How often does a polynomial hit a square?

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Algebraic Geometry

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- **Questions**: singular points, topology of the variety, how large the variety is.
- Complex points of the algebraic varieties; more generally, solutions with coordinates in an algebraically closed field.

Arithmetic Geometry

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- Intersection between Algebraic Geometry and Number Theory.

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- Let $f(x)$ be a polynomial with integer coefficients

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- How often does $f(x)$ take square values in $\mathbb{Q}$?

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  The equation $y^2 = x^2 + 1$ describes an algebraic variety.
- If you feel more comfortable with integers, then think of $Y^2 = X^2 + Z^2$. Pythagorean Triples!

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- Are they finite? Are they infinite? Why?
- Any solution to $X^2 + Z^2 = Y^2$ is given by

$$(X,Y,Z) = (s^2 - t^2, 2st, s^2 + t^2), \qquad s,t \in \mathbb{Z}.$$

Squares represented by sums of multiples of squares

Theorem

Let C be the conic described by $ax^2 + by^2 + c = 0$, where a, b, c are square free coprime integers. The set of **rational points** *is the set*

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This means that once we have a rational point on C, there exists infinitely many! But how would I find a rational point in the first place?

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Theorem (Legendre)

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1) *a, b, c do not all have the same sign, and*
2) *the congruences*

$$\begin{aligned} as^2 + b &\equiv 0 \pmod{c} \\ bt^2 + c &\equiv 0 \pmod{a} \\ cu^2 + a &\equiv 0 \pmod{b} \end{aligned}$$

have solutions $s, t, u \in \mathbb{Z}$.

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- The picture is complete here!

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- If so, then how large $C(\mathbb{Q})$ is? Finite (how finite?). Infinite (how infinite?)
- The answer is beautiful! It depends on the graph of C in $\mathbb{C}^2$.

Topology and rational points

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- They form one subfamily out of the family of algebraic curves described by $h(x, y) = 0$.
- The topology of a curve C defined by $h(x, y) = 0$ but thought of as a surface in $\mathbb{C}^2$ provides us with an answer to our previous question.

Recall

$$C(\mathbb{Q}) = \{(x, y) : h(x, y) = 0, \quad x, y \in \mathbb{Q}\}.$$

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- In fact the situation is even worse. We do not know how to decide whether a nontrivial rational point exists on C or not.

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- In fact the situation is even worse. We do not know how to decide whether a nontrivial rational point exists on C or not. Let alone finding an algorithm which lists all the rational points in $C(\mathbb{Q})$.

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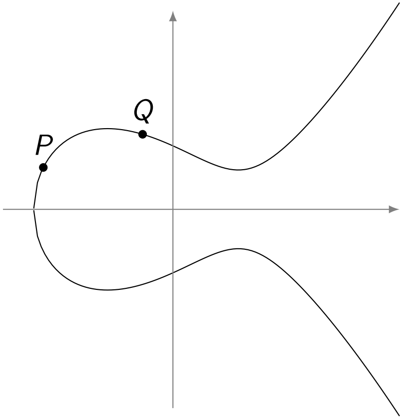
Elliptic curves



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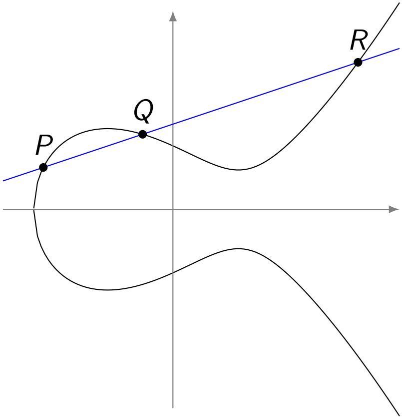

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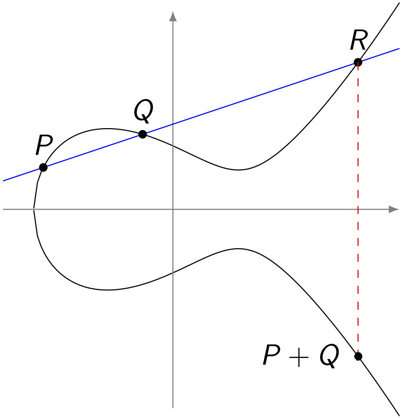

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Algebraic Formulas for addition in E

Suppose $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are points on the elliptic curve $E : y^2 = x^3 + Ax + B$.

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One has

$$P_1 + P_2 = (\lambda^2 - x_1 - x_2, -\lambda^3 + 2\lambda x_1 + \lambda x_2 - y_1).$$

An Example

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$$E(K) = \{(x, y) \in E : x, y \in K\} \cup \{O_E\}.$$

Then $E(K)$ is a subgroup of E.

History: how did they originate?

- Recall that the arc length of the upper half of $x^2 + y^2 = a^2$ is given by
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- The arc length of the upper half of $x^2/a^2 + y^2/b^2 = 1,\ b < a$, is given by
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- An *elliptic integral* is an integral of the form $\displaystyle\int R(x,y)\,dx$, where $R(x,y)$ is a rational function of the coordinates (x,y) on an elliptic curve $E : y^2 = f(x)$, $f(x)$ is a cubic or a quartic polynomial.

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Corollary

There exists a nonnegative integer r such that

$$E(\mathbb{Q}) \simeq \mathbb{Z}^r \times \mathbb{T}, \qquad |\mathbb{T}| < \infty.$$

r is the **rank** of $E(\mathbb{Q})$.

In other words, there exist finitely many points $P_1, \ldots, P_s \in E(\mathbb{Q})$, $s \geq r$, such that any point $P \in E(\mathbb{Q})$ can be written as

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This means that there exist finitely many points in $E(\mathbb{Q})$ that I can start from using the chord & tangent process and produce every single point in $E(\mathbb{Q})$.

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$\mathbb{T}$ *is one of the following fifteen groups:*

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This implies that $|\mathbb{T}| \leq 16$.

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$E(\mathbb{Z})$ is not a subgroup of E. The following finiteness theorem is due to Siegel, 1928.

Theorem

$E(\mathbb{Z})$ is finite.

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Theorem (Hasse, 1922)

$$||E(\mathbb{F}_p)| - (p + 1)| < 2\sqrt{p}.$$

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RSA: Factorization of integers.

The Discrete Logarithm Problem

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Example. Let $G = {\mathbb{F}_p}^{\times}$. (Diffie-Hellman)

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The best algorithms for solving the elliptic curve discrete logarithm problem (ECDLP) are much less efficient than the algorithms for solving DLP in ${\mathbb{F}_p}^\times$.

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Definition

A set of m positive integers (rationals) $\{a_1, a_2, \cdots, a_m\}$ is called a *(rational) Diophantine m-tuple* if $a_i \times a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$.

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- How large these Diophantine sets can be? In other words, how large m can be?

Diophantine m-tuples, $m \geq 5$

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There does not exist a Diophantine sextuple.

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Theorem (He, Togbé, Ziegler, 2018)

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Example.

$$\{19/12, 33/4, 52/3, 60/2209, -495/24964, 595/12\},$$

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Theorem (Dujella, Kazalicki, Mikic, Szikszai, 2017)

There exist infinitely many **rational** *Diophantine sextuples.*

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(5) Necessary and sufficient conditions were given so that (4) happens for three different rationals d, e, f, and such that (a, b, c, d, e, f) is a rational Diophantine sixtuple.

□

Rational **Diophantine** *m*-tuples

Conjecture

There are no rational Diophantine 9-tuples.

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- We still do not have a single example of a rational Diophantine 7-tuple.

Rational **Diophantine** *m*-tuples

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- We still do not have a single example of a rational Diophantine 7-tuple.
- Are they finite? infinite? Maybe there is no such 7-tuple!

Geometry of the Problem

$$S = \{1, 3, 8, 120\}$$

$$\begin{array}{ccc} 1 \times 3 + 1 = 2^2, & 1 \times 120 + 1 = 12^2, & 1 \times 8 + 1 = 3^2, \\ 3 \times 120 + 1 = 19^2, & 3 \times 8 + 1 = 5^2, & 8 \times 120 + 1 = 31^2. \end{array}$$

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$$x_1x_2 + 1 = y_1^2, \qquad x_1x_3 + 1 = y_2^2, \qquad x_1x_4 + 1 = y_3^2,$$
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Then we investigate $\mathcal{C}(\mathbb{Q})$.

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$$F(s_i, s_j) = \square_{ij} \qquad \text{for all } s_i, s_j \in S,\ s_i \neq s_j.$$

F-Diophantine sets

Definition

Let $F \in \mathbb{Z}[x, y]$. A set $A \subseteq \mathbb{Z}$ is called an F-**Diophantine set** if $F(a, b)$ is a perfect square for any $a, b \in A$ with $a \neq b$.

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So Diophantine tuples are F-Diophantine sets for $F(x, y) = xy + 1$.

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- Can we find a polynomial F such that there are **infinite** F-Diophantine sets?

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- Can we find a polynomial F such that there are **infinite** F-Diophantine sets?
- Think of $F(x, y) = xy$.
- Bérczes, Dujella, Hajdu and Tengely, 2017, gave a complete classification of all such polynomials F.
- For certain families of polynomials F, they found upper bounds on the size of F-Diophantine sets.

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Given a set of distinct integers

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- if such polynomials exist, how many are they?
- what is the smallest possible degree of such polynomial?

Results

Theorem (2018)

Given $S = \{x_1, \cdots, x_k\} \subset \mathbb{Z}$ where $x_i \neq x_j$ if $i \neq j$, there are **infinitely** *many polynomials $F \in \mathbb{Z}[x, y]$ with $\deg F = 2(k-2)$ such that the set S is an F-Diophantine set.*

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Given $S = \{x_1, \cdots, x_k\} \subset \mathbb{Z}$ *where* $x_i \neq x_j$ *if* $i \neq j$*, there are* **infinitely** *many polynomials* $F \in \mathbb{Z}[x, y]$ *with* $\deg F = 2(k-2)$ *such that the set* S *is an* F*-Diophantine set.*

Proof.

Studying Determinantal varieties that we may associate to F-diophantine sets. □

Refinement

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- *Strong* F-Diophantine sets. $F(x_i, x_i) = \square$.

Thank you!