo) restriction on preferences requires that negotiators are able to compare *all* logrolling bundles unambiguously in a certain way and guarantees that any adjacent rule is strategy-proof. While a sufficient condition, it is not necessary to obtain a strategy-proof mechanism. For example, if only negotiator 1's preferences satisfy the logrolling property but negotiator 2 cannot unambiguously compare all logrolling bundles, then the negotiator 1-optimal rule is still strategy-proof. Indeed, under the negotiator 1-optimal rule, negotiator 2 can only choose between getting either a logrolling bundle or the disagreement outcome. As another example, consider the CS rule with an odd number of alternatives. By similar reasoning, the CS rule remains strategy-proof so long as negotiator 1's preferences satisfy the logrolling property for the first half of the alternatives, i.e., alternatives x_1 through the median alternative x_n, and negotiator 2's preferences satisfy the logrolling property for the last half of the alternatives, i.e., alternatives x_m through the median alternative x_n.

In this section we ask how far we can weaken the logrolling property, and to this end, provide two types of equivalent conditions on preferences either of which is necessary and sufficient for strategy-proofness. The first condition, called *weak logrolling,* is a recursive and algorithmic definition on the set of logrolling bundles, whereas the second condition is based on an order-theoretic semilattice structure that the set of logrolling bundles must satisfy. Note that, by Theorem 1, any rule that survives strategy-proofness under these weaker conditions must still be a member of the class of adjacent rules.

The first condition is essentially an iterative process that requires us to consider a series of pairings of logrolling bundles, where it should always be possible for at least one negotiator to unambiguously compare the bundles in each pair. Before formally stating this condition, we find it useful to describe this process using an analogy of an "elimination tournament" among logrolling bundles. As an example, consider the mediation problem with three alternatives in each issue. Imagine that the tournament starts with all logrolling bundles ordered from b_1 to b_3 (see the left side of Figure 5.) In the first round of the tournament, each logrolling bundle matches up with all of its neighbors, i.e., both b_1 and b_3 match with b_2. In the matchup between b_k and b_{k+1}, the "winner" is b_{k+1} if negotiator 1 unambiguously ranks b_{k+1} at least as high as b_k whenever two of these bundles are acceptable to her. Similarly, the winner is b_k if negotiator 2 unambiguously ranks b_k at least as high as b_{k+1} whenever two of these bundles are acceptable to her. (This is formalized by the notion of concatenation below.) When both negotiators are able to compare these two bundles, then the winner can be any of these two bundles, but only one of them will proceed to next round. In the example below, we suppose that only negotiator 2 unambiguously compares b_1 with b_2, and so b_1 "eliminates" b_2 and moves on to the second round, whereas b_2 loses to b_3 because only negotiator 1 can unambiguously compare b_3 and b_2.[53] Then, in the second round (see the placements on the second diagonal), b_1 and b_3 will match up. Once again, the winner will be b_3 (respectively b_1) if negotiator 1 (respectively 2) can unambiguously rank b_3 over b_1 (respectively b_1 over b_3) whenever they are acceptable. Suppose both bundles are unambiguously ranked by both negotiators in the desired way. Then either bundle can be a winner of this round. In the illustration below, we have arbitrarily chosen b_3 between the two to proceed to the final round (where it wins the tournament).

[53] More precisely, 2 unambiguously ranks b_1 over b_2 and 1 unambiguously ranks b_3 over b_2.



b_1		
b_1	b_2	
b_3	b_3	b_3

Figure 5: *An example for the elimination tournament and the matrix representation for the corresponding mediation rule*

The main idea of our necessity result is that it is possible to obtain a strategy-proof mechanism for a given preference domain if and only if one can construct a tournament of the above form among the set of logrolling bundles where there is always a winner of each match of each round. In particular, each round of the tournament represents a corresponding diagonal of the matrix of the mediation rule to be constructed. The winners of each round fill up the corresponding entries of the next diagonal and the process repeats until only one logrolling bundle remains. For example, for the tournament described above, the order of the logrolling bundles in the first round gives the placement order of these bundles (from the top corner to the bottom corner) along the first diagonal, the order in the second round gives the placement order along the second diagonal, and the last winner, b_3, fills up the bottom left entry of this matrix (which is the last diagonal). The constructed rule is the adjacent rule $f^{\rhd}$ with $\rhd$: $b_3,\ b_1,\ b_2$ where the winner of a match gains precedence over the loser. This rule can be easily verified to be strategy-proof under the preference restrictions given in our example. The following series of definitions formalizes these insights.

Definition 9. *The extension map Λ is **consistent** if it is regular and if for all $i \in N$, $\theta_i \in \Theta_i$, and $b = (x, y), b' = (x', y') \in X \times Y$ with $x, x' \in A(\theta_i)$ we have*

1. $\Big[b\ I_i\ b' \text{ for all } R_i \in \Lambda(\theta_i)\Big] \Longrightarrow \Big[b = b'\Big]$, *and*

2. $\Big[b\ R_i\ b' \text{ for all } R_i \in \Lambda(\theta_i)\Big] \Longrightarrow \Big[b\ R_i\ b' \text{ for all } R_i \in \Lambda(\theta_i') \text{ and } \theta_i' \in \Theta_i\Big]$.[54]

The first condition implies that if negotiators are indifferent between two bundles at all admissible preferences, then these two bundles must be the same. The second condition implies that if some types of a negotiator can unambiguously rank two acceptable bundles, then all types of that negotiator who deem these two bundles acceptable should unambiguously and similarly rank them. Recalling our tournament analogy, which logrolling bundle will be matched with which bundle at what round is formalized by the following adjacency requirement.

Definition 10. *Let B be a nonempty subset of $\mathbf{B}$ and the bundles $b = (x_\ell, y_{m-\ell+1}), b' = (x_j, y_{m-j+1})$ are in B. Then b' is **adjacent to bundle** b **in** B if there exists no bundle*

[54] Let I_i denote the indifference part of R_i, i.e., $b\ I_i\ b'$ if and only if $b\ R_i\ b'$ and $b'\ R_i\ b$.

$(x_k, y_{m-k+1}) \in B$ *with* $\ell < k < j$ *(or* $j < k < \ell$*) whenever* $\ell < j$ *(or* $j < \ell$*). We call such two bundles adjacent in* B *and denote* $b' \in B(b)$.

Note that adjacency is symmetric, that is, if bundle b is adjacent to b' in B, then b' is also adjacent to b in B. We formalize the requirement that there exists a winner in every match at any round of the tournament by iterating the following completeness requirement.

Definition 11. *A binary relation* $\rhd$ *on* $\mathbf{B}$ *is* ***complete with respect to adjacency on*** $B \subseteq \mathbf{B}$ *if for any two distinct and adjacent bundles* $b, b' \in B$*, we have either* $b \ \rhd \ b'$ *or* $b' \ \rhd \ b$.

Let $\rhd$ be a binary relation over the set of logrolling bundles. Set $B^0_{\rhd} = \mathbf{B}$ and recursively define $B^k_{\rhd} = \{b \in B^{k-1}_{\rhd} \mid b \ \rhd \ b' \text{ for some } b' \in B^{k-1}_{\rhd}(b)\}$ for $k = 1, 2, ...$ If the binary relation $\rhd$ is antisymmetric, then the number of elements in each $B^k_{\rhd}$ is at most $m - k$ for $k < m$ and 1 for all $k \geq m$. We next formalize the requirement that there exists a winner of each match of each round by the following connectedness requirement.

Definition 12. *The binary relation* $\rhd$ *on* $\mathbf{B}$ *is called* ***connected*** *if it is complete with respect to adjacency on all* $B^k_{\rhd}$, $k = 0, 1, ...$

Finally, the existence of at least one such elimination tournament among the set of logrolling bundles, in which there always exists a winner of all matches at all rounds, is guaranteed by the following requirements.

Definition 13. *The extension map* Λ *admits a binary relation* $\rhd$ *on* $\mathbf{B}$ *that* ***concatenates negotiators' preferences*** *if for any two distinct logrolling bundles* $b = (x_\ell, y_{m-\ell+1}), b' = (x_j, y_{m-j+1}) \in \mathbf{B}$, $b \ \rhd \ b'$ *implies*

1. $b \ R_1 \ b'$ *for all* $R_1 \in \Lambda(\theta_1)$ *and all* $\theta_1 \in \Theta_1$ *with* $x_\ell, x_j \in A(\theta_1)$ *whenever* $j < \ell$,
2. $b \ R_2 \ b'$ *for all* $R_2 \in \Lambda(\theta_2)$ *and all* $\theta_2 \in \Theta_2$ *with* $x_\ell, x_j \in A(\theta_2)$ *whenever* $j > \ell$.

Definition 14. *The extension map* Λ *satisfies* ***weak logrolling*** *if it admits a transitive, antisymmetric, and connected binary relation* $\rhd$ *over the set of logrolling bundles that concatenates negotiators preferences. In that instance we call* $\rhd$ ***admissible with respect to*** Λ.

We are now ready to introduce our first necessary and sufficient condition.

Theorem 5. *Suppose that* Λ *is consistent. The mediation rule* f *is strategy-proof, efficient, and individually rational if and only if* Λ *satisfies weak logrolling and* $f = f^{\rhd}$ *for some binary relation* $\rhd$ *that is admissible with respect to* Λ.

Checking whether an extension map satisfies the weak logrolling condition essentially requires a recursive algorithmic procedure. We next provide a more concise necessary and sufficient condition for a possibility result using order theory.

Recall that the full logrolling property implies that the set of logrolling bundles forms a simple lattice with respect to either negotiator's preferences. Our next result shows that while a lattice structure is sufficient, one needs only the set of logrolling bundles to form a type of semilattice to guarantee the existence of a strategy-proof rule. Specifically, a **join-semilattice** is a partially ordered set that has a *join* (a least upper bound) for any nonempty finite subset.

The partial order over $\mathbf{B}$ that we will use in our semilattice construction is based on a reflexive completion of the above defined binary relation on $\mathbf{B}$ that concatenates negotiators' preferences. We say that a partial order $\trianglerighteq$ on $\mathbf{B}$ *reflexively concatenates negotiators' preferences* if it concatenates negotiators' preferences and is reflexive, namely $b \trianglerighteq b$ for any $b \in \mathbf{B}$.

We say that a subset of the set of logrolling bundles is *linked* if it is the hypotenuse of some triangle. Put differently, there are no "gaps" in a linked set; it consists of logrolling bundles that are adjacent to each other.

Definition 16. *A set $B \subset \mathbf{B}$ is **linked** if $B = \{(x_k, y_{m-k+1}) \in \mathbf{B} | j \leq k \leq \ell\} \equiv \mathbf{B}_{\ell j}$ for some ℓ, j with $1 \leq j \leq \ell$.*[55]

The next result shows that a strategy-proof, efficient, and individually rational mediation rule induces a join-semilattice structure on the set of logrolling bundles as well as on any of its linked subsets. Conversely, if all linked subsets of $\mathbf{B}$ form join-semilattices, then a mediation rule satisfying the three properties always exists.

Theorem 6. *Given the consistent extension map Λ, there exists a mediation rule f satisfying strategy-proofness, efficiency, and individual rationality if and only if Λ admits a partial order $\trianglerighteq$ on $\mathbf{B}$ that reflexively concatenates negotiators' preferences such that for any linked set $B \subset \mathbf{B}$, $(B, \trianglerighteq)$ is a join-semilattice.*

Figure 6 illustrates a join-semilattice that corresponds to the adjacent rule in Figure 2. This semilattice essentially delineates the full set of restrictions on the extension map Λ that is necessary and sufficient for the particular adjacent rule to be strategy-proof. It is worth observing that in addition to $\mathbf{B}$, all of its linked subsets, e.g., $\mathbf{B}_{13}, \mathbf{B}_{48}$, can be seen to form a join-semilattice when equipped with the partial order $\trianglerighteq$) whereas a nonlinked set, e.g. $\mathbf{B}_{45} \cup \mathbf{B}_{78}$, does not necessarily form a join-semilattice.

[55] By this definition, a linked set is nonempty. Note that set $\mathbf{B}$ itself is linked and so is any singleton subset of $\mathbf{B}$.

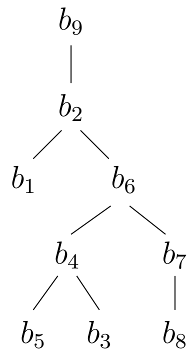


Figure 6: *The join-semilattice structure that is essential for the strategy-proofness of the adjacent rule in Figure 2*

6. Mediation with Continuum of Alternatives

Our main model assumes a discrete set of alternatives for both issues. In this section we extend the characterization of the class of adjacent rules to a continuous analogue of our model.[56]

Suppose now that the issues X and Y are two closed and convex intervals of the real line. The outside options, o_X and o_Y, may or may not be the elements of these sets. We assume, without loss of generality, that $X = Y = [0, 1]$, with the interpretation that the negotiators aim to divide a unit surplus in each issue. To keep the notation consistent with the previous section, let a bundle $b = (x, y)$ indicate what negotiator 2 gets in the two issues, i.e., negotiator 2 gets $x \in X$ and $y \in Y$, and thus, negotiator 1 gets the remaining $1 - x$ and $1 - y$ in issues X and Y. Agents having diametrically opposed preferences on each issue means that for any issue $Z \in \{X, Y\}$ and two alternatives $z, z' \in Z$, negotiator 1 (respectively 2) prefers z to z' whenever $z < z'$ (respectively $z > z'$). The value/ranking of the outside option o_X in issue X is each negotiator's private information. However, the value/ranking of the outside option o_Y in issue Y is common knowledge, and both negotiators prefer all $y \in Y$ to o_Y.

For any $\ell \in [0, 1]$, type ℓ of negotiator 1 (respectively 2), denoted by θ_1^ℓ (respectively θ_2^ℓ), prefers the outside option o_X to all alternatives $k \in [0, 1]$ with $\ell < k$ (respectively $\ell > k$).[57] Parallel to the discrete case, we denote the mediation rule $f = [f_{\ell,j}]_{(\ell,j)\in[0,1]^2}$ where $f_{\ell,j} = f(\theta_1^\ell, \theta_2^j)$ for all $0 \le \ell, j \le 1$.[58] The negotiators have no mutually acceptable

[56] Matsuo (1989) shows that it is possible to overcome the impossibility in the bilateral exchange model of Myerson and Satterthwaite (1983) by restricting to a finite number of types. This section also shows that the possibility results in our context are not driven by the finiteness of the number of types in our main model.

[57] In other words, $1 - \ell$ (respectively ℓ) is the least acceptable amount of X for type θ_1^ℓ (respectively θ_2^ℓ). Therefore, all k with $\ell \ge k$ (respectively $\ell \le k$) are deemed acceptable by type θ_1^ℓ of negotiator 1 (respectively type θ_2^ℓ of negotiator 2).

[58] We assume, without loss of generality, that each negotiator has at least one acceptable alternative. Therefore, there is no type profile where a negotiator deems all alternatives unacceptable.

alternative in issue X at type profile $(\theta_1^\ell, \theta_2^j)$ when $\ell < j$. The set of mutually acceptable alternatives is $A(\theta_1^\ell, \theta_2^j) = [j, \ell]$ whenever $\ell \geq j$. We use Θ_i as the set of all types of negotiator i and $\theta_i \in \Theta_i$ as the generic element whenever there is no need to specify the type's least acceptable alternative. The monotonicity and the deal-breakers assumptions of the regularity condition in the previous sections directly apply here. The same is true for the definitions of strategy-proofness, efficiency, and individual rationality. We need a slight modification in the logrolling property for the continuous model.

Definition 19. *The extension map Λ satisfies logrolling if there exists a unique function $t : X \to [0,1]$ such that for all i, $\theta_i \in \Theta_i$, $R_i \in \Lambda(\theta_i)$ and all $x, x' \in A(\theta_i)$ with $x \; \theta_i \; x'$, we have $\big(x', t(x')\big) \; R_i \; \big(x, t(x)\big)$.*

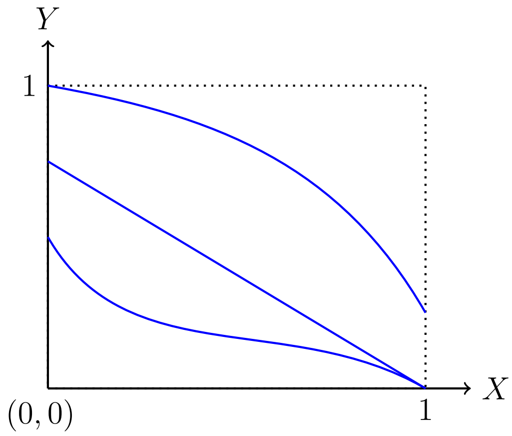


Figure 7: A possible set of logrolling bundles

Each t function in Figure 7 (in fact any such decreasing function) could be used to represent the set of logrolling bundles. In keeping with our main model, uniqueness of the function t in this definition ensures a unique set of logrolling bundles. However, as we elaborate in the discussion in Section 8, the uniqueness of this function is not essential for a possibility result in either the discrete or the continuous model.[59] Again let $\mathbf{B}$ denote the set of logrolling bundles. For all values of $\ell, j \in [0,1]$ with $j \leq \ell$, let $\mathbf{B}_{\ell j} = \{(x,y) \in \mathbf{B} \mid j \leq x \leq \ell\}$ denote the set of all mutually acceptable logrolling bundles at type profile $(\theta_1^\ell, \theta_2^j)$.

Define $\rhd$ to be a complete, transitive, and antisymmetric binary relation over the set of logrolling bundles. When $(\mathbf{B}, d)$ is a metric space with a proper metric d, $\mathbf{B}_{\ell j}$ with $\ell \geq j$ is a nonempty and compact subset of the set of logrolling bundles.

Definition 20. *The binary relation $\rhd$ is said to be* ***quasi upper-semicontinuous over*** $\mathbf{B}_{\ell j}$ *with $\ell \geq j$ if for all $a, c \in \mathbf{B}_{\ell j}$ with $a \neq c$, $a \rhd c$ implies that there exists a bundle $a' \in \mathbf{B}_{\ell j}$ and a neighborhood $\mathcal{N}(c)$ of c such that $a' \rhd b$ for all $b \in \mathcal{N}(c) \cap \mathbf{B}_{\ell j}$.*[60]

[59] In fact, if there were multiple such functions satisfying the logrolling property, then each one would generate a different set of logrolling bundles and a separate family of strategy-proof, efficient, and individually rational rules of the form that we characterize in the next result.

[60] This is Definition 2 in Tian and Zhoub (1995).

The binary relation $\rhd$ is quasi upper-semicontinuous if it is quasi upper-semicontinuous over all compact subsets $\mathbf{B}_{\ell j}$ of $\mathbf{B}$. A bundle $b^\star \in \mathbf{B}_{\ell j}$ is said to be a maximal element of the binary relation $\rhd$ on $\mathbf{B}_{\ell j}$, i.e., $b^\star \in \underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}}$ if $b^\star \rhd b$ for all $b \in \mathbf{B}_{\ell j}$. Theorem 1 of Tian and Zhoub (1995) proves that quasi upper-semicontinuity is both necessary and sufficient for $\rhd$ to attain its maximum over all compact subsets $\mathbf{B}_{\ell j}$ of $\mathbf{B}$. Therefore, the analogous version of Theorem 2 in the continuous case reads as follows.

Theorem 7. *Suppose that the regular extension map Λ satisfies logrolling. The mediation rule f is strategy-proof, efficient, and individually rational if and only if there exists a complete, transitive, antisymmetric, and quasi upper-semicontinuous binary relation $\rhd$ over the set of logrolling bundles $\mathbf{B}$ and some $y \in Y \setminus \{o_Y\}$ such that*

$$f_{\ell,j} = \begin{cases} (o_X, y), & \textit{if } \ell < j, \\ \underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}} \rhd, & \textit{oth.} \end{cases}$$

Analogous to the discrete case, we use the following continuously indexed matrix to describe a mediation rule f.

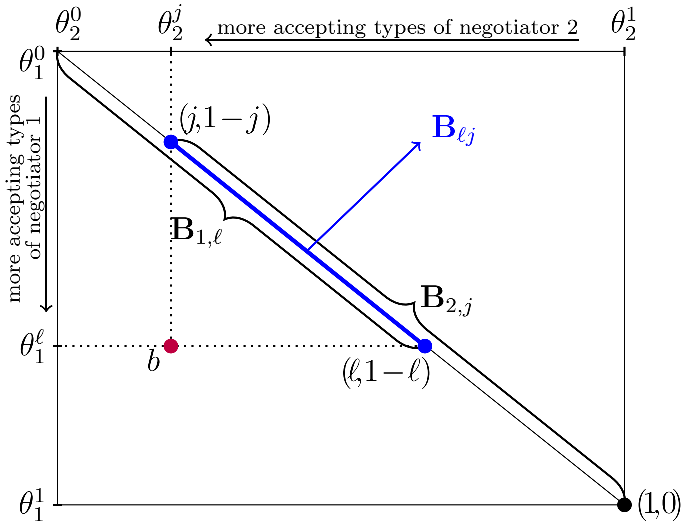


Figure 8: Adjacent rules in the continuous case when $t(x) = 1 - x$

The rows, i.e., the vertical axis, correspond to the types of negotiator 1 and columns, i.e., the horizontal axis, indicate all possible types of negotiator 2. Each point on the main diagonal represents a logrolling bundle for the mediation rule that is described in Theorem 7, and each logrolling bundle appears only once on this diagonal. The bundle b, for example, represents the value of f when the types of negotiator 1 and 2 are θ_1^ℓ and θ_2^j, respectively. When the type profile is (θ_1^1, θ_2^1), negotiator 1 finds all alternatives acceptable and negotiator 2 deems all alternatives except 1 unacceptable.

Let us assume, without loss of generality, that $t(x) = 1 - x$. This implies that the set of logrolling bundles is $\mathbf{B} = \{(x, y) \in [0, 1]^2 \mid y = 1 - x\}$. Then the only mutually acceptable logrolling bundle is $(1, 0)$ at type profile (θ_1^1, θ_2^1). The set of all acceptable logrolling bundles for type θ_1^ℓ of negotiator 1 is denoted by $\mathbf{B}_{1,\ell}$, which consists of all the logrolling bundles on the upper portion of the main diagonal, starting from the top-left corner bundle, $(0,1)$, and goes all the way down to the bundle $(\ell, 1-\ell)$. That is, $\mathbf{B}_{1,\ell} = \{(k, 1-k) \in \mathbf{B} \mid 0 \leq k \leq \ell\}$. Similarly, the set of all acceptable logrolling bundles for type θ_2^j of negotiator 2 is represented by $\mathbf{B}_{2,j}$ and consists of all the bundles on the lower portion of the main diagonal, i.e., all bundles from $(j, 1-j)$ to $(1,0)$. Namely, $\mathbf{B}_{2,j} = \{(k, 1-k) \in \mathbf{B} \mid j \leq k \leq 1\}$. Thus, the set of mutually acceptable logrolling bundles at the type profile $(\theta_1^\ell, \theta_2^j)$ is the intersection of these two sets, i.e., $\mathbf{B}_{\ell j} = \mathbf{B}_{1,\ell} \cap \mathbf{B}_{2,j}$. Theorem 7 says that bundle $b = f_{\ell,j}$ is the logrolling bundle that maximizes $\rhd$ over the set $\mathbf{B}_{\ell j}$ (see Figure 8). Such a maximal bundle is always unique because $\rhd$ is antisymmetric.

7. Discussion and Extensions

In this section, we provide a general discussion of our main model in light of the results obtained so far. To this end, first, we elaborate on some of our essential modeling assumptions, discuss the role they play in driving the positive results of our paper, and offer directions in which they can be extended to cases not covered in the main exposition. Second, drawing on our findings, we continue the discussion we started in the related literature section on the comparison between the cardinal/Bayesian and the ordinal mechanism design approaches. Specifically, we consider how one can go about formulating the mediation problem in a standard Bayesian setting such as that of Myerson and Sattertwaite (1983) [MS] and offer a reconciliation of the possibility results in our setup with the impossibility result in the MS setting.

Symmetric treatment of the outside options: We start by exploring how our results would change if the outside option in issue Y were also treated as each negotiator's private information. In particular, we consider a relaxation of the assumption that $y\ \theta_i^Y o_Y$ for all $i \in \mathrm{I}$ and $y \in Y \setminus \{o_Y\}$. Now the ranking of each of her outside options is a negotiator's private information. Let $\Theta_i = \Theta_i^X \times \Theta_i^Y$ denote the set of all **types** of negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ the set of all type profiles. We now also need to adjust the regularity assumption concerning the negotiators' preferences over bundles. Specifically, we need to modify the monotonicity and the deal-breakers conditions since both issues can now potentially have unacceptable alternatives.

Definition 18. *Under the symmetric treatment of the outside options, the extension map Λ is regular if the following hold for all i, $\theta_i \in \Theta_i$ and all $R_i \in \Lambda(\theta_i)$:*

i. [Monotonicity] For any $x, x' \in X$ and $y, y' \in Y$ with $(x,y) \neq (x', y')$,

$$(x,y)\ P_i\ (x',y') \text{ whenever } \left[x\ \theta_i^X\ x' \text{ or } x = x'\right] \text{ and } \left[y\ \theta_i^Y\ y' \text{ or } y = y'\right].$$

ii. [Deal-breakers] $(o_X, o_Y)\ P_i\ (x,y)$ whenever $o_X\ \theta_i^X\ x$ or $o_Y\ \theta_i^Y\ y$.

Proposition 2. *Under the symmetric treatment of the outside options, there is no mediation rule f that is strategy-proof, individually rational, and efficient.*

Proposition 2 is a fairly straightforward extension of the impossibility result we discussed in case of single issue mediation. Having multiple issues alone is not sufficent to offset the tension between strategy-proofness and efficiency unless the two issues are treated asymmetrically. It is also easy to see the logical independence of the properties in this impossibility. A rule that always picks the pair (o_X, o_Y) is strategy-proof but not efficient. A dictatorship rule is strategy-proof and efficient but not individually rational. A constrained dictatorship rule, where one negotiator always chooses her favorite bundle from the set of individually rational and efficient bundles, violates strategy-proofness despite satisfying the remaining two properties.

More than two issues or negotiators: As we argued earlier, the two-issue model is without loss of generality. If there are more than two issues in the mediation problem, then we can regroup these issues under two types of categories depending on whether negotiators are assumed to have private outside options in an issue. In particular, let category-X be the collection of issues for which a negotiator's least acceptable alternative is her private information and category-Y be the collection of issues in which it is common knowledge that all alternatives are acceptable to both negotiators and the outside option is the least desirable (inefficient) outcome. Under this regrouping, each negotiator now faces a vector of alternatives for each category. The negotiators' preferences over these vectors (of alternatives) need not be diametrically opposed in general. However, as long as the negotiators' preferences are monotonic, by applying the idea in Proposition 1, one can eliminate all the inefficient vectors and find ourselves back in an environment analogous to our main model, in which preferences over vectors are diametrically opposed. When there are multiple parties involved in a dispute as would be the case for community/public disputes, we can similarly regroup them to be represented by either negotiator, effectively treating them as clones of the two negotiators.

Issue-wise voting in the direct mechanism with veto rights: In our main model, we assumed that the negotiators decide to accept or veto the proposed bundle as a whole in the ratification stage of the direct mechanism representing the mediation protocol. An alternative consideration would be to allow the negotiators to vote separately for each individual alternative in the proposed bundle such that unless both negotiators accept the alternative being voted on, the outside option is chosen as the final outcome in the corresponding issue. In this case, revealing one's type truthfully in the announcement stage may no longer be an optimal strategy even if the mediation rule is strategy-proof. Consider, for example, the negotiator-1 optimal rule. Suppose negotiator 1 reports a type

$\theta_1^{x_\ell}$ with $\ell < m$. Suppose negotiator 2's true type is $\theta_2^{x_j}$ with $j > \ell$. When negotiator 2 reports truthfully, the rule picks the disagreement bundle (o_X, y_1) as the recommendation in the announcement stage of the direct mechanism. Both o_X and y_1 prevail in the ratification stage when voted on individually. Suppose negotiator 2 were instead to report type $\theta_2^{x_{j'}}$ with $j' \leq \ell$, in which case the rule picks the logrolling bundle $(x_\ell, y_{m-\ell+1})$ as the recommendation in the announcement stage. In the ratification stage, negotiator 2 vetoes the unacceptable alternative x_ℓ and the outcome of the mechanism is $(o_X, y_{m-\ell+1})$, i.e., 2 gains by misreporting in the announcement stage. All adjacent rules can similarly be shown to be manipulable under issue-wise voting. We conclude that there is no dominant strategy incentive compatible and efficient direct mechanism with veto rights under issue-wise voting.

The general impossibility of truthfully eliciting negotiators' private information under issue-wise voting underlines the importance of jointly resolving the two issues. In particular, bundling alternatives from different issues allows the negotiators to trade favors, which our analysis reveals to be manifested by the logrolling bundles. Consequently, to achieve dominant strategy incentives together with efficiency, it is paramount that the ratification stage only allows for voting on proposed bundles as a whole.

More alternatives in issue Y: We assume that the number of alternatives in issue Y is no less than the number of alternatives in issue X, i.e., $\#Y \geq \#X$. When there are more alternatives in issue Y, then Theorem 1 needs to be modified slightly to allow for all one-to-one t functions since there can now be multiple possible pairings between the alternatives in the two issues, i.e., multiple sets of logrolling bundles. Specifically, at any given pair of bundles in the set of logrolling bundles (that comprise the diagonal), the more preferred alternative from X must always paired up with the less preferred alternative from Y, i.e., for any pair $b = (x, y), b' = (x', y') \in \mathbf{B}$, for all i, $\theta_i \in \Theta_i$ and all $x, x' \in A(\theta_i)$, we have $x \ \theta_i^X \ x' \implies y' \ \theta_i^Y \ y$. Any such function that permits reversal of preferences as in Definition 5 suffices to generate a class of strategy-proof, efficient, and individually rational mediation rules as decsribed in Theorem 2. In this case, there can be multiple adjacent rules families depending on which set of logrolling bundles is chosen to form the diagonal of the matrix.

Possibility without deal-breakers: The deal-breakers assumption provides tractability in our analysis. One might wonder what role this assumption plays in obtaining a possibility result. We provide an example (also see Example 5) to show that the deal-breakers property is not necessary for a possibility result.

Example 4 (Possibility despite the failure of deal-breakers): Suppose that $m = 3$ and consider our model with monotonic preferences satisfying logrolling. We expand this domain of preferences by adding preferences that satisfy the following rankings:

$\theta_1^{x_1}$	$\theta_1^{x_2}$	$\theta_1^{x_3}$	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$[(x_1, y_3)]$	$[(x_2, y_2)]$	$[(x_3, y_1)]$	$[(x_1, y_3)]$	$[(x_2, y_2)]$	$[(o_X, y_1)]$
$[(o_X, y_1)]$	$[(o_X, y_1)]$	$\begin{bmatrix}(x_2, y_2)\\(x_1, y_3)\\(o_X, y_1)\end{bmatrix}$	$[(x_2, y_2)]$	$[(o_X, y_1)]$	(o_X, o_Y)
$\begin{bmatrix}(x_3, y_1)\\(o_X, y_2)\\(x_2, y_1)\end{bmatrix}$	(x_1, y_3)		(x_3, y_1)	(x_3, y_1)	$\begin{bmatrix}(x_1, y_3)\\(x_2, y_2)\end{bmatrix}$
	(x_3, y_1)		(o_X, y_3)		
	(o_X, y_2)	(o_X, o_Y)	$[(o_X, y_1)]$	$[(x_1 y_3)]$	
(o_X, o_Y)	(o_X, o_Y)		(o_X, o_Y)	(o_X, o_Y)	

While the above is one specific preference profile where the deal-breakers assumption is violated, one can include as many preference profiles of this form to our domain as long as the relative rankings of the bundles in the brackets are preserved. (The relative rankings of the bundles within the same brackets can be chosen arbitrarily.) It is easy to verify that negotiator 1-optimal rule is still strategy-proof, efficient, and individually rational.

8. Conclusion

Mediation is a preferred alternative dispute resolution method thanks to the cost-effectiveness, speed, and convenience it affords to all parties involved. The need for structured and rigorous mediation protocols in practice has often been stressed by researchers and practitioners alike. Taking a market design approach to this problem, we sought systematic rules mediators can rely on to for delivering consistent, transparent, and objective recommendations. We considered rules that have a simple preference reporting language: negotiators only report their least acceptable alternatives in the main issue. It turns out that complementing the main issue with a second issue—a piece of advice often voiced by pioneers in the field—is key to achieving strategy-proof, efficient, and individually rational rules. Any such rule belongs to the family of adjacent rules, which require that the mediator's recommendation must always be a logrolling bundle when a mutual agreement is feasible, i.e., a bundle that complements a (rank-wise) more preferred alternative in one issue with a (rank-wise) less preferred alternative in the other. A sufficient (and necessary) condition for strategy-proofness is the logrolling (weak logrolling) property of preferences that necessitates the alternatives in the second issue to be interesting enough relative to those in the main issue. The constrained shortlisting rule is the central member within the characterized class and aims to make recommendations as close to the median logrolling bundle as possible. We contend that this rule is intuitive and simple enough to be used as a standardized protocol for finding the middle ground between disputing parties in practice.

Our approach marries the two distinct literatures of bargaining and matching. The former literature emphasizes the role of private information and outside options in mechanism design with transferable utility. In this literature the optimal mechanism is rarely first-best. The latter literature offers blueprints for designing robust protocols in assignment problems that often arise in practice. The multiple-assignment nature of the problem at hand in our study, however, is less than encouraging in light of the abundance of negative results in that literature. Our analysis confirms these challenges in that possibility results in our framework are also elusive unless the outside options in the two issues are treated asymmetrically. We argued that ordinal mechanisms coupled with strategy-proofness can help obtain detail-free and genuinely simple mechanisms for mediating disputes in practice. Notwithstanding our emphasis on ordinality, the framework developed in this paper can accommodate both transferable and nontransferable utility settings.

SUPPLEMENTARY APPENDIX (Not for Publication)

Relating to the impossibility result of MS

The influential work of MS is an important milestone for bargaining problems with asymmetric information. It is useful to discuss the underlying factors that are absent in the MS model, which may account for the possibility results in our model. Briefly, the mechanism design problem in that model concerns a bilateral trade between a buyer and a seller, who have private information about their valuations of a good. The mechanism has two components: the probability of trade, p, and the transfer, x, both of which are functions of the traders' reports. If no trade occurs, then $x = p = 0$ (the outside option), and so both traders receive zero utility. The utility functions are $U_b = v_b p - x$ for the buyer and $U_s = x - v_s p$ for the seller, where the valuations v_b, v_s are the traders' private information.

It is initially tempting to view the MS model as a two-issue mediation problem, where the probability of trade is one issue and the size of the transfer is the other issue. It is clear from the utility functions that the traders' preferences over the individual issues are indeed diametrically opposed. That is, for any fixed value of x, as p increases, the buyer is better off and the seller is worse off. Similarly, for any fixed value of p, as (any positive value of) x increases, the buyer is worse off. Moreover, the quasi-linear utility functions in MS also satisfy the monotonicity and the logrolling assumptions of our model.

It is already well known in the literature that the denseness of the type space is one of the reasons for the impossibility result in MS.[61] However, this is not the driving force

[61] For example, Matsuo (1989) shows the feasibility of efficient mechanisms in the MS setup when each trader has only two types.

for our possibility result, as shown by the continuous version of our model in Section 6. A notable difference between the two models is that MS takes agents' preferences over bundles as a primitive in the model, whereas we start with preferences over alternatives for each issue, i.e., the marginal preferences, and then generate the set of all possible preferences over bundles that are compatible with the reported marginals. This implies that a trader's outside option in the MS model, i.e., no trade, is defined in conjunction with the two issues as opposed to the issue-wise outside options defined separately in our model, e.g., individual rationality in the MS model implies that $p = 0$ if and only if $x = 0$, whereas in our model an individually rational bundle may choose the outside option for one issue but not the other. Given this difference in how outside options are defined, it is elusive to translate the MS model directly into our framework.

Nevertheless, it may be plausible to draw a rough parallel between the two models. If we think of the two issues in the MS model as sharing a single, joint outside option for each trader, then the two issues can effectively be viewed as one combined issue. In this case, the MS model would correspond to our single-issue mediation model, where strategy-proofness, efficiency, and individual rationality are incompatible. Alternatively, if we think of $p = 0$ and $x = 0$ as two separate outside options, the set of acceptable alternatives in issue x for the buyer must satisfy $pv_b \geq x$ for any fixed value of p, and this set is the buyer's private information as v_b is not common knowledge. The same is true for the set of acceptable alternatives in issue p, and for the seller. Therefore, the set of acceptable outcomes (or the rankings of the outside options in each individual issue) are the traders' private information.[62] In this case, the MS model would correspond to our two-issue model that treats the outside options symmetrically, i.e., the model considered at the beginning of this section. Regardless of whether the outside options in the MS model are viewed to be joint or separate, our conclusions are in agreement with that of MS: for both models, there is no strategy-proof, efficient, and individually rational mechanism. What is needed for a possibility is a new issue which treats the outside option asymmetrically as in the case of issue Y in our model.

To provide an illustration of the above points, in the following example we offer a simple adaptation of the MS setup in our model and demonstrate how one can overcome the impossibility by adding an extra issue:

[62]Furthermore, efficiency in MS implies that the probability of trade is generically either 0 or 1, depending on whether or not the buyer's valuation is higher than the seller's valuation. Namely, for any type profile, a bundle (x, p) is inefficient if $p \in (0, 1)$, and so in line with Proposition 1, we can eliminate all the alternatives $p \in (0, 1)$ from issue p. This suggests that the second issue essentially contains only two alternatives while there is a continuum of alternatives in issue X. Importantly, whether $p = 1$ or $p = 0$ is efficient depends on the traders' reports unlike the case with the alternatives in issue Y of our model.

Example 5 (Possibility in the augmented MS framework): Suppose that the seller and the buyer now negotiate not only over the terms of trade but also over the division of a unit surplus. We refer to the latter as issue Y. The valuations of the good to the buyer and the seller are v_b and v_s, respectively. We assume that each negotiator knows her valuation and believes that the opponent's valuation is distributed over $[0, 1]$ with some probability distribution. The mediator privately solicits the traders' valuations and recommends a quadruple (p, x, y_s, y_b), where p denotes the probability of trade, x is the transfer, and y_s and y_b are the seller's and the buyer's (respectively) share of the unit surplus. The preferences of the two traders are as follows:

$$U_b = pv_b - x + u_b(y_b) \qquad U_s = x - pv_s + u_s(y_s).$$

For simplicity, suppose that $u_b(y) = u_s(y) = y$ and each trader has only two types, $v_b, v_s \in \{0.2, 0.6\}$.

Efficiency implies that $p = 1$ if $v_b \geq v_s$, $p = 0$ if $v_s < v_b$, and $y_b + y_s = 1$. Individual rationality implies that the traders' utilities are nonnegative. Therefore, the following mechanism is strategy-proof, efficient, and individually rational:[63]

	$v_b = 0.6$		$v_b = 0.2$	
$v_s = 0.6$	$p = 1$ $x = 0.6$	$y_s = 0.3$ $y_b = 0.7$	*No trade*	$y_s = 0.5$ $y_b = 0.5$
$v_s = 0.2$	$p = 1$ $x = 0.4$	$y_s = 0.5$ $y_b = 0.5$	$p = 1$ $x = 0.2$	$y_s = 0.7$ $y_b = 0.3$

When issue Y is absent, this mechanism is not strategy-proof. In fact, it is easy to show that there is no strategy-proof, efficient, and individually rational mechanism in that case.

Two observations about this example are worth noting. First, this example also serves to show that the deal-breakers assumption is not necessary for a possibility result in our setup.[64] Second, it hints at a certain similarity between our mechanisms and the celebrated VCG mechanisms. Restoring strategy-proofness in the MS setup comes to the mediator at the cost of an additional unit of surplus. This is much like a budget imbalance that may arise in a VCG mechanism, in which case the planner may be compelled to subsidize the trade.

[63] The seller of type $v_s = 0.2$ has no incentive to mimic type $v_s = 0.6$. This is true because the seller's payoff under truth-telling (which is 0.7 regardless of the buyer's type) is higher than or equal to her deviation payoffs 0.7 (if the buyer is of type $v_b = 0.6$) and 0.5 (if the buyer is of type $v_b = 0.2$). Similarly, the seller of type $v_s = 0.6$ has no incentive to mimic type $v_s = 0.2$. Her payoff under truth-telling is either 0.3 (if the buyer is type $v_b = 0.6$) or 0.5 (if the buyer is type $v_b = 0.2$). However, her deviation payoffs are 0.3 regardless of the buyer's type. Symmetric arguments apply for the buyer.

[64] Indeed, regardless of the realization of their types, either negotiator would rather make no trade, i.e., $p = x = 0$, but get the whole surplus than make a trade via this mechanism.

Logrolling (Quid Pro Quo)

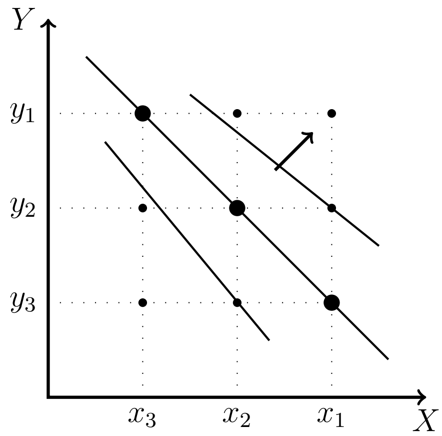
A critical requirement for our possibility results is (weak) logrolling. It requires that the issue Y has sufficient appeal for the negotiators so that they are willing to trade an acceptable alternative in issue X with a less preferred (yet still acceptable) alternative in X. In this supplementary appendix, we provide geometric illustrations of the logrolling condition and further explore its implications. In particular, we seek conditions on negotiators' utility functions that would make the underlying preferences compatible with this condition. To this end, we progressively consider two scenarios. First, we reconsider the setting of our main model, where the set of alternatives is finite but does not necessarily represent the division of a divisible commodity. Next, we consider settings where the set of alternatives may be finite or a continuum and issues represent division of a divisible commodity (as in Example 3 or the model in Section 6).

Let us reconsider the main model. Suppose, for an example, that each issue X and Y has three alternatives, i.e., $m = 3$. Let $U(x, y) = u(x) + v(y)$ be the additively separable the utility function of negotiator 1, where $u(.)$ and $v(.)$ represent preferences over issues X and Y, respectively. The utility specifications are given in the table below. Observe that these preferences are compatible with the logrolling assumption.[65]

X	$u(.)$	Y	$v(.)$	$X \times Y$	$U(.)$
x_1	100	y_1	20	(x_3, y_1)	110
x_2	98	y_2	12	(x_2, y_2)	110
x_3	90	y_3	10	(x_1, y_3)	110

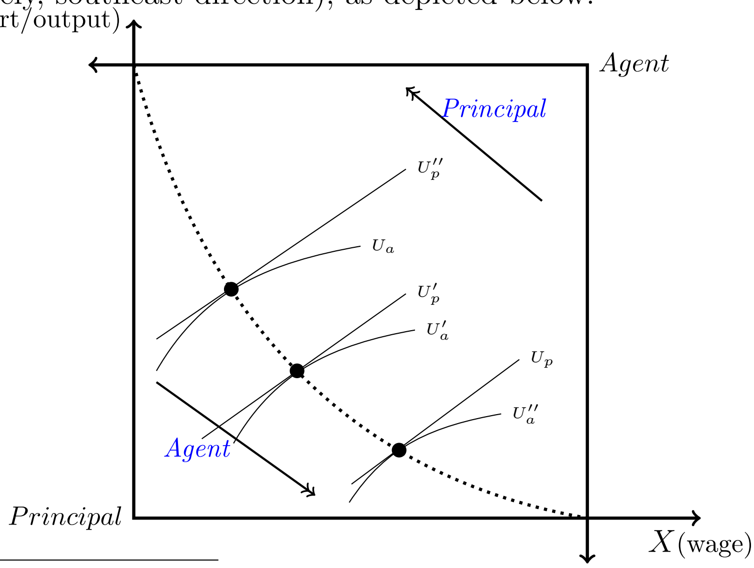
In standard consumer theory, it is customary to represent preferences through indifference curves on a commodity space. In our framework, issues play the same role as commodities so we can still invoke indifference curves. One caveat of such representation is that when the alternatives in a finite set do no correspond to commodity consumption levels, the distance between any two alternatives is immaterial. Therefore, without loss of generality, we position all alternatives equidistantly in the following representation. By convention, we also place less preferred alternatives in each issue closer to the origin, which by monotonicity implies higher indifference curves as we move to the northeast. Some of the indifference curves of negotiator 1, who deems all alternatives acceptable in the above example, can be depicted as follows:

[65]Note that although the worst alternative in issue X is 4.5 times more valuable, in absolute terms, than the best alternative in issue Y, this does not contradict our requirement since logrolling concerns only relative differences in utility.



As evident from this graph, logrolling is a property of the bundles that are placed on the *diagonal*. Logrolling then requires that negotiator 1 is weakly better off as we move along the diagonal to the northwest and, respectively, negotiator 2 (whose indifference curves are not depicted) is weakly better off in the southeast direction.[66]

Suppose now the alternatives (of which there is a finite number) in the two issues represent quantities. For an illustration, let us consider a moral hazard situation between a principal and an agent, where X denotes the set of possible wage levels and Y denotes the set of possible effort/output levels. Let the agent's and the principal's utility functions respectively be $U_a(x, y) = \sqrt{x} - y^2$ and $U_p = Ky - x$ for some large enough $K > 0$. Suppose that the logrolling bundles are chosen from the contract curve (Pareto set) of this problem. Then the logrolling condition is satisfied since the principal (respectively, the agent) becomes better off as we move along the contract curve in the northwest direction (respectively, southeast direction), as depicted below.



[66]For this discrete representation, suppose we define the notion of *marginal rate of substitution (MRS)* at a bundle as the relative gain vs. loss ratio of the number of ranks in issue X vs. Y that keeps the negotiator indifferent, e.g., MRS at bundle (x, y) is the ratio of the number of ranks she should go up in issue Y to the number of ranks she should go down in issue X to remain indifferent. Clearly, the notion may not be well defined at each bundle. Although the indifference curve through the logrolling bundles in this example is linear, i.e., constant MRS of 1, logrolling generally puts no restriction on the shape of the indifference curves, e.g. logrolling is compatible with both convex and concave indifference curves.

Given these illustrations, we are now ready to formalize the sufficiency conditions on negotiators' utility functions that guarantee the logrolling property. For tractability, we now assume a continuum of alternatives in each issue.

Let $\langle X, Y, U_1, U_2 \rangle$ represent the mediation problem with continuum of alternatives, where $X = [\underline{x}, \bar{x}] \subset \mathbb{R}$, $Y = [\underline{y}, \bar{y}] \subset \mathbb{R}$, and $U_i : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable and increasing utility function that represents negotiator i's preferences over the bundles. In this environment, a bundle (x_i, y_i) indicates that negotiator i gets x_i and y_i in issues X and Y, respectively. Thus, an outcome in mediation is a pair of bundles, (x_1, y_1) and (x_2, y_2) one for each negotiator, such that $x_1 + x_2 = \bar{x}$ and $y_1 + y_2 = \bar{y}$. Therefore, in this context, the pair of utility functions U_1 and U_2 satisfies the logrolling property if there exists a set of logrolling bundles, i.e., a pair of functions (t_1, t_2) where $t_i : X \to Y$ for each i and $t_1(x) + t_2(x) = \bar{x}$ for all $x \in X$, such that $U_i(x, t_i(x)) \leq U_i(x', t_i(x'))$ for all i and all $x, x' \in X$ with $x > x'$.

The following lemma provides sufficiency conditions (for logrolling) on the negotiators' utility functions in conjunction with the negotiators' t functions that govern the set of logrolling bundles.

Lemma S.1. *Given any pair of strictly decreasing functions (t_1, t_2), where $t_i : X \to Y$ for each i and $t_1(x) + t_2(x) = \bar{x}$ for all $x \in X$, any pair of increasing and differentiable utility functions U_1 and U_2 satisfy the logrolling property if*

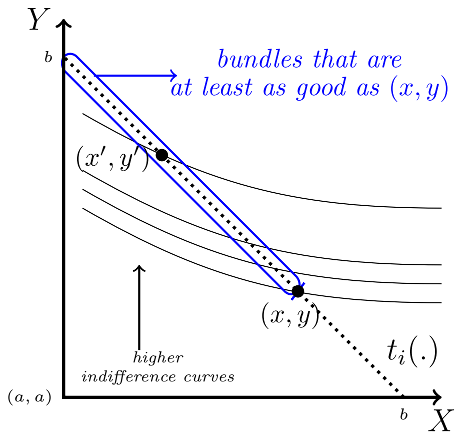
$$\left.\frac{\partial U_i / \partial x}{\partial U_i / \partial t_i(x)}\right|_{(x, t_i(x))} \leq \left.\frac{\partial t_i / \partial x}{\partial t_i / \partial t_i(x)}\right|_{(x, t_i(x))} \tag{1}$$

holds for all i and all $x \in X$. Alternatively, given any pair of increasing and differentiable utility functions U_1 and U_2, any pair of strictly decreasing functions (t_1, t_2), where $t_i : X \to Y$ for each i and $t_1(x) + t_2(x) = \bar{x}$ for all $x \in X$, induce a set of logrolling bundles, and so U_1 and U_2 satisfies the logrolling property, if the inequality (1) holds for all i and all $x \in X$.

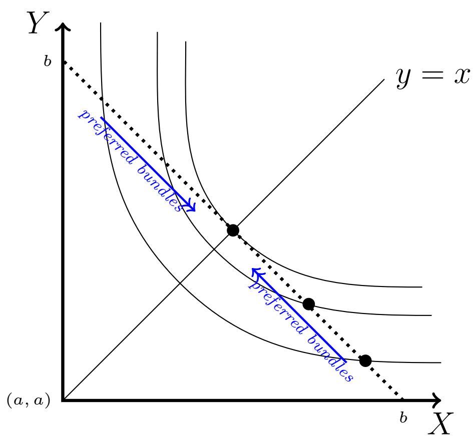
Proof. Let t_1 and t_2 be any two strictly decreasing functions, where $t_i : X \to Y$ for each i and $t_1(x) + t_2(x) = \bar{x}$ for all $x \in X$, and U_1 and U_2 be increasing and differentiable. Because the inequality 1 holds for any $x \in X$, the indifference curve of negotiator i that passes from the point $(x, t_i(x))$ will lie below the curve t_i and never cross it again for points x' located left of x, i.e., $x' < x$. Therefore, for any $x, x' \in X$ with $x > x'$, the indifference curve for negotiator i that passes from the point $(x, t_i(x))$ lies below the indifference curve that passes from the point $(x', t_i(x'))$ because U_i is increasing and t_i is decreasing. Thus, $U_i(x, t_i(x)) \leq U_i(x', t_i(x'))$, and thus, the pair (t_1, t_2) induces a set of logrolling bundles and U_1 and U_2 satisfy the logrolling property. □

Lemma S.1 suggests that if the mediator knows the underlying utility functions of the negotiators, then this information can guide his choice of the set of logrolling bundles. In particular, the t function that characterizes a negotiator's set of logrolling bundles must be steeper than the corresponding indifference curve of that negotiator at these bundles. Conversely, if the mediator has determined the set of logrolling bundles but is uncertain about the negotiators' preferences, then the same restriction essentially reveals what type of utility functions would be compatible with such a choice of logrolling bundles.

In light of Lemma S.1, many well-behaved preferences, e.g., all constant elasticity of substitution (CES) utility functions and quasi linear utility, are compatible with logrolling. For example, in an environment where $X = Y = [a, b] \subseteq \mathbb{R}^2$ and the diagonal consists of all the logrolling bundles (see below), the utility function $U(x, y) = \sqrt{x} + y$ is consistent with logrolling whenever $1/4 \leq a$. This is true because the upper counterset of a bundle (x, y) that is on the diagonal includes all the other bundles (x', y') on the diagonal that are situated northwest of the original bundle (x, y).



Finally, the necessary and sufficient condition weak logrolling is compatible with even larger classes of utility functions. An example is Cobb-Douglass utility. Suppose, for example, that both negotiators' preferences are of the form $U(x, y) = xy$ and the diagonal consists of all the logrolling bundles (see below). Then the logrolling property for the first half of the alternatives in issue X holds for negotiator 1, and the logrolling property for the second half of the alternatives holds for negotiator 2, and thus, preferences are compatible with weak logrolling.



A mediation mechanism with veto rights $\Gamma = \big(S_1, S_2, g(.)\big)$ is a collection of action sets (S_1, S_2) and an outcome function $g : S_1 \times S_2 \to X \times Y$. The mechanism Γ combined with possible types (Θ_1, Θ_2) and preferences over bundles (R_1, R_2) with $R_i \in \Lambda(\theta_i)$ for all i defines a game of incomplete information. A strategy for negotiator i in the game of incomplete information created by a mechanism Γ is a function $s_i : \Theta_i \to S_i$.

Lemma S.2 (Revelation Principle in Dominant Strategies). *Suppose that there exists a mechanism $\Gamma = \big(S_1, S_2, g(.)\big)$ that implements the mediation rule f in dominant strategies. Then f is strategy-proof and individually rational.*

Proof. If Γ implements f in dominant strategies, then there exists a profile of strategies $s^*(.) = \big(s_1^*(.), s_2^*(.)\big)$ such that $g\big(s^*(\theta)\big) = f(\theta)$ for all $\theta \in \Theta$, and for all $i \in I$ and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}(\theta_{-i})\big)\ R_i\ g\big(s_i'(\theta_i'), s_{-i}(\theta_{-i})\big) \tag{2}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$ and all $s_i'(.), s_{-i}(.)$. Condition 2 must also hold for s^*, meaning that for all i and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big)\ R_i\ g\big(s_i^*(\theta_i'), s_{-i}^*(\theta_{-i})\big) \tag{3}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last inequality implies that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i})\ R_i\ f(\theta_i', \theta_{-i}) \tag{4}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$.

Moreover, because mechanism Γ allows each negotiator to veto the proposed bundle and receive the outside options in each issue, there also exists a deviation strategy $\hat{s}_i(.)$ for any strategy $s_i(.)$ such that $g(\hat{s}_i(\theta_i), s_{-i}) = \big(o_X, o_Y\big)$ for all $\theta_i \in \Theta_i$ and all $s_{-i} \in S_{-i}$. The idea is that the negotiator i plays in $\hat{s}_i(.)$ exactly the same way in $s_i(.)$ (for all θ_i's) until the ratification stage and vetoes the proposed bundle.

Therefore, if $\hat{s}_i(.)$ is such a deviation strategy for $s_i^*(.)$, then condition 2 must also hold for $\hat{s}_i(.)$, implying that for all i and $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big)\ R_i\ g\big(\hat{s}_i(\theta_i'), s_{-i}^*(\theta_{-i})\big) = \big(o_X, o_Y\big)$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last condition means that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i})\ R_i\ \big(o_X, o_Y\big) \tag{5}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta'_i \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Hence, conditions 4 and 5 imply that f is strategy-proof and individually rational.[67]

$\square$

APPENDIX

Proof of Proposition 1: Let $\widetilde{A} \subseteq A$ be the set of alternatives that survive the elimination of Pareto inefficient alternatives. That is, none of the alternatives in $\widetilde{A}$ is Pareto inefficient. Renumber the elements in $\widetilde{A}$, and suppose, without loss of generality, that $\widetilde{A} = \{x_1, ..., x_m\}$ where $m \geq 2$, and negotiator 1 ranks alternatives as $x_k \ \widetilde{\theta}_1 \ x_{k+1}$. If x_m is not the best alternative for $\widetilde{\theta}_2$ on $\widetilde{A}$, then there must exist some x_k where $k < m$ such that $x_k \ \widetilde{\theta}_2 \ x_m$. But this contradicts the assumption that x_m is not Pareto inefficient. Thus, negotiator 2 must rank x_m as the top alternative. With similar reasoning, if x_{m-1} is not negotiator 2's second-best alternative, then it must be Pareto inefficient, contradicting the assumption that x_{m-1} survives after the deletion of Pareto inefficient alternatives. Iterating this logic implies that the rankings of the negotiators must be diametrically opposed. $\blacksquare$

Proof of Theorem 1: Now suppose that Λ is regular and the mediation rule f is strategy-proof, efficient, and individually rational.

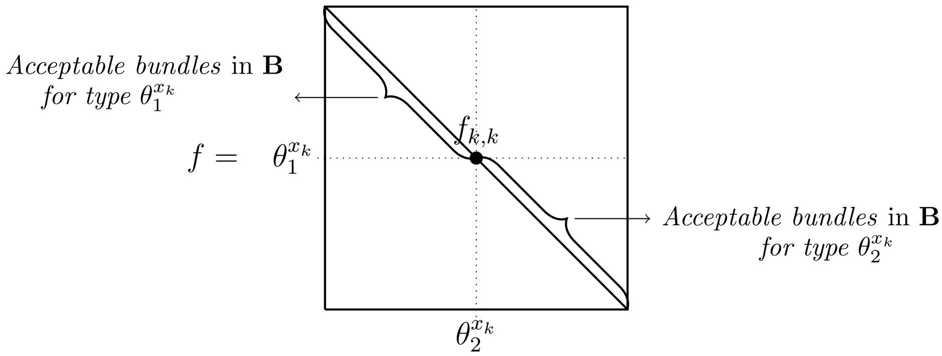
***Proof of Part i*:** By individual rationality and regularity of preferences, the alternative for issue X must be o_X whenever $\ell < j$. Then by regularity and efficiency, $f_{\ell,j} = (o_X, y)$ for some $y \in Y \setminus \{o_Y\}$. By strategy-proofness and monotonicity, we must have $f_{\ell',j} = (o_X, y)$ for all $\ell' < j$. Similarly, $f_{\ell,j'} = (o_X, y)$ for all $\ell < j'$. Fixing j (and ℓ) and applying the same argument for all remaining rows and columns yields $f_{\ell,j} = (o_X, y)$ whenever $\ell < j$.

***Proof of Part ii and existence of t*:** Consider the main diagonal where $\ell = j = k$. Now, we want to show that $f_{k,k} = (x_k, t(x_k))$ for every $k = 1, ..., m$ and $t(x_k) = y_{m+1-k}$. Row and column k correspond to preference profile $(\theta_1^{x_k}, \theta_2^{x_k})$ where the only mutually acceptable alternative in issue X is x_k. Therefore, for any $1 \leq k \leq m$ efficiency and individual rationality of f and regularity of preferences imply $f^X_{k,k} = x_k$ and $f^X_{k+1,k} \in \{x_k, x_{k+1}\}$ whenever $k \neq m$. We claim that $f^Y_{k+1,k+1} \ \theta^Y_1 \ f^Y_{k,k}$ for each $k = 1, ..., m-1$. If this statement is correct, then we have the desired result (including t being unique and one-to-one) because the number of alternatives in issue X and Y is the same.

[67] If negotiators were to approve or veto each issue separately, then we would have $f(\theta_i, \theta_{-i}) \ R_i \ \left(f_X(\theta'_i, \theta_{-i}), o_Y\right)$ and $f(\theta_i, \theta_{-i}) \ R_i \ \left(o_X, f_Y(\theta'_i, \theta_{-i})\right)$, where $f_Z(.)$ denotes the suggested alternative by f in issue Z, together with conditions 4 and 5.

Consider any $1 \le k \le m-1$. If $f^X_{k+1,k} = x_{k+1}$, then strategy-proofness and monotonicity of preferences of negotiator 2 imply that $f^Y_{k+1,k} = f^Y_{k+1,k+1}$. Similarly, strategy-proofness and monotonicity of preferences of negotiator 1 require that $f^Y_{k+1,k+1}\ \theta^Y_1\ f^Y_{k,k}$. On the other hand, if $f^X_{k+1,k} = x_k$, then strategy-proofness and monotonicity of preferences of negotiator 1 imply that $f^Y_{k+1,k} = f^Y_{k,k}$. But then strategy-proofness and monotonicity of preferences of negotiator 2 require that $f^Y_{k,k}\ \theta^Y_2\ f^Y_{k+1,k+1}$, which implies $f^Y_{k+1,k+1}\ \theta^Y_1\ f^Y_{k,k}$ as the negotiators' preferences over the alternatives in issue X are diametrically opposed.

An important implication of part (ii) is that for type $\theta_2^{x_k}$ of negotiator 2 all the bundles in $\mathbf{B}$ that appear below the bundle $f_{k,k}$ on the first diagonal are acceptable, i.e., strictly better than the bundle of outside options. Likewise, for type $\theta_1^{x_k}$ of negotiator 1 all the bundles in $\mathbf{B}$ that appear above the bundle $f_{k,k}$ on the first diagonal are acceptable.



***Proof of Part iii*:** We refer to bundles $\{f_{k,1}, f_{k+1,2}, ..., f_{m,m-k+1}\}$ where $k = 1, ..., m$ as those on the $k-$th diagonal. Note that each diagonal has one less bundle than its immediate predecessor and the $m-$th diagonal consists of a single bundle, namely $f_{m,1}$.

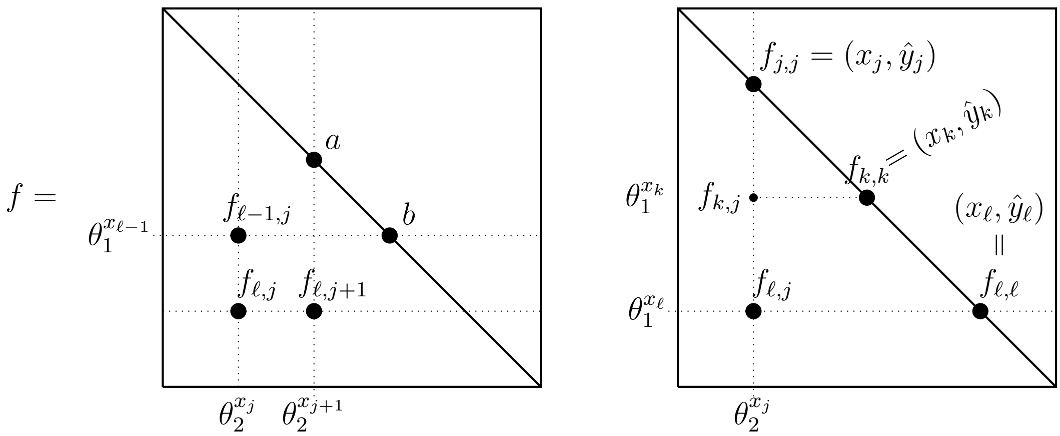
Lemma 1. *Suppose that adjacency holds for all bundles on all diagonals $t = 2, ..., k$ where $k \le m$. That is, for all $t \in \{2, ..., k\}$ and $m \ge \ell > j$ with $\ell = j + t - 1$, $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$. Consider two bundles $a, b \in \mathbf{B}$ that appear on some diagonal $t \in \{2, ..., k\}$. If bundle a lies on a higher row than b on the first diagonal, then a also lies on a higher row than b on all diagonals up to (and including) diagonal t.*

Proof. Since both a and b appear on diagonal t, by adjacency, they must also both appear on every diagonal from the second through $(t-1)-$st diagonal. Suppose that a lies above b on the first diagonal. From the first diagonal to the second, adjacency implies that a bundle can either move one cell horizontally to the left or drop one cell down. If a moves horizontally, clearly it will remain above b on the second diagonal. If a drops down one cell, it remains above b or on the same row with b (which happens when a and b are diagonally adjacent on the first diagonal). In the former case, b is clearly below a on the second diagonal. In the latter case, for b to also appear on the second diagonal it must also have dropped one cell below, in which case it is again below a on the second diagonal. Iterating this argument for rows 3 through t yields the desired result. $\square$

STEP 1 (Adjacency): We first show the following: Take a bundle on some diagonal except the first one. This bundle is equal to the bundle immediately above it or immediately to its right. Lemma 2 states this more formally.

Lemma 2. *$f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ for all $j < \ell \leq m$.*

Proof. In the proof of part (ii) we showed that the set of bundles on the first diagonal is equal to the set of bundles $\mathbf{B}$. Consider any $j < \ell \leq m$ and suppose for a contradiction that $f_{\ell,j} \notin \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$. Note that strategy-proofness implies $f_{\ell,j}\ R_1\ f_{\ell-1,j}$ for all admissible R_1 and all types that find bundles $f_{\ell,j}$ and $f_{\ell-1,j}$ acceptable, and $f_{\ell,j}\ R_2\ f_{\ell,j+1}$ for all admissible R_2 and all types that find bundles $f_{\ell,j}$ and $f_{\ell,j+1}$ acceptable.



The bundles $f_{\ell,j+1}$ and $f_{\ell-1,j}$ appear on the first diagonal because they are both in $\mathbf{B}$. Furthermore, $f_{\ell,j+1}$ appears on or below bundle a while $f_{\ell-1,j}$ appears on or above bundle b because f is individually rational (see the last figure). If $f_{\ell,j} \in \mathbf{B}$, then it should also appear on the main diagonal. If it is located below a, then it means type $\theta_2^{x_{j+1}}$ finds both $f_{\ell,j}$ and $f_{\ell,j+1}$ acceptable, which means she has incentive to deviate to $\theta_2^{x_j}$ to get $f_{\ell,j}$, contradicting the strategy-proofness of f. On the other hand, if it is located above b, then type $\theta_1^{x_{\ell-1}}$ finds both $f_{\ell,j}$ and $f_{\ell-1,j}$ acceptable, meaning that she has incentive to deviate to $\theta_1^{x_\ell}$ to get $f_{\ell,j}$, again contradicting the strategy-proofness of f. Thus, $f_{\ell,j}$ has to be equal to one of these two bundles if $f_{\ell,j}$ is in $\mathbf{B}$.

Suppose now that $f_{\ell,j} \notin \mathbf{B}$. Note that row ℓ and column j correspond to the profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$ and by efficiency and individual rationality $f_{\ell,j}^X \in \{x_j, x_{j+1}, ..., x_\ell\}$. By monotonicity and strategy-proofness $f_{\ell,j}^X$ cannot be x_j because $f_{\ell,j} \notin \mathbf{B}$ and so $f_{\ell,j}^Y \neq \hat{y}_j$. Similarly, $f_{\ell,j}^X \neq x_\ell$. Now suppose that $f_{\ell,j}^X = x_k$ where $j < k < \ell$. Again by monotonicity and strategy-proofness we must have $f_{k,j} = f_{\ell,j}$: this is true because any bundle acceptable to type $\theta_1^{x_k}$ is also acceptable to type $\theta_1^{x_\ell}$ and $f_{\ell,j}$ is acceptable to $\theta_1^{x_k}$, and thus if $f_{k,j} \neq f_{\ell,j}$ one of these types would have incentive to deviate. But then again by monotonicity and strategy-proofness (regarding negotiator 2) we have $f_{k,j} = f_{\ell,j} = f_{k,k}$, which contradicts the presumption that $f_{\ell,j} \notin \mathbf{B}$. $\square$

STEP 2 (Construction of a precedence order $\rhd$): By step 1, we know that $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ for all $\ell > j$. To construct $\rhd$, perform a pairwise comparison for all the entries $f_{\ell,j}, f_{\ell-1,j}, f_{\ell,j+1}$. More formally, $f_{\ell-1,j} \ \rhd \ f_{\ell,j+1}$ whenever $f_{\ell,j} = f_{\ell-1,j}$ and $f_{\ell,j+1} \ \rhd \ f_{\ell-1,j}$ whenever $f_{\ell,j} = f_{\ell,j+1}$. We obtain a partial order $\rhd$ on $\mathbf{B}$, which may not be complete at this point. Next, we will show that $\rhd$ is antisymmetric and transitive.

Lemma 3. *Order $\rhd$ is antisymmetric. That is, for any $a, b \in \mathbf{B}$, $a \rhd b$ implies $\neg b \rhd a$.*

Proof. Suppose for a contradiction that there are $a, b \in \mathbf{B}$ such that both $a \rhd b$ and $b \rhd a$ hold. Let $t \geq 1$ be the smallest diagonal on which a and b are diagonally adjacent and a is "chosen" according to $\rhd$. That is, let $f_{\ell-1,j} = a$, $f_{\ell,j+1} = b$, and so $f_{\ell,j} = a$. Because f is efficient, individually rational, and taking values a and b when negotiator 1 announces her type as $\theta_1^{x_\ell}$, both bundles must be acceptable to all types $\theta_1^{x_k}$ where $k \geq \ell$. Moreover, because $b \rhd a$ by assumption, there must exist another diagonal $t' > t$ in which a and b are diagonally adjacent and b is chosen. By Lemma 1, bundles a and b cannot be adjacent to one another more than once on the same diagonal, and thus $t' > t$. Therefore, let $f_{s-1,r} = a$, $f_{s,r+1} = b = f_{s,r}$. By Lemma 1 and Lemma 2, we have $s > \ell$. Strategy-proofness implies that $b \ R_1 \ a$ for all admissible $R_1 \in \Lambda(\theta_1^{x_s})$, and so we must have $b \ R_1 \ a$ for all admissible $R_1 \in \Lambda(\theta_1^{x_k})$ where $k \geq \ell$, including type $\theta_1^{x_{s-1}}$. But the last observation contradicts the strategy-proofness of f as type $\theta_1^{x_{s-1}}$ would profitably deviate to $\theta_1^{x_s}$ and get bundle b rather than a. $\square$

Let two bundles a and b be diagonally adjacent. If a lies on a higher row than b, then we say that a is *diagonally adjacent to b from below.* Equivalently, we say that b is *diagonally adjacent to a from above.*

Lemma 4. (*i*) *Let bundle $a = f_{\ell,j} \in \mathbf{B}$ be diagonally adjacent to some bundle $b \in \mathbf{B}$ from below and $a \rhd b$. Then, bundle b never appears on or below row ℓ, i.e., there is no $k \geq \ell$ and r such that $f_{k,r} = b$. Additionally, bundle a never appears (strictly) above row ℓ and (strictly) to the left of column j, i.e., there is no $\ell' < \ell$ and $j' < j$ such that $f_{\ell',j'} = a$.*

(*ii*) *Let bundle $c = f_{\ell,j} \in \mathbf{B}$ be diagonally adjacent to some bundle $d \in \mathbf{B}$ from above and $c \rhd d$. Then, bundle d never appears on column j or any lower column, i.e., there is no $k \leq j$ and r such that $f_{r,k} = d$. Additionally, bundle c never appears (strictly) below row ℓ and (strictly) to the right of column j, i.e., there is no $\ell' > \ell$ and $j' > j$ such that $f_{\ell',j'} = c$.*

Proof. We prove part (*i*), as symmetric arguments suffice to prove part (*ii*). First part of (*i*): The bundle b must be above a on the first diagonal because b is above a at some diagonal. Moreover, negotiator 2 may receive bundles a and b (depending on negotiator 1's type) when she declares her type as $\theta_2^{x_{j-1}}$, and so by efficiency and individual rationality of f, both these bundles must be acceptable for type $\theta_2^{x_{j-1}}$ of negotiator 2. Suppose for a contradiction that b occurs below row ℓ. By the adjacency property, this b should be coming all the way from the main diagonal, and so b must also appear on row ℓ. Let $f_{\ell,k} = b$ for some $k \neq j, j-1$. Strategy-proofness implies $b \; R_2 \; a$ for all admissible R_2 and all types of negotiator 2 that deem both bundle a and b acceptable. But then type $\theta_2^{x_{j-1}}$ would profitably deviate to type $\theta_2^{x_k}$ in order to get the bundle b rather than a, contradicting the strategy-proofness of f.

Second part of (i): Because $f_{\ell-1,j-1} = b$, the bundle b must appear on the first diagonal on column j or higher. Because a is below b on main diagonal as well, it can also appear on the main diagonal on column $j+1$ or higher. Therefore, if bundle a appears in the region, for a contradiction, then by the adjacency property bundle a must appear on column $j+1$ as well. Let $f_{k,j+1} = a$ for some $k \leq \ell - 1$. But if a is acceptable for type $\theta_1^{x_k}$ of negotiator 1, it must also be acceptable for type $\theta_1^{x_{\ell-1}}$ of negotiator 1, when he gets the bundle b. Therefore, because $f_{\ell,j-1} = a$ and $f_{\ell-1,j-1} = b$, strategy-proofness implies $b \; R_1 \; a$ for all admissible preferences and all types of negotiator 1 that deem both bundles acceptable. But then type $\theta_1^{x_\ell}$ deems both bundles acceptable and prefers to deviate to type $\theta_1^{x_{\ell-1}}$ to get b rather than a, contradicting the strategy-proofness of f. □

Lemma 5. *Order $\rhd$ is transitive. That is, for any triple $a, b, c \in \mathbf{B}$ such that $a \rhd b$ and $b \rhd c$, we have $\neg c \rhd a$.*

Proof. Suppose, for a contradiction, that $a \rhd b$ and $b \rhd c$, but $c \rhd a$. Without loss of generality, suppose b is diagonally adjacent to a from above. Let $t \geq 1$ be the smallest diagonal on which a and b are adjacent where $f_{\ell,j} = a$, $f_{\ell-1,j-1} = b$ and $f_{\ell,j-1} = a$ because $a \rhd b$. By Lemma 4 part (i), b never appears on row ℓ or below. Let t' be the smallest diagonal on which b and c are adjacent. We consider two cases:

Case 1: $t' \geq t$: This case has two subcases:

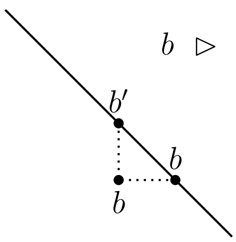
Case 1A: Suppose first that c is adjacent to b from below on diagonal t': Consider diagonal t. Clearly, c should also lie on this diagonal, for otherwise, by Lemma 1 it cannot be on diagonal $t' \geq t$. Then by Lemma 2, since c is adjacent to b from below on diagonal t', it must appear below b on row $\ell+1$ or below on diagonal t. Then by Lemma 1 and adjacency, c can appear only on $\ell+1$ or below on diagonal $t' \geq t$ as well. However, by Lemma 4 part (i), $f_{\ell,j} = a \rhd b$ implies that b can never appear on row ℓ or below. This means that b and c cannot be adjacent on diagonal $t' \geq t$, a contradiction.

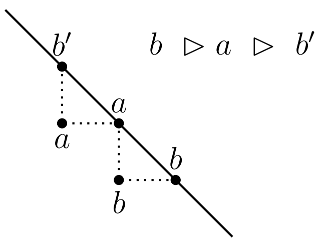
Case 1B: Suppose now that c is adjacent to b from above on diagonal t'. Let $f_{p,q} = b$ and $f_{p-1,q-1} = c$. Because b never appears on row ℓ or below, $p \leq \ell - 1$. By Lemma 3, $b \rhd c$ implies $f_{p,q-1} = b$. By Lemma 4 part (i), $b \rhd c$ implies that c never appears on row p or below. Because b is diagonally adjacent to a from above and c is adjacent to b from above, by Lemma 2, $c \rhd a$ implies that c must be adjacent to a from above on some diagonal t''. By Lemma 2, there is no b on diagonal t'', for otherwise it would be either below a or above c. Then $t'' > t'$. Thus, let $f_{r,s} = a$ and $f_{r-1,s-1} = c$ on diagonal t'', and so $f_{r,s-1} = c$ by $c \rhd a$. Because there is no c on or below row $p \leq \ell - 1$, a and c must then be adjacent above row p on diagonal $t'' > t'$. That is, $r < p$. Then $t'' > t'$ implies that $s \leq q-2$. Because there is no c on row $p \leq \ell - 1$ or below and $t' \geq t$, $f_{r,s} = a$ lies on row above row p, i.e., $r < p$ and on column $j - 2$ or to the left, i.e., $s \leq j - 2$. However, by Lemma 4 part (i), $f_{\ell,j} = a \rhd b$ implies that bundle a should never appear in the box (strictly) above row ℓ and (strictly) to the left of column j, a contradiction.

Case 2: $t' < t$: This case also has two subcases.

Case 2A: Suppose c is adjacent to b from above on diagonal t'. Consider diagonal t'. Clearly, a should also lie on this diagonal, for otherwise, by Lemma 1 it cannot be on diagonal $t > t'$. Since a lies below b on diagonal t, it must again be below b on diagonal t'. Let k be the row on which b lies on diagonal t'. Clearly, a lies below row k on diagonal t' or any other diagonal $t'' > t$. Since c is adjacent to b from above on diagonal t' and $b \rhd c$, Lemma 4 part (i) implies that c never appears on row k or below. Thus, a and c cannot be diagonally adjacent on any diagonal $t'' > t'$. But they cannot be diagonally adjacent on any diagonal $t''' < t'$ either because that would mean that there is no b on diagonal t''', for otherwise b would be above c or below a, contradicting Lemma 2, a contradiction.

Case 2B: Suppose c is adjacent to b from below on diagonal t'. Consider diagonal t'. Clearly, a should also lie on this diagonal, for otherwise, by Lemma 1 it cannot be on diagonal $t > t'$. Because a lies below b on diagonal t, it must lie below both b and c on diagonal t'. Suppose a and c are diagonally adjacent on some diagonal t''. Let $f_{p,q} = c$ on diagonal t''. Clearly, c must lie above a on diagonal t''. Because b is diagonally adjacent to a from above on diagonal t, there is no c on diagonal t (or on any higher numbered diagonal), for otherwise c would be above b or below a on diagonal t, contradicting Lemma 1. Thus, $t'' < t$. Since $a = f_{p+1,q+1}$ and $c = f_{p,q}$ are diagonally adjacent on t'' and $c \rhd a$, Lemma 4 part (ii) implies that a never appears on column q or any lower numbered column. Since $f_{\ell,j} = a$, we need $q < j-1$. Since $t'' < t$ and $q < j-1$, bundle $a = f_{p+1,q+1}$ must lie above row ℓ. But recall that Lemma 4 part (i) and $f_{\ell,j} = a \rhd b$ imply that a should never appear in the box (strictly) above row ℓ and (strictly) to the left of column j, a contradiction. $\square$





Finally, we stipulate that any incomplete portions of partial order $\rhd$ are chosen in any arbitrary manner without violating transitivity. This and Lemmas 4-6 give us a complete, transitive, and antisymmetric precedence order $\rhd$. ■

Proof of Theorem 2: Theorem 2 is a corollary to Theorem 1 and Theorem 5 when Λ is regular and satisfies logrolling. Indeed, logrolling implies weak logrolling, and so, all complete, transitive, and antisymmetric binary relations over the set of logrolling bundles are admissible with respect to Λ. Thus, the mediation rule f is efficient, individually rational, and strategy-proof if and only if $f = f^{\rhd}$ for any $\rhd$ that is complete, transitive, and antisymmetric over $\mathbf{B}$. ■

Proof of Theorem 3:

We start with (1) $\Rightarrow$ (2). Suppose f satisfies parts (ii) and (iii) of Theorem 1. The first diagonal contains all the bundles in $\mathbf{B}$, which is the hypotenuse of $\triangle_{m,1}$. First, consider the highest-ranked logrolling bundle on the hypotenuse of $\triangle_{m,1}$ and let $f_{r_1,r_1} = b_{r_1} = \underset{\mathbf{B}}{\boldsymbol{max}} \ \rhd$ where $1 \leq r_1 \leq m$. Iteratively applying the adjacency requirement, starting from the hypotenuse, implies that all the entries on row r_1 to the left of entry f_{r_1,r_1}, all the entries on column r_1 below entry f_{r_1,r_1} and all the entries in between must fill up with bundle b_{r_1} because b_{r_1} has the highest rank. Thus, the rectangle $\Re^{b_{r_1}}_{m,1}$ fills up with b_{r_1}. Let $\Re^{b_{r_1}}_{m,1}$ be the first element of the rectangular partition of $\triangle_{m,1}$. Note that, when $m \geq 3$, the so-far-unfilled $\triangle_{m,1} \backslash \Re^{b_{r_1}}_{m,1}$ consists of at least one triangle (if $r_1 \in \{1, m\}$) and at most two triangles (if $r_1 \notin \{1, m\}$).

Next, take an arbitrary triangle $\triangle_{a,b} \in \triangle_{m,1} \backslash \Re^{b_{r_1}}_{m,1}$. Note that either $a = r_1$ and $b = 1$, or $a = m$ and $b = r_1 + 1$. Let $f_{r_2,r_2} = b_{r_2}$, where $r_2 \neq r_1$, denote the highest-ranked logrolling bundle on the hypotenuse of $\triangle_{a,b}$. Then iteratively applying the adjacency requirement, starting from the hypotenuse of $\triangle_{a,b}$, implies that all the so-far-unfilled entries on row r_2 to the left of entry f_{r_2,r_2}, all the so-far-unfilled entries on column r_2 below entry f_{r_2,r_2}, and all entries in between must fill up with bundle b_{r_2} because b_{r_2} has the highest rank among the bundles on the hypotenuse of $\triangle_{a,b}$. Thus, let $\Re^{b_{r_2}}_{a,b}$ denote the second element of the rectangular partition of $\triangle_{m,1}$.

Note that the so-far-unfilled set $\triangle_{m,1} \backslash \left\{ \Re^{b_{r_1}}_{m,1} \cup \Re^{b_{r_2}}_{a,b} \right\}$ consists of at least one triangle. Iterate this reasoning and at each step pick a triangle from the so-far-unfilled subset of $\triangle_{m,1}$ and fill its corresponding rectangle with the highest-ranked bundle on its hypotenuse. By the finiteness of the problem, the rectangular partition is obtained in m steps.

We next show (2) $\Rightarrow$ (1). Consider a rectangular partition $\mathcal{P}^1$ of $\triangle_{m,1} (\equiv \triangle^1)$ such that for any $\Re \in \mathcal{P}^1$, $a, b \in \Re$ implies $a = b$. Let $\Re^{b_{r_1}} \subset \triangle^1$ be the rectangle that includes the entry at the bottom left corner of triangle $\triangle^1$, i.e., $f_{m,1}$. We construct a precedence order $\rhd$ as follows. Let b_{r_1} have a higher precedence rank than any other bundle on the hypotenuse of $\triangle^1$. Namely, let $b_{r_1} \rhd b$ for all $b \in \mathbf{B} \backslash \{b_{r_1}\}$. Next consider $\triangle^1 \backslash \Re^{b_{r_1}}$ which has a triangular partition $\mathcal{P}^2$ that consists of at most two triangles.

Take an arbitrary triangle $\triangle^2 \in \mathcal{P}^2$ and let $\Re^{b_{r_2}} \subset \triangle^2$ denote the rectangle that includes the entry at the bottom left corner of triangle $\triangle^2$. Then let b_{r_2} have a higher precedence rank than any other bundle on the hypotenuse of $\triangle^2$, i.e., if $r_2 > r_1$, then let $b_{r_2} \rhd b$ for all $b \in \{b_1, ..., b_{r_2-1}, b_{r_2+1}, ..., b_{r_1-1}\}$, and if $r_1 > r_2$, then let $b_{r_2} \rhd b$ for all $b \in \{b_{r_1+1}, ..., b_{r_2-1}, b_{r_2+1}, ..., b_m\}$. Iterate in this fashion by considering an arbitrary triangle from the remaining partition $\triangle^1 \backslash \{\Re^{b_{r_1}}, \Re^{b_{r_2}}\}$. At the end of this finite procedure (consisting of exactly m steps), we obtain a transitive but possibly incomplete strict precedence order $\rhd$ on $\mathbf{B}$: Consider an arbitrary strict completion of $\rhd$. It is easy to verify that the adjacency requirement is compatible with the constructed binary relation $\rhd$. Namely, we have $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ whenever $\ell > j$, and $f_{\ell,j} = f_{\ell-1,j}$ if $f_{\ell-1,j} \rhd f_{\ell,j+1}$ and $f_{\ell,j} = f_{\ell,j+1}$ otherwise.

Next, we prove (1) $\Rightarrow$ (3). Suppose that f satisfies the adjacency condition in part (*iii*) of Theorem 1. Let $b_k = \underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}} \rhd$ where $j < \ell$ and $\mathbf{B}_{\ell j} = \{b_j, ..., b_\ell\}$. By adjacency, any entry in the rectangle $\Re^{b_k}_{\ell,j} \ni f_{\ell,j}$ must be a bundle from $\mathbf{B}_{\ell j}$. Because b_k has the highest precedence rank over $\mathbf{B}_{\ell j}$, adjacency implies $f_{k,j'} = b_k$ for all $j \leq j' < k$ and $f_{\ell',j} = b_k$ for all $k < \ell' \leq \ell$. Thus, $f_{\ell,j} = b_k$.

Finally, we show that (3) $\Rightarrow$ (1). Suppose $f_{\ell,j} = \underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}} \rhd$ whenever $j < \ell$. Clearly $\mathbf{B}_{\ell j} = \mathbf{B}_{(\ell-1)j} \cup \mathbf{B}_{\ell(j+1)}$. Then $\underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}} \rhd = \underset{\{f_{\ell-1,j}, f_{\ell,j+1}\}}{\boldsymbol{max}} \rhd$, where we have $f_{\ell-1,j} = \underset{\mathbf{B}_{(\ell-1)j}}{\boldsymbol{max}} \rhd$ and $f_{\ell,j+1} = \underset{\mathbf{B}_{\ell(j+1)}}{\boldsymbol{max}} \rhd$ by (3). Namely, we have $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\}$, and $f_{\ell,j} = f_{\ell-1,j}$ if $f_{\ell-1,j} \rhd f_{\ell,j+1}$ and $f_{\ell,j} = f_{\ell,j+1}$ otherwise. The precedence order $\rhd$ is strict by (3) because the max operator always has a unique value. If it is not transitive, one can easily construct a transitive (and complete) precedence order $\rhd'$ by using the adjacency of f, where $\underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}} \rhd = \underset{\mathbf{B}_{\ell j}}{\boldsymbol{max}} \rhd' = f_{\ell,j}$ for all $\ell, j \in M$ with $j < \ell$. This completes the proof. ■

Proof of Theorem 4: Clearly, a CS rule belongs to the adjacent rules family. To see that the rank variance of a CS rule is lower than any other member of the adjacent rules family, we simply consider two cases about the number of possible alternatives. First, when m is odd, $var(b_n) = (m+1)^2$. For any $b_{n-t}, b_{n+t} \in \mathbf{B}$ with $t < n$, we have $var(b_{n-t}) = var(b_{n+t}) = 2(\frac{(m+1)}{2} - t)^2 + 2(\frac{(m+1)}{2} + t)^2 = (m+1)^2 + 4t^2$. Thus, $var(b_n) < var(b)$ for any $b \in \mathbf{B} \setminus \{b_n\}$.

Since any member of the adjacent rules family must pick an element of $\mathbf{B}$ whenever there is a mutually acceptable alternative in issue X (by cases (*ii*) and (*iii*) of Theorem 1), minimization of rank variance requires that $b_n \rhd b$ for any $b \in \mathbf{B} \setminus \{b_n\}$. Also observe that $var(b_n) < var(b_{n-1}) < \ldots < var(b_1)$ and $var(b_n) < var(b_{n+1}) < \ldots < var(b_m)$. Thus, minimization of rank variance subsequently requires that $b_{n-1} \rhd \ldots \rhd b_1$ and $b_{n+1} \rhd \ldots \rhd b_m$. By case (i) of Theorem 1, the outcome for issue X is fixed to o_X whenever there is no mutually acceptable alternative in this issue. Therefore, (o_X, y_n) is the rank-variance-minimizing bundle. Note that when m is odd, rank variance of the unique CS rule is strictly less than any other member of the adjacent rules family.

On the other hand, when m is even, $var(b_{\overline{n}}) = var(b_{\underline{n}}) = \frac{1}{2}(m^2 + (m+2)^2)$. For any $b_{\underline{n}-t}, b_{\overline{n}+t} \in \mathbf{B}$ with $t < n$, we have $var(b_{\underline{n}-t}) = var(b_{\overline{n}+t}) = 2(\frac{m}{2} - t)^2 + 2(\frac{(m+2)}{2} + t)^2 = \frac{1}{2}(m^2 + (m+2)^2) + 4t^2$. Hence, $var(b_{\overline{n}}) = var(b_{\underline{n}}) < var(b)$ for any $b \in \mathbf{B}\backslash\{b_{\overline{n}}, b_{\underline{n}}\}$. Note that we also have $var(b_{\underline{n}}) = var(b_{\overline{n}}) < var(b_{\underline{n}-1}) < \ldots < var(b_1)$ and $var(b_{\underline{n}}) = var(b_{\overline{n}}) < (b_{\overline{n}+1}) < \ldots < var(b_m)$. Then, minimization of rank variance subsequently requires that either $b_{\overline{n}} \rhd b_{\underline{n}}$ or $b_{\underline{n}} \rhd b_{\overline{n}}$ together with $b_{\underline{n}-1} \rhd \ldots \rhd b_1$ and $b_{\overline{n}+1} \rhd \ldots \rhd b_m$. By case (i) of Theorem 1, the outcome for issue X is o_X and both $(o_X, y_{\overline{n}})$ and $(o_X, y_{\underline{n}})$ are rank-variance-minimizing bundles. Consequently, any one of the four types of CS rules are rank variance minimizing. Note that when m is even, rank variance of a CS rule is weakly less than any other member of the adjacent rules family. ■

Proof of Theorem 5:

Proof of 'if':

Suppose that Λ is consistent and satisfies weak logrolling, and the mediation rule f satisfies following:

(i) If $\ell < j$, then $f_{\ell,j} = (o_X, y)$ for some $y \in Y$.

(ii) If $\ell = j$, then $f_{\ell,j} = (x_\ell, y_{m+1-\ell})$.

(iii) (Adjacency) If $\ell > j$, then $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ and

$$f_{\ell,j} = \begin{cases} f_{\ell-1,j}, & \text{if } f_{\ell-1,j} \rhd f_{\ell,j+1} \\ f_{\ell,j+1}, & \text{oth.} \end{cases}$$

where $\rhd$ is a complete, transitive, and antisymmetric precedence order on $\mathbf{B}$ that is admissible with respect to Λ. Then, we want to prove that f is individually rational, efficient, and strategy-proof.

It is relatively easy to verify that an adjacent rule f is individually rational: it never suggests an alternative for an issue that is worse than the outside option of that issue, and thus, it is individually rational by the regularity of preferences. To show efficiency, first consider the type profile where both negotiators deem all alternatives acceptable in issue X. At that profile, an adjacent rule proposes a bundle from the set of logrolling bundles $\mathbf{B}$. Let us call this bundle b. If instead the negotiators receive another bundle from $\mathbf{B}$ at that profile, one of the negotiators will certainly be worse off. Suppose for a contradiction that there is another logrolling bundle, say, a in which a is unambiguously better than b for both negotiators, namely $a\ R_i\ b$ for $i = 1, 2$ and for all admissible preferences R_i. On the other hand, transitivity of $\rhd$ implies $b \rhd a$ because the adjacent rule f suggests the bundle b when both a and b are mutually acceptable. Furthermore, because Λ satisfies weak logrolling and $\rhd$ is admissible with respect to Λ, $b \rhd a$ implies either $b\ R_1\ a$ or $b\ R_2\ a$

for all admissible preferences (as $\rhd$ concatenates negotiators preferences), contradicting the presumption that bundle a is unambiguously better than b for both negotiators.

At that type profile, if the negotiators receive a bundle with the outside option in issue X, rather than b, then both negotiators would be worse off because of the deal-breakers assumption. Finally, if the negotiators would receive any other bundle, say c, which is neither a logrolling bundle nor a bundle with the outside option in issue X, then there exists at least one negotiator, i, and an admissible preference ordering, R_i, such that $b \; P_i \; c$, namely the bundle c makes negotiator i worse off at some admissible preference ordering. This is true because neither regularity nor the weak logrolling assumption puts a restriction on how negotiators compare bundle b with c.

Thus, no other bundle makes one negotiator better off without hurting the other when both of the negotiators deem all alternatives acceptable. We can directly apply the same logic to all type profiles that the negotiators deem less alternatives acceptable. Finally, for those type profiles where there is no mutually acceptable alternative in issue X, in which case the rule suggests (o_X, y) for some $y \in Y \setminus \{o_Y\}$, any other bundle will include an alternative that is unacceptable in issue X by at least one of the negotiators because their preferences over each individual issue are diametrically opposed. Thus, by regularity, at least one negotiator would be worse off if f proposes something other than (o_X, y). Hence, the adjacent rule f is efficient.

We next prove that adjacent rules are strategy-proof, but first we establish some facts about the structure of these rules. Let $a = f_{\ell,j}$ and $b = f_{r,s}$ be two bundles, namely bundle a appears on row ℓ and column j whereas bundle b appears on row r and column s. We say bundle a appears above (below) bundle b whenever $\ell < r$ ($\ell > r$). Likewise, bundle a appears the right (left) of bundle b whenever $j > s$ ($j < s$).

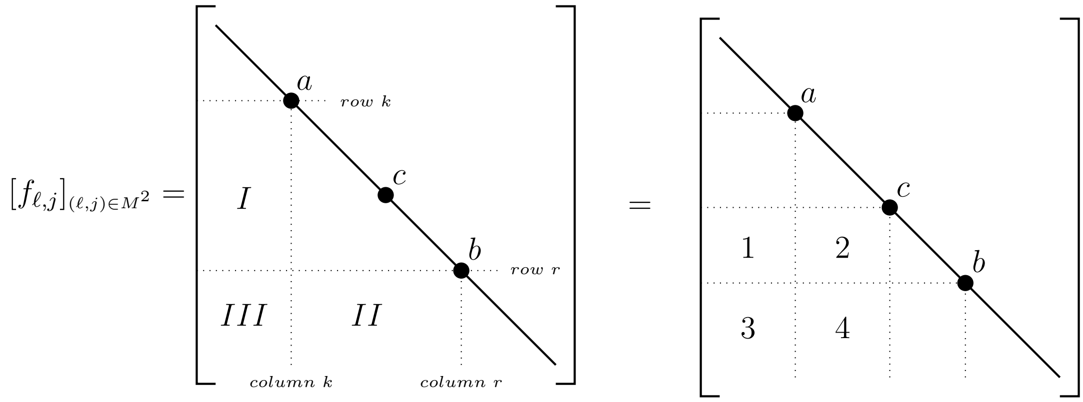
Given a mediation rule f and a bundle a that appears on the main diagonal, i.e., $a = f_{k,k}$ for some $k \in M$, define $V(a)$ to be the **value region of bundle** a, which is the submatrix of $[f_{\ell,j}]_{(\ell,j)\in M^2}$ excluding all the rows lower than row k and all the columns higher than column k. Namely, $V(a) = [f_{\ell,j}]_{(\ell,j)\in(M^k,M_k)}$ where $M^k = \{k, ..., m\}$ and $M_k = \{1, ..., k\}$. Furthermore, if bundle $b = f_{r,r}$ appears on the main diagonal with $r \in M$ and $r > k$, then $V(a) \cap V(b) = [f_{\ell,j}]_{(\ell,j)\in(M^r,M_k)}$ where $M^r = \{r, ..., m\}$. In the following figure, the value region of bundle a is region I and III, the value region of bundle b, $V(b)$, is region II and III, and $V(a) \cap V(b)$ is region III.

Lemma 6. *If the mediation rule f is an adjacent rule that is described in Theorem 1, then for any two bundles $a, b \in \mathbf{B}$*

(i) a never appears outside of its value region $V(a)$,

(ii) a and b both never appear in $V(a) \cap V(b)$, and

(*iii*) *if both a and b appear on the same column (or row), where a is above b (or a is on the left of b), then on the main diagonal, bundle a appears above bundle b.*



Proof. The first claim follows directly from the last two conditions of Theorem 1. The existence of a complete, transitive, and strict order $\rhd$ on $\mathbf{B}$ implies the second claim but deserves a proof. Suppose first that a and b appear on the same column in region III, say column s, and a is located above bundle b on this column, namely a is on row r_a and b is on row r_b where $r \leq r_a < r_b \leq m$. Starting from column and row r, i.e., from bundle b, as we move from column r to column s along the row r, adjacency and transitivity of $\rhd$ imply that the bundles on the row r are either ranked higher than b (with respect to $\rhd$) or equal to b, which includes the bundle $f_{r,s}$. Now starting from column s and row r, i.e., the bundle $f_{r,s}$, and move toward row r_a along column s. Adjacency and transitivity of $\rhd$ imply that the bundle on the row r_a and column s, i.e., the bundle a is ranked higher than b with respect to $\rhd$. Namely, $a \rhd b$ must hold.

Continue iterating from where we left off. Starting from column s and row r_a, i.e., the bundle a, as we move from row r_a to r_b along the column s, adjacency and transitivity of $\rhd$ imply that all the bundles are either ranked above a or equal to a, including the bundle at row r_b, i.e., b. Thus, we must have $b \rhd a$, contradicting the fact that $\rhd$ is strict. If bundle b is above bundle a on column s, then we start the iteration from $f_{k,k} = a$. Therefore, a and b cannot appear on the same column in region III. Symmetric arguments suffice to show that they cannot appear on the same row in region III either.

Therefore, suppose that a and b appear on different rows and columns. With similar arguments as above, if we start iteration from $f_{r,r} = b$ and go left on the same row and then go down to bundle a in region III, we conclude that $a \rhd b$ by adjacency and transitivity of $\rhd$. However, when we start iteration from $f_{k,k} = a$ and move down the same column and then go left to bundle b in region III, we conclude that $b \rhd a$, which yields the desired contradiction. Hence, either bundle a or b, whichever is ranked first with respect to $\rhd$, may appear in region III, but not both.

The proof of condition (iii) uses (ii). Suppose for a contradiction that a and b appear on the same column s, where b is above a (i.e., $r_b < r_a$) and a appears above b on the main diagonal. If we refer back to the previous figure, a and b can appear on the same column with $r_b < r_a$ only in region III, which contradicts what we just proved above. We can make symmetric arguments for rows as well. $\square$

We are now ready to show that an adjacent rule $f = [f_{\ell,j}]_{(\ell,j)\in M^2}$ is strategy-proof. Consider, without loss of generality, deviations of negotiator 1 only. If $\ell < j$, then $A(\theta_1^{x_\ell}, \theta_2^{x_j}) = \emptyset$. Negotiator 1 may receive a different bundle by deviating to a type that is represented by a higher (numbered) row, say $\theta_1^{x_k}$ where $k > \ell$. $A(\theta_2^{x_j})$ is fixed because negotiator 2's type is fixed. Because the negotiators' preferences over issue X are diametrically opposed and f is individually rational, the alternative in issue X at type profile $(\theta_1^{x_k}, \theta_2^{x_j})$ will be unacceptable for negotiator 1's true type, $\theta_1^{x_\ell}$. Thus, by the deal-breakers property, negotiator 1 has no profitable deviation from a type profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$ with $\ell < j$.

On the other hand, if $\ell = j$, then negotiator 1 can deviate to (1) a lower row and receive (o_X, y), which is worse than $f_{\ell,i} = (x_\ell, y_{m-\ell+1})$ by deal-breakers, or (ii) a higher row and receive a bundle that suggests an unacceptable alternative in issue X. Thus, the deal-breakers property implies that negotiator 1 has no profitable deviation in that case either.

Finally, suppose that $\ell > j$. Let $c \in \mathbf{B}$ denote the bundle negotiators get under truthful reporting. If negotiator 1 deviates to a row where f takes the value (o_X, y), then he clearly is worse off, by deal-breakers property. If he deviates to a lower numbered row and receives, say, bundle a, then a appears above bundle c on the first diagonal, by the third condition of Lemma 6. Therefore, we must have $c \ \rhd \ a$ because f suggests c at some type profile where both a and c are acceptable and f is an adjacent rule. The fact that a appears above bundle c on the first diagonal and $\rhd$ concatenates negotiators' preferences imply that $c \ R_1 \ a$ for all admissible R_1. Thus, there is no profitable deviation for negotiator 1 by declaring a lower numbered row and getting a instead of c. However, if he declares a higher numbered row and gets a different bundle, say, b, then c appears on the first diagonal above bundle b, again by the third condition of Lemma 6. As it is clearly visible in the last figure, Lemma 6 implies that negotiator 1's true preferences must give him the bundle c in region 1 or 2 and the deviation bundle b must be in region 3 or 4 because they cannot coexist in region 3 or 4. However, bundle b includes alternative x_r from issue X, which is an unacceptable alternative for all types that lie above row r, including negotiator 1's true type. Thus, by the deal-breakers property, negotiator 1 has no profitable deviation in that case either. Hence, f is strategy-proof.

Proof of 'only if': Now suppose that Λ is consistent and the mediation rule f is strategy-proof, efficient, and individually rational. We want to prove that Λ satisfies weak logrolling and f satisfies the following:

(i) If $\ell < j$, then $f_{\ell,j} = (o_X, y)$ for some $y \in Y$.

(ii) If $\ell = j$, then $f_{\ell,j} = (x_\ell, y_{m+1-\ell})$.

(iii) (Adjacency) If $\ell > j$, then $f_{\ell,j} \in \{f_{\ell-1,j}, f_{\ell,j+1}\} \subset \mathbf{B}$ and

$$f_{\ell,j} = \begin{cases} f_{\ell-1,j}, & \text{if } f_{\ell-1,j} \rhd f_{\ell,j+1} \\ f_{\ell,j+1}, & \text{oth.} \end{cases}$$

where $\rhd$ is a complete, transitive, and antisymmetric precedence order on $\mathbf{B}$ that is admissible with respect to Λ. Proof of Part $(i) - (iii)$ follows from Theorem 1.

Lemma 7. *Order $\rhd$ is admissible with respect to Λ.*

Proof. We need to prove that the order $\rhd$, which we constructed (by using the strategy-proof, efficient, and individually rational f) is a transitive, antisymmetric, and connected binary relation that concatenates negotiators' preferences. We already proved the first two properties. Connectedness of $\rhd$ is simple. Let all logrolling bundles on diagonal k of f constitute the set $B^k_\rhd$. Then by construction of $\rhd$, it is complete with respect to adjacency on all $B^k_\rhd$ where $k = 1, ..., m$. Thus, $\rhd$ is connected.

To show that $\rhd$ concatenates negotiators' preferences, take any $b, b' \in \mathbf{B}$ where b' is located above b on the first diagonal and $b \ \rhd \ b'$ (symmetric arguments work when b is located above b' on the first diagonal). If these two bundles are ever adjacent on some diagonal, then $b \ \rhd \ b'$ means b is located below b' on some column, and so by strategy-proofness we must have $b \ R_1 \ b'$ for all admissible R_1's, as required. If, however, these two bundles are never adjacent but $b \ \rhd \ b'$ is the result of the transitivity of $\rhd$, then adjacency of f, i.e., part (iii), implies that there must exist at least one $a \in \mathbf{B}$ such that a appears on the main diagonal above b and below b', a is adjacent to b on some diagonal and $b \ \rhd \ a$, and a is adjacent to b' on some (other) diagonal and $a \ \rhd \ b'$.

Adjacency of a and b and $b \ \rhd \ a$ imply that b is located below a on some column, and so by strategy-proofness we must have $b \ R_1 \ a$ for all admissible R_1's. Similarly, adjacency of a and b' and $a \ \rhd \ b'$ imply that a is located below b' on some column, and so by strategy-proofness we must have $a \ R_1 \ b'$ for all admissible R_1's. Thus, by transitivity of admissible preferences, we have $b \ R_1 \ b'$ for all admissible R_1's, as required. If there were multiple bundles like a in between b and b', then we repeat these transitivity arguments multiple times and reach the same conclusion. □

Theorem 1 and Lemma 7 give us a complete, transitive, and antisymmetric precedence order $\rhd$ that is admissible with respect to Λ. Thus, Λ satisfies weak logrolling. ■

Proof of Theorem 6:

We first show the necessity part. By Theorem 1, f must be an adjacent rule. Let f be some member of the adjacency family with precedence order $\rhd$. We invoke the proof of Theorem 3 to construct the partial order $\unrhd$ over $\mathbf{B}$. In particular, construct $\unrhd$ such that it coincides with the partial order $\rhd$ obtained in the proof of part (2) $\Rightarrow$ (1) of Theorem 3, i.e., $b \rhd b' \Rightarrow b \unrhd b'$ for any distinct pair $b, b' \in \mathbf{B}$. Furthermore, let $\unrhd$ be reflexive, i.e., $b \unrhd b$ for any $b \in \mathbf{B}$. In the first step of the construction, suppose some $b_{k_1} \in \mathbf{B}$ fills up the first rectangle $\Re^{b_{k_1}}$ and so let $b_{k_1} \unrhd b$ for any $b \in B$, i.e., b_{k_1} has (weakly) higher $\unrhd$-precedence than any bundle on the hypotenuse of $\triangle_{m,1} = \triangle^1$. Moreover, consider the entries on any row of $\Re^{b_{k_1}}$ each of which is filled with b_{k_1}. Note that each such entry lies on the same column with a distinct logrolling bundle from the set $\mathbf{B}_{k_1 1} = \{b_1, ..., b_{k_1}\}$ that lie weakly above b_{k_1} on the hypotenuse of $\triangle^1$. Because f is strategy-proof, it must be that $b_{k_1}\ R_1\ b_j$ for all $R_1 \in \Lambda(\theta_1)$ and all $\theta_1 \in \Theta_1$ with $x_{k_1}, x_j \in A(\theta_1)$ and $j \leq k_1$. Symmetrically, consider the entries on any column of $\Re^{b_{k_1}}$ each of which is filled with b_{k_1}. Note that each such entry lies on the same row with a distinct logrolling bundle from $\mathbf{B}_{mk_1} = \{b_{k_1}, ..., b_m\}$ that lie weakly below b_{k_1} on the hypotenuse of $\triangle^1$. Because f is strategy-proof, it must be that $b_{k_1}\ R_2\ b_j$ for all $R_2 \in \Lambda(\theta_2)$ and all $\theta_2 \in \Theta_2$ with $x_{k_1}, x_j \in A(\theta_2)$ and $j \geq k_1$.

In the second step, consider $\triangle^1 \backslash \Re^{b_{k_1}}$ which has a triangular partition $\mathcal{P}^2$ that consists of at most two triangles. Take an arbitrary triangle $\triangle^2 \in \mathcal{P}^2$. Let $\Re^{b_{k_2}} \subset \triangle^2$ be the rectangle that is filled up with b_{k_2}. Then let b_{k_2} have a (weakly) higher $\unrhd$-precedence than any other bundle on the hypotenuse of $\triangle^2$, i.e., if $k_2 < k_1$, then let $b_{k_2} \unrhd b$ for all $b \in \mathbf{B}_{(k_1-1)1} = \{b_1, ..., b_{k_1-1}\}$, and if $k_1 < k_2$, then let $b_{k_2} \unrhd b$ for all $b \in \mathbf{B}_{m(k_1+1)} = \{b_{k_1+1}, ..., b_m\}$. Without loss of generality, suppose $k_2 < k_1$. Consider the entries on any row of $\Re^{b_{k_2}}$. Each such entry lies on the same column with a distinct logrolling bundle from $\mathbf{B}_{k_2 1} = \{b_1, ..., b_{k_2}\}$, which lie weakly above b_{k_2} on the hypotenuse of $\triangle^2$. Because f is strategy-proof, it must be that $b_{k_2}\ R_1\ b_j$ for all $R_1 \in \Lambda(\theta_1)$ and all $\theta_1 \in \Theta_1$ with $x_{k_2}, x_j \in A(\theta_1)$ and $j \leq k_2$. Symmetrically, consider the entries on any column of $\Re^{b_{k_2}}$ each of which is filled with b_{k_1}. Note that each such entry lies on the same row with a distinct logrolling bundle from $\mathbf{B}_{(k_1-1)k_2} = \{b_{k_2}, ..., b_{k_1-1}\}$ which lie weakly below b_{k_2} on the hypotenuse of $\triangle^2$. Because f is strategy-proof, it must be that $b_{k_2}\ R_2\ b_j$ for all $R_2 \in \Lambda(\theta_2)$ and all $\theta_2 \in \Theta_2$ with $x_{k_2}, x_j \in A(\theta_2)$ and $k_2 \leq j < k_1$.

Iteratively continuing in this fashion, in the third step we consider an arbitrary triangle from the partition of $\triangle^1 \backslash \{\Re^{b_{k_1}}, \Re^{b_{k_2}}\}$ and so on. By construction, this procedure leads, in m steps, to a transitive and antisymmetric partial order $\unrhd$ on $\mathbf{B}$ that reflexively concatenates the negotiators' preferences. Moreover, $\rhd$ is a strict completion of $\unrhd$.

We next show that for any linked set $B \subset \mathbf{B}$, the partially ordered set $(B, \unrhd)$ is a join-semilattice. Take an arbitrary linked set $B \subset \mathbf{B}$. Given any pair of bundles $b_k, b_{k'} \in B$, let $b_k \vee b_{k'} \in B$ denote the sup (or the least upper bound) of b_k and $b_{k'}$ under $\trianglerighteq$. To show that $b_k \vee b_{k'}$ uniquely exists, it suffices to show that every linked set $\mathbf{B}_{\ell j} \subset B \subset \mathbf{B}$ with some ℓ, j with $1 \leq j \leq \ell \leq m$ has a unique maximal element, i.e., there exists $b \in \mathbf{B}_{\ell j}$ such that $b \trianglerighteq b'$ for all $b' \in \mathbf{B}_{\ell j}$ and there is no $b' \neq b$ such that $b' \trianglerighteq b$. We use the above geometric construction of $\trianglerighteq$ to show that this statement holds. Note that later steps of the above construction lead to finer partitions of $\mathbf{B}$ and any such partition of $\mathbf{B}$ consists of linked sets, e.g., the partition induced at the end of step $1 < k \leq m$ is finer than that induced at any earlier step $k' < k$. Let $\triangle_{r,s}$ with $r \leq j$ and $\ell \leq s$ be the *smallest* triangle that was considered in the above m-step construction that contains $\mathbf{B}_{\ell j}$ on its hypotenuse (possibly together with other bundles from $\mathbf{B} \backslash \mathbf{B}_{\ell j}$). Such a triangle always exists since $\mathbf{B}_{\ell j} \subset \mathbf{B}$ where $\mathbf{B}$ forms the hypotenuse of $\triangle^1$. Let $H \supset \mathbf{B}_{\ell j}$ be the set of bundles on the hypotenuse of $\triangle_{r,s}$. We claim that when triangle $\triangle_{r,s}$ is considered in the above construction, bundle b that fills the bottom left corner of $\triangle_{r,s}$ belongs to $\mathbf{B}_{\ell j}$. Suppose, for a contradiction, that $b \in H \backslash \mathbf{B}_{\ell j}$ where $b \trianglerighteq b'$ for all $b' \in H$. In that case, the bottom left corner of $\triangle_{r,s}$ contains bundle b which also fills up the corresponding rectangle $\Re^b \subset \triangle_{r,s}$. But then, $\triangle_{r,s} \backslash \Re^b$ contains a triangle whose hypotenuse also contains $\mathbf{B}_{\ell j}$, contradicting $\triangle_{r,s}$ being the smallest triangle containing $\mathbf{B}_{\ell j}$ on its hypotenuse in the above construction. Hence, $b \trianglerighteq b'$ for all $b' \in \mathbf{B}_{\ell j}$. Note that $\mathbf{B}_{\ell j} \backslash \{b\}$ is either a linked set (when $b \in \{b_j, b_l\}$), or consists of two disjoint linked sets. Since $\mathbf{B}_{\ell j}$ was an arbitrary linked set, any linked set in $\mathbf{B}_{\ell j} \backslash \{b\}$ also has a unique maximal element and so on. In particular, given any pair $b_k, b_{k'} \in B$, $b_k \vee b_{k'}$ is the maximal element in $\mathbf{B}_{kk'} \subset B$. Hence, $(B, \trianglerighteq)$ is a join-semilattice.

We finally show the sufficiency part. Suppose Λ admits a partial order $\trianglerighteq$ on $\mathbf{B}$ that reflexively concatenates negotiators' preferences and for any linked set $B \subset \mathbf{B}$, the partially ordered set $(B, \trianglerighteq)$ is a join-semilattice. We define the least upper bound (or the maximal element) of a linked set $B \subset \mathbf{B}$ as a bundle $b \in B$ such that $b \trianglerighteq b'$ for all $b' \in B$ and let $\sup B$ denote the *least upper bound* of B, i.e., $\sup B = \{b \in B \,\big|\, b \trianglerighteq b' \text{ for all } b' \in B\}$. Since $(B, \trianglerighteq)$ is a join-semilattice, $\sup B$ uniquely exists for all linked $B \subset \mathbf{B}$. Let f be a member of the adjacent family of rules associated with an arbitrary strict completion of $\trianglerighteq$. Note that all arbitrary strict completions of $\trianglerighteq$ lead to the same rectangular partition of $\triangle_{m,1}$. Specifically, $\sup \mathbf{B}$ has the highest precedence and so the bottom left corner of $\triangle^1 = \triangle_{m,1}$ contains $\sup \mathbf{B}$ which also fills up $\Re^{\sup \mathbf{B}}$. The hypotenuse of any triangle $\triangle^2$ in $\triangle_{m,1} \backslash \Re^{\sup B}$ is a linked set. The least upper bound of the set of bundles on the hypotenuse of $\triangle^2$, denoted as $\sup^2 \mathbf{B}$, has higher precedence than any other bundle on the hypotenuse of $\triangle^2$. Hence, the bottom left corner of $\triangle^2$ contains $\sup^2 \mathbf{B}$ which also fills up $\Re^{\sup^2 \mathbf{B}}$ and so on.

We next argue that f is individually rational, efficient, and strategy-proof. The argument for individually rationality is identical to that in Theorem 5, since the logrolling assumption (or whether negotiators' preferences are concatenated or not) plays no role in the proof. The argument for efficiency is also identical to that in Theorem 5, since no logrolling bundle is unambiguously better than any other logrolling bundle for both negotiators.

Consider, without loss of generality, negotiator 1 and fix the type of negotiator 2 as $\theta_2^{x_j}$. Take a type profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$ with $\ell < j$, for which f chooses outcome (o_x, y) for some $y \in Y$. As in the proof of Theorem 5, by the deal-breakers property, negotiator 1 has no profitable deviation to another type that leads to an outcome that is a logrolling bundle when her true type is $\theta_1^{x_\ell}$, nor does she have a profitable deviation to misreport her type as $\theta_1^{x_\ell}$ when revealing her true type leads to an outcome that is a logrolling bundle. Therefore, take a type profile $(\theta_1^{x_\ell}, \theta_2^{x_j})$ with $\ell > j$, for which $f_{\ell,j} \in \mathbf{B}$. We refer again to the construction of the rectangular partition induced by an arbitrary strict completion of $\trianglerighteq$. Let $f_{\ell,j} = b_t$ for some $b_t \in \mathbf{B}$ with $1 \leq t \leq m$, where b_t fills up the rectangle $\Re^{b_t}$ at some step k of the construction. Clearly, $\ell \geq t$ and $j \leq t$. Let $\triangle^k$ be the triangle that envelopes rectangle $\Re^{b_t}$ at step k. In particular, $\triangle^k$ and $\Re^{b_t}$ share the same bottom left corner entry. Moreover, $f_{t,t} = b_t$ lies on the the hypotenuse of $\triangle^k$ and is also the top right corner entry of $\Re^{b_t}$ (see Figure 9). Let $\mathbf{B}_{sr}$ be the hypotenuse of $\triangle^k$ where $r \leq t \leq s$. Note that triangle $\triangle_{t-1,r}$ (which is contained in $\triangle^k$ and lies immediately above $\Re^{b_t}$) contains bundles only from the set $\mathbf{B}_{tr}$. By construction, $b_t \trianglerighteq b$ for any $b \in \mathbf{B}_{tr}$. Since $\trianglerighteq$ reflexively concatenates negotiators' preferences, $b_t \; R_1 \; b_p$ for all $R_1 \in \Lambda(\theta_1^{x_\ell})$ and all $b \in \mathbf{B}_{tr}$. Consequently, type $\theta_1^{x_\ell}$ cannot profitably mimic a less accepting type $\theta_1^{x_p}$ where $p \leq t$. We next consider type $\theta_1^{x_\ell}$ mimicking a more accepting type. This would lead to an outcome that is contained in a rectangle below rectangle $\Re^{b_t}$ (see Figure 9). In particular, the outcome will be from the set $\mathbf{B}_{m(s+1)}$. Recall that $f_{\ell,j} = b_t$. Since $\mathbf{B}_{sr}$ is the hypotenuse of $\triangle^k$, we have $\ell \leq s$. But then, no bundle in $\mathbf{B}_{m(s+1)}$ is acceptable for type $\theta_1^{x_\ell}$. Hence, f is strategy-proof. ■

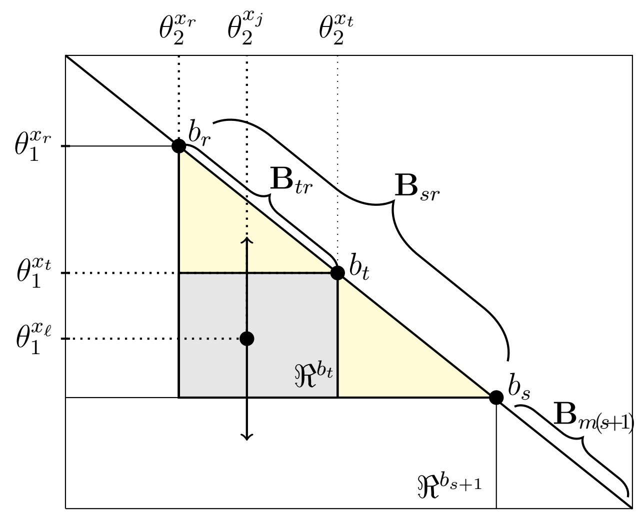


Figure 9

Proof of Proposition 2: Consider the preference profile $(\theta_1, \theta_2) = (\theta_1^{x_m}, \theta_1^{y_m}, \theta_2^{x_1}, \theta_2^{y_1})$. That is, both negotiators find all alternatives acceptable. Let $(x, y) = f(\theta_1, \theta_2)$. Because negotiators preferences over alternatives are diametrically opposed for each single issue, there is at least one negotiator $i \in \mathrm{I}$ and an issue for which negotiator i does not get her top alternative for that issue. Suppose, without loss of generality, that this negotiator is 1 and the issue is X: that is, $x \neq x_1$. Consider the new profile where only negotiator 1's preferences are different, $(\theta_1', \theta_2) = (\theta_1^{x_1}, \theta_1^{y_1}, \theta_2^{x_1}, \theta_2^{y_1})$.

We claim that $f(\theta_1', \theta_2) = (x_1, y_1)$. Suppose for a contradiction that $f(\theta_1', \theta_2) = (x', y') \neq (x_1, y_1)$. I will only show that $x' = x_1$ because similar arguments also prove $y' = y_1$, yielding the desired contradiction. To show $x' = x_1$, suppose for a contradiction that $o_X\ \theta_1^{x_1}\ x'$. Since Λ satisfies DB, $(o_X, o_Y)\ P_1\ (x', y')$ for all $R_1 \in \Lambda(\theta_1')$, and thus $f(\theta_1', \theta_2) = (x', y')$ contradicts with the individual rationality of f. Now suppose for a contradiction that $x' = o_X$. Then, since Λ satisfies monotonicity, $(x_1, y')\ P_i\ (x', y')$ for $i = 1, 2$ and all $R_1 \in \Lambda(\theta_1^{x_1})$ and all $R_2 \in \Lambda(\theta_2^{x_1})$. Therefore, (x', y') is an inefficient bundle at (θ_1', θ_2), and thus $f(\theta_1', \theta_2) = (x', y')$ contradicts with the efficiency of f. Hence, we must have $x' = x_1$.

To conclude, we already know that $f(\theta_1, \theta_2) = (x, y)$ and $x \neq x_1$, which implies $x_1\ \theta_1^{x_m}\ x$. Because y_1 is negotiator 1's best alternative in issue Y, either $y = y_1$ or $y_1\ \theta_1^{y_1}\ y$ is true. In either case, Monotonicity and transitivity of preferences imply $(x_1, y_1)\ P_1\ (x, y)$ for all $R_1 \in \Lambda(\theta_1)$. Finally, we showed in the previous paragraph that by misrepresenting her preferences at profile (θ_1, θ_2), negotiator 1 can achieve the bundle (x_1, y_1), which is strictly better than (x, y) for all $R_1 \in \Lambda(\theta_1)$, contradicting the strategy-proofness of f. ■

Proof of Theorem 7:

Proof of 'if': The same arguments in the proof of Theorem 4 suffice to verify that the mediation rule described in Theorem 7 is individually rational and efficient. Lemma 6 also holds in the continuous case. The proof of part (i) of Lemma 6 is straightforward; given the location of a logrolling bundle a on the main diagonal, $f_{\ell,j}$ can be a only if $a \in \mathbf{B}_{\ell j}$, and so, a can never appear outside of its value region $V(a)$. To prove part (ii), let $f_{\ell,j} = a$ and $f_{s,r} = b$ and suppose for a contradiction that $a, b \in V(a) \cap V(b)$. Therefore, we have $a, b \in \mathbf{B}_{\ell j} \cap \mathbf{B}_{sr}$. The bundle a beats b with respect to $\rhd$ because a wins over $\mathbf{B}_{\ell j}$. Likewise, b beats a with respect to $\rhd$ because b wins over $\mathbf{B}_{sr}$. The last two observations contradict with the assumption that $\rhd$ is strict. To prove part (iii), suppose that $f_{\ell,s} = a$ and $f_{j,s} = b$ where $\ell < j$, whereas a appears below b on the main diagonal. This is possible only when $a, b \in V(a) \cap V(b)$, contradicting the second part. Similar arguments prove the claim when bundles a and b are on the same row.

Now we prove that f is strategy-proof. It suffices to consider the deviations of one negotiator to prove that f is strategy-proof. Take any $\ell, j \in [0, 1]$ such that $f(\theta_1^{\ell}, \theta_2^{j}) = f_{\ell,j} = (o_X, y)$ (see figure 10-a). Deviating from θ_1^{ℓ} does not benefit negotiator 1 if he deviates to θ_1^s where $s < j$ because the outcome of f will not change. However, if negotiator 1 deviates to some $s \geq j$ and get some b, we know that b is one of the logrolling bundles in $\mathbf{B}_{sj}$. However, all of the bundles in $\mathbf{B}_{sj}$ are unacceptable for type θ_1^ℓ of negotiator 1 since $\ell < s$, and so, not preferable to (o_X, y) by the deal-breaker property.

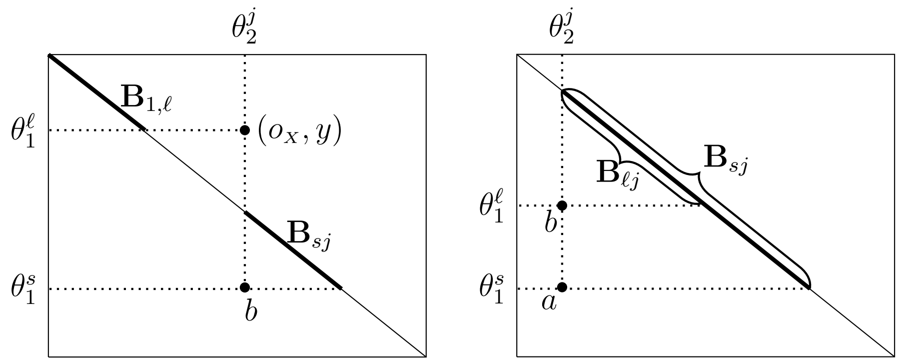


Figure 10-a   Figure 10-b

Now take any $\ell, j \in [0,1]$ such that $\ell \geq j$ and $f(\theta_1^\ell, \theta_2^j) = f_{\ell,j} = b \in \mathbf{B}$. Deviating from θ_1^ℓ does not benefit negotiator 1 if he deviates to θ_1^s where $s < j$ because the outcome of f would be (o_X, y), which is not better than $b \in \mathbf{B}$ by the deal-breaker property. If negotiator 1 deviates to some $\ell > s \geq j$ and get some a, then a must appear above b on the main diagonal (part (ii) of Lemma 6). Logrolling implies that negotiator 1 finds b at least as good as a at all admissible preferences, and thus, deviating to s is not profitable.

Finally, suppose that negotiator 1 deviates to some $s > \ell \geq j$ and get some a (see figure 10-b). Therefore, a beats b with respect to $\rhd$ because both a and b are in $\mathbf{B}_{sj}$ and a is chosen. Thus, a cannot be an element of $\mathbf{B}_{\ell j}$ as b is the maximizer of $\rhd$ over this set. Thus, $a \in \mathbf{B}_{sj} \setminus \mathbf{B}_{\ell j}$, implying that a is not acceptable for type θ_1^ℓ, and so, deviating to θ_1^s is not profitable by the deal-breaker property. Hence, f is strategy-proof.

Proof of 'only if': The same arguments in the proof of Theorem 1 suffice to show that there must exist some $y \in Y \setminus \{o_Y\}$ such that $f_{\ell j} = (o_X, y)$ for all $\ell, j \in [0,1]$ with $\ell < j$. Consider now for $\ell \geq j$.

STEP 1 (Adjacency):

Lemma 8. *If f is strategy-proof, efficient, and individually rational, then $f_{\ell,j} \in \mathbf{B}_{\ell j}$ for all $\ell \geq j$.*

Proof. We first show that $f_{kk} = (k, 1-k) \in \mathbf{B}_{kk}$ for all $k \in [0,1]$. Suppose for a contradiction that there is some $\ell \in [0,1]$ such that $f_{\ell,\ell} = (x,y) \neq (\ell, 1-\ell)$, and so $f_{\ell,\ell} \notin \mathbf{B}$. By individual rationality and the deal-breaker property, we have $x = \ell$ because ℓ is the only mutually acceptable alternative in X at type profile $(\theta_1^\ell, \theta_2^\ell)$. Next, we show that $f_{\ell,k} = (\ell, y)$ for any $k < \ell$. Suppose not, i.e., there is some $j < \ell$ such that $f_{\ell,j} = (x', y') \neq (\ell, y)$. Individual rationality implies $x' \leq \ell$, and so, there are three exhaustive cases we need to consider:

1. If $x' = \ell$ and $y' \geq y$, then by monotonicity θ_2^ℓ profitably deviates to θ_2^j, contradicting strategy-proofness.

2. If $x' \leq \ell$ and $y' \leq y$, then by monotonicity θ_2^j profitably deviates to θ_2^ℓ, contradicting strategy-proofness.

3. If $x' < \ell$ and $y' > y$, then bundles (ℓ, y) and (x', y') are not unambiguously comparable, i.e., there exists an admissible preference ordering of negotiator 1 where the bundle (ℓ, y) is preferred to the bundle (x', y') and another admissible ordering where (x', y') is preferred to (ℓ, y). Therefore, type θ_1^ℓ would profitably deviate to θ_1^j, again contradicting strategy-proofness.

Thus, we must have $f_{\ell,j} = (\ell, y)$. Given that $f_{\ell,j} = (\ell, y)$, symmetric arguments suffice to prove that $f_{j,j} = (\ell, y)$ as well, which contradicts individual rationality because $\ell > j$ is not acceptable by type θ_1^j of negotiator 1. Thus, we have $y = 1 - \ell$, and so $f_{\ell,\ell} \in \mathbf{B}_{\ell\ell}$.

Now consider the case where $\ell > j$ and suppose for a contradiction that $f_{\ell,j} = (x, y) \notin \mathbf{B}$. By individual rationality we have $x \in [j, \ell]$. Moreover, strategy-proofness implies $x = j$. Suppose not, i.e., $x > j$. If $y \geq 1 - j$, then there is an admissible preference ordering of negotiator 1 such that the bundle $f_{j,j} = (j, 1 - j)$ is preferred to the bundle $f_{\ell,j} = (x, y)$ by monotonicity, and so type θ_1^ℓ would profitably deviate to type θ_1^j, contradicting with strategy-proofness. On the other hand, if $y < 1 - j$, then bundles $f_{j,j}$ and (x, y) are not unambiguously comparable, namely there exists an admissible preference ordering of negotiator 1 where the bundle $f_{j,j}$ is preferred to the bundle $f_{\ell,j}$ and another admissible ordering where $f_{\ell,j}$ is preferred to $f_{j,j}$. Therefore, type θ_1^ℓ would profitably deviate to θ_1^j, again contradicting strategy-proofness. Symmetric arguments suffice to prove that strategy-proofness implies $x = \ell$, because otherwise negotiator 2 would profitably deviate. The last two claims lead to the desired contradiction because we must have $x = j$ and $x = \ell$, but $\ell > j$. Finally, given that $f_{\ell,j} \in \mathbf{B}$, individual rationality requires $f_{\ell,j} \in \mathbf{B}_{\ell j}$. $\square$

STEP 2 (Construction of a precedence order): To construct $\triangleright$, we perform the following pairwise comparison: Let $f_{\ell,\ell} = a \in \mathbf{B}$ and $f_{j,j} = b \in \mathbf{B}$ for some $\ell, j \in [0, 1]$ with $\ell > j$ and define $a \triangleright b$ whenever $f_{\ell,j} = a$ and $b \triangleright a$ whenever $f_{\ell,j} = b$. The binary relation $\triangleright$ is asymmetric by definition because the logrolling bundles a and b can appear on the main diagonal only once. However, it may not be complete. Lemma 9 below shows that there exists some a and b such that either $a \triangleright b$ or $b \triangleright a$.

Lemma 9. *Let f be strategy-proof, efficient, and individually rational, and $f_{\ell,j} = a \in \mathbf{B}$ where $\ell > j$. Then there exists some $k \geq j$ such that $f_{k,k} = a$ and $f_{\ell,k} = a$.*

Proof. Given that $f_{\ell,j} = a \in \mathbf{B}$ where $\ell > j$, Lemma 8 implies that $a \in \mathbf{B}_{\ell j}$, and so there is some $k \in [j,\ell]$ such that $f_{k,k} = a$. To prove the second part, suppose for a contradiction that $f_{\ell,k} = z$ where $z \neq a$. Again by Lemma 8, we know that $z \in \mathbf{B}_{\ell k}$, and so there is some $k' \in [k,\ell]$ such that $f_{k',k'} = z$. By the way the logrolling bundles are ranked by negotiator 2, $f_{k,k} = a$ is preferred to $f_{k',k'} = z$ because $k < k'$. Therefore, given that the type of negotiator 1 is θ_1^ℓ, type θ_2^k of negotiator 2 would profitably deviate to θ_2^j to get a instead of z, contradicting strategy-proofness. □

Lemma 10. *Let f be strategy-proof, efficient, and individually rational. Then the order $\rhd$ is transitive. That is, for any triple $a, b, c \in \mathbf{B}$ such that $a \rhd b$ and $b \rhd c$, we have $\neg c \rhd a$.*

Proof. Suppose for a contradiction that there exists $a, b, c \in \mathbf{B}$ such that $a \rhd b$, $b \rhd c$ and $c \rhd a$. There are six possible cases to consider regarding how these three bundles are placed on the main diagonal, and symmetric arguments suffice to prove them all. Therefore, we present only the proof of one of these cases.

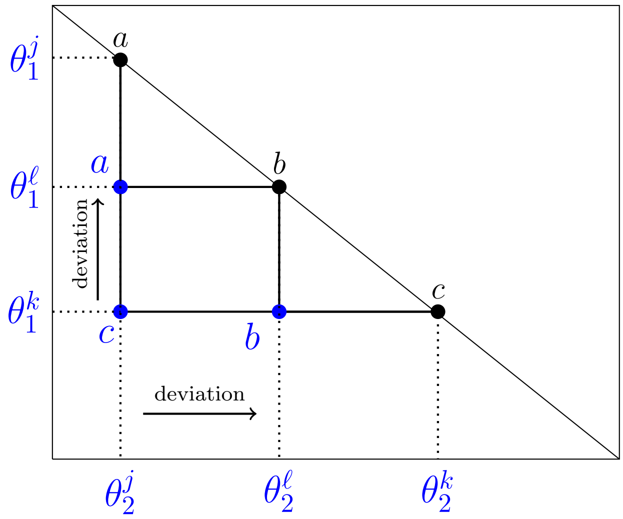


Figure 11

Suppose, without loss of generality, that a appears above bundle b and b appears above bundle c on the main diagonal. (See Figure 11.) Therefore, negotiator 2 prefers a to b and b to c, and type θ_2^j finds all three bundles acceptable. Moreover, $a \rhd b$, $b \rhd c$ and $c \rhd a$ imply that $f_{\ell,j} = a$, $f_{k,j} = c$, and $f_{k,\ell} = b$. Given that negotiator 1 is of type θ_1^k, θ_2^j would profitably deviate to type θ_2^ℓ because b is more preferred than c, contradicting with strategy-proofness.

□

Lemma 11. *Let f be strategy-proof, efficient, and individually rational, and $f_{\ell,j} = a \in \mathbf{B}$ for some $\ell, j \in [0,1]$ with $\ell > j$. Then, $a \rhd b$ for all $b \in \mathbf{B}_{\ell j}$ with $b \neq a$.*

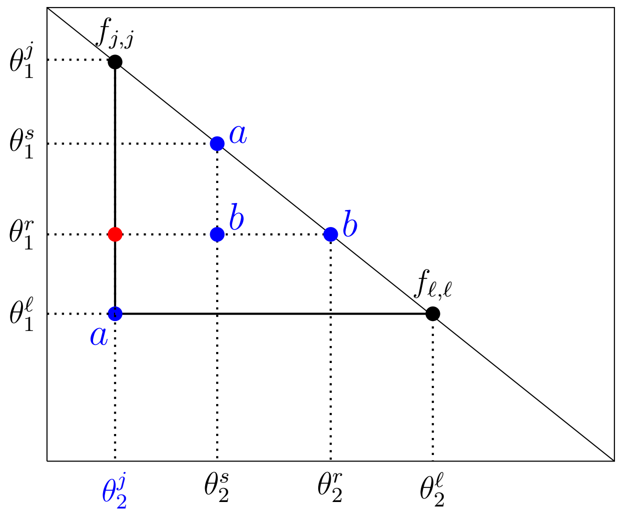


Figure 12

Proof. Suppose for a contradiction that there exists some $b \in \mathbf{B}_{\ell j}$ with $b \neq a$ such that $b \rhd a$. Consider the case where the bundle a is located above the bundle b on the main diagonal. Symmetric arguments will yield a contradiction when a is located below the bundle b on the main diagonal. Suppose that $f_{s,s} = a$ and $f_{r,r} = b$, and so $f_{r,s} = b$. Strategy-proofness and individual rationality imply that $f_{r,j} = a$: This is true because if $f_{r,j} \in \mathbf{B}_{sr} \setminus \{a\}$, then type θ_1^ℓ would profitable deviate to θ_1^r, and if $f_{r,j} \in \mathbf{B}_{js} \setminus \{a\}$, then θ_1^r would deviate to θ_1^ℓ, all of which contradict with strategy-proofness. With a similar reasoning, we must have $f_{r,s} = a$ given that $f_{r,j} = a$, which contradicts with $a \neq b$: This is true because when $f_{r,s} \in \mathbf{B}_{rs} \setminus \{a\}$, then type θ_2^s would deviate to θ_2^j, contradicting with strategy-proofness. Thus, $a \rhd b$ for all $b \in \mathbf{B}_{\ell j}$ with $b \neq a$.

□

The last lemma proves that a strategy-proof, efficient, and individually rational rule picks the maximal element of $\rhd$ on $\mathbf{B}_{\ell j}$ for all $0 \leq \ell, j \leq 1$ with $\ell \geq j$. Namely, $f_{\ell,j} = \max\limits_{\mathbf{B}_{\ell j}} \rhd$ for all $\ell \geq j$. By the Szpilrajn's extension theorem (Szpilrajn 1930), one can extend $\rhd$ to a complete order. This extension will clearly preserve the maximal elements in every compact subset $\mathbf{B}_{\ell j}$ because the maximal elements in every set $\mathbf{B}_{\ell j}$ already have a complete relation with all the elements in that set. Finally, Theorem 1 in Tian and Zhoub (1995) proves that quasi upper-semicontinuity is both necessary and sufficient for $\rhd$ to attain its maximum on all compact subsets $\mathbf{B}_{\ell j}$, which completes the proof. ■

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