A THESIS TITLED

Theory of Polyomino Ideals

Submitted to GC University Lahore
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By

Rizwan Jahangir

Registration No.

2016-MP-ASSMS-432

ABDUS SALAM SCHOOL OF MATHEMATICAL SCIENCES,
GC UNIVERSITY LAHORE

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Date	Supervisor	Co-supervisor

Submitted Through

Prof. Dr. G. Murtaza
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Certified that the research work contained in this thesis titled **Theory of Polyomino Ideals** was conducted by **Mr. Rizwan Jahangir** Registration No. **2016-MP-ASSMS-432** under the supervision of **Dr. Imran Anwar** and co-supervision of **Dr. Ayesha Asloob Qureshi.**

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Dedicated to my parents and teachers

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Abstract

We have studied the characterization of binomial ideal attached to a combinatorial object polyomino. We reviewed the papers of Ayesha Asloob Qurehsi mentioned in the references [8], [9], [12] and [13].

Contents

Introduction

Polyominoes are two dimensional figures which are originally rooted in recreational mathematics and combinatorics, and extensively discussed in connection with tiling problems of the plane. A polyomino is plane figure obtained by joining squares of equal sizes, which are known as cells. In other words, a cell is a unit square of $\mathbb{R}^2$ whose corners are elements in $\mathbb{N}^2$, and a polyomino is a finite union of cells. The first connection of polyominoes with commutative algebra appeared in [13] by assigning each polyomino the ideal of its inner 2-minors or the *polyomino ideal*.

To each polyomino $\mathcal{P} \subset \mathbb{N}^2$, we attach an ideal $I_{\mathcal{P}}$ as follows. Let K be a field and S be the polynomial ring over K in the variables x_a with $a \in V(\mathcal{P})$. To each proper interval $[a, b]$ of $\mathbb{N}^2$, we assign the binomial $f_{a,b} = x_a x_b - x_c x_d$, where c and d are the anti-diagonal corners of $[a, b]$. A proper interval $[a, b]$ is called an inner interval of $\mathcal{P}$ if all cells of $[a, b]$ belong to $\mathcal{P}$. The binomial $f_{a,b}$ is called an inner 2-minor of $\mathcal{P}$, if $[a, b]$ is an inner interval of $\mathcal{P}$. We denote by $I_{\mathcal{P}} \subset S$ the ideal generated by the inner 2-minors of $\mathcal{P}$ and by $K[\mathcal{P}]$ the quotient ring $S/I_{\mathcal{P}}$.

A collection of cells $\mathcal{P}$ is called a polyomino if it is a connected collection of cells which means that for any two cells $A, B \in \mathcal{P}$ there exists a sequence of cells $C_1, \ldots, C_m$ with $C_1 = A$, $C_m = B$, and for all i, the cells C_i and C_{i+1} have an edge in common.

In Chapter 2, we introduce some basic concepts related to collection of cells. In particular we introduce column convex, row convex and convex collection of cells. The first main result in this direction is stated in Section 2.3 where it is shown

that $K[\mathcal{P}]$ is a normal Cohen–Macaulay domain of dimension $|V(\mathcal{P})| - |\mathcal{P}|$, if $\mathcal{P}$ is convex. We define for any collection of cells $\mathcal{P}$ a natural toric ring $T_{\mathcal{P}}$ and a natural K-algebra homomorphism $K[\mathcal{P}] \to T_{\mathcal{P}}$.

In the same chapter, We gave a result from the paper of Qureshi that $I_{\mathcal{P}}$ is prime ideal if $\mathcal{P}$ is a simple collection of cells. Roughly speaking $\mathcal{P}$ is simple if it is connected and has no holes, see Section 2.1 for the precise definition. In the Chapter 3 we study a case of non simple polyomino with prime ideal and in Chapter 4 we study the Gröbner basis of our binomial ideal. The main result in this direction is that the set of inner 2-minors of $\mathcal{P}$ form a reduced Gröbner basis under the certain conditions on polyomino, see Theorem 4.1.1.

Chapter 1

Preliminaries

This chapter comprises of some basic concepts that we will use in next chapters. In this dissertation all rings considered are commutative and K denotes field.

1.1 Binomial ideals

Definition 1.1.1. Assume we have a field K and $K[x_1, \ldots, x_n]$ be a ring of polynomials in n variables, denoted by S. The product of the form $x_1^{a_1} \cdots x_n^{a_n}$ is called a *monomial* in S, where $a_i \in \mathbb{N}$. If $s = x_1^{a_1} \cdots x_n^{a_n}$, then we write $s = \boldsymbol{x}^{\boldsymbol{a}}$, with $\mathbf{a} = (\mathbf{a_1}, \ldots, \mathbf{a_n}) \in \mathbb{N}^{\mathbf{n}}$.

The set of all monomials in S is denoted by $\mathrm{Mon}(S)$. It is well known that the ring of polynomials S has a structure of K-vector space and $\mathrm{Mon}(S)$ forms the K-basis of S. Therefore, any polynomial $g \in S$ can be uniquely written as

$$g = \sum_{\boldsymbol{x}^{\boldsymbol{a}} \in \mathrm{Mon}(S)} c_a \boldsymbol{x}^{\boldsymbol{a}} \quad \textit{where} \quad c_a \in K.$$

Moreover, the set

$$\mathrm{supp}(g) = \{\boldsymbol{x}^{\boldsymbol{a}} \in \mathrm{Mon}(S) | c_a \neq 0\}$$

is called *support* of the polynomial g.

Example 1.1.2. For a field K, $g(y,z) = yz + y^3z^2 - yz^4 \in K[y,z]$ and $\operatorname{supp}(g) = \{yz, y^3z^2, yz^4\}$.

An ideal $I \subset S$ generated by monomials in S, is called a *monomial ideal*. In all the ideals of ring S, the monomial ideal have a main role because they have natural connection with many combinatorial objects such as graphs and simplicial complexes.

Definition 1.1.3. A polynomial g in S is a *binomial* if g is of the form $g = c_a\boldsymbol{x}^{\boldsymbol{a}} - c_b\boldsymbol{x}^{\boldsymbol{b}}$ with $c_a, c_b \neq 0$. An ideal $I \subset S$ is called *binomial ideal* if it is generated by binomials in S. For example, the ideal $J = (x^2y - y^3z, xyz^4 - x^3y^3) \in K[x,y,z]$ is a binomial ideal. Binomial ideals are well studied in detail in [14].

1.2 Toric ring and defining ideal

It is a well known fact about monomial ideal that it is a prime if it is generated by variables. However, classification for arbitrary prime ideal in a ring of polynomials is a hard problem. In case of binomial ideals, it is known that they are prime if and only if they are defining ideals of a toric ring.

Let A be an affine semigroup of $\mathbb{Z}^n$ generated by $\{a_1, a_2, \ldots, a_n\}$, that is

$$A = \mathbb{N}a_1 + \mathbb{N}a_2 + \ldots + \mathbb{N}a_n.$$

Then $u_i = \boldsymbol{x}^{\boldsymbol{a}_i}$ are the generators of subring $K[A]$ of Laurent polynomial ring $T = K[x_1^{\pm}, \ldots, x_n^{\pm}]$ and $K[A]$ is a K-subalgebra of the polynomial ring $S = K[x_1, \ldots, x_n]$. Let $R = K[t_1, \ldots, t_n]$ be another polynomial ring and we define $\phi : R \longrightarrow K[A]$ the $K-$algebra homomorphism defined as $t_i \mapsto u_i$ for $i = 1, \ldots, n$. Clearly ϕ is surjective so, $K[A] \cong R/\operatorname{Ker}\phi$. The ideal J is called the *toric ideal* or defining ideal of A.

The most distinguish property of toric ideal is that they are generated by binomials. In particular, they are prime binomial ideals, e.g. see [4].

Definition 1.2.1. Let us consider a commutative ring R and m is the maximum length of regular sequence in maximal ideal of R, this m is called the *depth* of ring R. The number of inclusion of a chain of prime ideals in a ring R is called the *length of chain*. e.g. a chain of prime ideals in R, $Q_0 \subsetneq Q_1 \subsetneq ... \subsetneq Q_l$ has length l. We define the *Krull dimension* of ring R to be the supremum of the lengths of all chains of prime ideals in R. In general for any ring we have depth of ring is less or equal to the dimension of ring and the ring R is called *Cohen-Macaulay* if its depth is equal to its dimension.

Example 1.2.2. The following are examples of Cohen-Macaulay ring.

1. Ring of integers $\mathbb{Z}$.

2. The ring $K[x]/(x^2)$.

1.3 Gröbner basis of binomial ideal

In 1965, Bruno Buch-berger developed the theory of Gröbner basis for polynomial rings and he named them after his advisor Wolfgang Gröbner. This theory has become fundamental field in algebra. An intensive research on this theory is developed e.g see [4].

Definition 1.3.1. Let S be a ring of polynomials, a total order $\leq$ on $\mathrm{Mon}(S)$ with the properties:

- $1 \leq t$ for all monomials t in S.

- if $t < u$ and $v \in \mathrm{Mon}(S)$, then $tv < uv$.

is called a *monomial order*.

Examples 1.3.2. Let $u = \boldsymbol{y}^{\boldsymbol{c}}, v = \boldsymbol{y}^{\boldsymbol{d}}$ be two monomials in S, where $\boldsymbol{c} = (c_1, \ldots, c_n)$ and $\boldsymbol{d} = (d_1, \ldots, d_n)$ are vectors in $\mathbb{Z}_+^n$. The variables are ordered as $y_1 > y_2 > \ldots > y_n$ in all the followings monomial orders.

Degree lexicographic order: $u < v$, if either $\sum_{k=1}^{n} c_k < \sum_{k=1}^{n} d_k$ or $\sum_{k=1}^{n} c_k = \sum_{k=1}^{n} d_k$, and the left most non-zero component of $\boldsymbol{c} - \boldsymbol{d}$ is less than zero.

Pure lexicographic order: in this monomial order $u < v$, if from the left side the first nonzero of $\boldsymbol{c} - \boldsymbol{d}$ is less than zero.

Degree reverse lexicographic order: $u < v$, if either $\sum_{k=1}^{n} c_k < \sum_{k=1}^{n} d_k$ or $\sum_{k=1}^{n} c_k = \sum_{k=1}^{n} d_k$, and the right most nonzero element of $\boldsymbol{c} - \boldsymbol{d}$ is positive.

Definition 1.3.3. Let $<$ a monomial order on S. for $0 \neq g \in S$, we set $\mathrm{in}_<(g)$ to be the largest monomial s in the support of g with respect to $<$, and call it the *initial monomial* for g. The coefficient c of $\mathrm{in}_<(g)$ is called the *leading coefficient* of g with respect to monomial order $<$, and $c\,\mathrm{in}_<(g)$ is called *leading term* of g. Now we are able to define initial ideal. Let $I \subset S$ be a nonzero ideal. The initial ideal of I is the monomial ideal

$$\mathrm{in}_<(I) = (\mathrm{in}_<(f) | f \in I, f \neq 0).$$

A sequence $g_1, ..., g_m$ of elements in I with $\mathrm{in}_<(I) = (\mathrm{in}_<(g_1), ..., \mathrm{in}_<(g_m))$ is called a *Gröbner bases* of I w.r.t. the monomial order $<$.

Note that every ideal $I \subset S$ has Gröbner basis because $\mathrm{in}_<(I)$ is finitely generated. Here we give some important properties of $\mathrm{in}_<(I)$ as below. In Section 3.3 of [6], there is comparison of $\mathrm{in}_<(I)$ and the ideal I is given in detail.

Theorem 1.3.4. [4, Theorem 3.3.4] Let $I \subset S$ be a graded ideal and $<$ a monomial order on S. Then

1. $\dim(S/I) = \dim(S/\mathrm{in}_<(I))$
2. $\mathrm{proj\,dim}(S/I) \leq \mathrm{proj\,dim}(S/\mathrm{in}_<(I))$
3. $\mathrm{reg}(S/I) \leq \mathrm{reg}(S/\mathrm{in}_<(I))$
4. $\mathrm{depth}(S/I) \geq \mathrm{depth}(S/\mathrm{in}_<(I))$
5. S/I is Cohen-Macaulay $\Rightarrow S/\mathrm{in}_<(I)$ is Cohen-Macaulay.

Theorem 1.3.5. [4, Macaulay Theorem] Let $<$ be a monomial order on S and let $I \subset S$ an ideal. Then the monomials in S which do not belong to $\mathrm{in}_<(I)$ form a K-basis of S/I.

For an ideal I, every set of generators not need to be a Gröbner bases. While the converse of this statement is true as given in the following theorem.

Theorem 1.3.6. [4, Theorem 2.8] Let $I \subset S$ be an ideal and $\mathcal{G} = \{g_1, \ldots, g_m\}$ be a Gröbner basis of I with respect to a monomial order $<$. Then $\mathcal{G}$ is a system of generators of I.

Now we will write an algorithm to find the Gröbner basis of ideal $I \subset S$. To explain this criterion we have to introduce the S-polynomials first. Suppose we have $J = (g, h) \in S$, while $g, h \neq 0$. Now we want to compute Gröbner bases of J. Certainly $\mathrm{in}_<(g)$, $\mathrm{in}_<(h) \in \mathrm{in}_<(J)$. A candidate of a polynomial $f \in J$ whose initial monomial does not belong to $(\mathrm{in}_<(g), \mathrm{in}_<(h))$ is a linear combination of g and h such that their initial terms cancel. This leads to define

$$S(g,h) = \frac{\mathrm{lcm}(\mathrm{in}_<(g), \mathrm{in}_<(h))}{c\ \mathrm{in}_<(g)} g - \frac{\mathrm{lcm}(\mathrm{in}_<(g), \mathrm{in}_<(h))}{d\ \mathrm{in}_<(h)} h$$

where c is the leading coefficient of g and d is the leading coefficient of h. The polynomial $S(g,h)$ is called the *S-polynomial* of g and h with respect to $<$. The following theorem gives us idea to get Gröbner basis with the help of S-polynomial.

Theorem 1.3.7. [4, Theorem 2.14] Let $<$ be a monomial order on S, and let $I = (g_1, \ldots, g_m)$ be an ideal in S with $g_i \neq 0$ for all i. Then the following conditions are equivalent:

(a) $g_1, \ldots, g_m$ is a Gröbner basis of I with respect to $<$.

(b) $S(g_i, g_j)$ reduces to 0 with respect to $g_1, \ldots, g_m$ for all $i < j$.

Checking whether a set of generators $h_1, \ldots, h_m$ of an ideal is a Gröbner bases can be rather cumbersome since we have to compute the remainder of $\binom{m}{2}$ S-polynomials. The proposition given below is often used to lessen the manipulation appropriately.

Proposition 1.3.8. [4, Proposition 2.15] S-polynomial of g and h reduce to zero if their initial monomials are relatively prime w.r.t. a monomial order $<$.

1.3.1 Buchberger's criterion

Assume I to be an ideal of the ring S with a finite generating set, say $\mathcal{H}$. The *Buchberger algorithm* is the following:
Step 1: For every distinct pair in $\mathcal{H}$ we compute the $S-$polynomial and corresponding remainder.
Step 2: If the remainder of all these such polynomials is 0 then the algorithm terminates and $\mathcal{H}$ is so called Gröbner basis of I, else one of the non-zero remainders is affixed in our system, this obtained system of generators can again be called $\mathcal{H}$ and reiterate.

This algorithm does terminate after finite steps. Indeed, each time when we add a nonzero remainder of an S-polynomial to $\mathcal{H}$, the initial ideal of $\mathcal{H}$ becomes strictly larger.

Definition 1.3.9. Assume that S has an ideal J. Then the Gröbner basis $\mathcal{H} = h_1, \ldots, h_m$ is called *reduced Gröbner basis* of J w.r.t. $<$, if it satisfy the following:
(i) the leading coefficient of each h_i is 1;
(ii) for all $i \neq j$ no $u \in \operatorname{supp}(h_j)$ is divisible by $\operatorname{in}_<(h_i)$.

Theorem 1.3.10. [4, Theorem 2.17] Each ideal $I \subset S$ has a unique reduced Gröbner basis.

Definition 1.3.11. Let J be a toric ideal. A binomial $\boldsymbol{x^a} - \boldsymbol{x^b} \in J$ is called *primitive* if $\boldsymbol{x^c} - \boldsymbol{x^d} \notin J$ such that $\boldsymbol{x^c} | \boldsymbol{x^a}$ and $\boldsymbol{x^d} | \boldsymbol{x^b}$.

It can be noted that if $\boldsymbol{x^a} - \boldsymbol{x^b}$ is a primitive binomial, then the monomials $\boldsymbol{x^a}$ and $\boldsymbol{x^b}$ have disjoint supports.

Proposition 1.3.12. [4, Proposition 5.8] Let J be a toric ideal and $\mathcal{G}$ the reduced Gröbner basis of J w.r.t. $<$. Then any binomial of $\mathcal{G}$ is primitive.

Proof. Let $u = \boldsymbol{x^a} - \boldsymbol{x^b} \in \mathcal{G}$ with $in_<(u) = \boldsymbol{x^a}$. Since $\mathcal{G}$ is reduced, it follows that $\boldsymbol{x^b} \notin in_<(J)$. Assume that there exist a binomial $v = \boldsymbol{x^c} - \boldsymbol{x^d} \in J, u \neq v$, such that $\boldsymbol{x^c}|\boldsymbol{x^a}$ and $\boldsymbol{x^d}|\boldsymbol{x^b}$. If $\mathrm{in}_<(v) = \boldsymbol{x^c}$, then we must have $\boldsymbol{x^c} = \boldsymbol{x^a}$. It will follow that $v' = \boldsymbol{x^b} - \boldsymbol{x^d} \in J$ and $in_<(v') = \boldsymbol{x^b}$ since $\boldsymbol{x^d}|\boldsymbol{x^b}$, a contradiction. Therefore $in_<(v) = \boldsymbol{x^d}$, which is again contradiction since $\boldsymbol{x^d}|\boldsymbol{x^b}$. $\square$

1.4 Lattice basis ideal

A well known class of binomial ideals is known as lattice ideals. In the following text, we give definition and some known facts about lattice ideals.

Definition 1.4.1. Let A be a commutative group and $b_1, ..., b_m \in A$ be distinct elements in A. Then $\phi : \mathbb{Z}^m \to A$ defined by $e_i \mapsto b_i$ is a group homomorphism and the subgroup $L = \operatorname{Ker} \phi \subset \mathbb{Z}^m$ is a *lattice* in $\mathbb{Z}^m$. The *lattice ideal* $I_L \subseteq K[y_1, \ldots, y_m]$ associated to L is the ideal

$$I_L = (\boldsymbol{y^s} - \boldsymbol{y^t} \mid \boldsymbol{s}, \boldsymbol{t} \in \mathbb{N}^m \text{ with } \boldsymbol{s} - \boldsymbol{t} \in L)$$

Example 1.4.2. Here L is sublattice of $\mathbb{Z}^3$ and $I_L \subset K[s, t, u]$ is a lattice ideal.

1. $L = \mathbb{Z}\{(4, 5, 6)\}, I_L = \langle s^4t^5u^6 - 1\rangle, A = \mathbb{Z}^2$

1.5 Edge ring of bipartite graphs and their defining ideals

To begin with this, lets recall some basic definitions from graph theory.

Definition 1.5.1. A graph G is defined by its vertex set, say $[m] = \{1, \ldots, m\}$ and of edge set $E(G)$.

Each edge is a subset $e \subset [n]$ with exactly two distinct elements. In Figure 1.1,

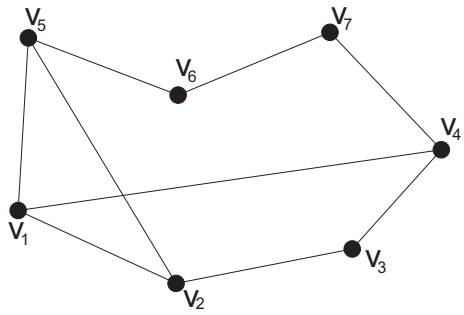

Figure 1.1: A simple graph with 7 vertices

there is a graph on seven vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ and the edges of this graph are $\{v_1, v_2\}, \{v_1, v_5\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_7\}, \{v_6, v_7\}, \{v_5, v_6\}$.

Definition 1.5.2. A subgraph W of a graph G with the edge set $E(W) = \{\{i_0, i_1\}, \{i_1, i_2\}, \ldots, \{i_{r-1}, i_r\}\} \subset E(G)$, where $i_0, i_1, \ldots, i_r$ are vertices of G. A walk W is closed if $i_r = i_1$. A *cycle* is a closed walk where the vertices $i_0, i_1, \ldots, i_{r-1}$ are pairwise distinct. A *path* in G is a walk with pairwise distinct vertices.

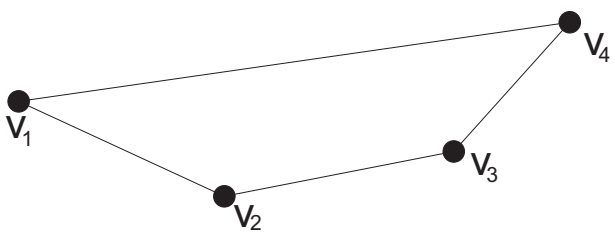

Figure 1.2: A walk in G

Figure 1.2 is a walk for figure 1.1. this can be seen that it is a closed walk and hence it is cycle too.

Definition 1.5.3. A graph is called *bipartite* if its vertex set can be partionized into two non-empty disjoint sets V_1 and V_2 such that any edge of G connects a vertex of V_1 to a vertex of V_2.

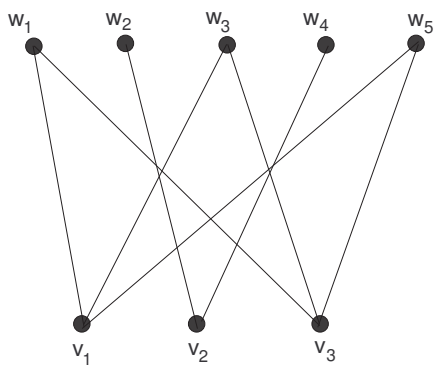

Figure 1.3: Bipartite graph

In figure 1.3, the graph is bipartite graph with two vertex sets $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{w_1, w_2, w_3, w_4, w_5\}$.

Proposition 1.5.4. [16] For a graph G, G is bipartite $\Leftrightarrow$ cycles of G have even length.

Proof. Consider a bipartite graph G with V_1 and V_2 the partition of vertices of G. So according to definition every edge of G joins the vertices of V_1 to the vertices of V_2. Let $\{v_0, v_1, \ldots, v_m\}$ be a cycle with $v_0 = v_m$. We may assume that $v_0 \in V_1$ then $v_1 \in V_2$ and $v_2 \in V_1$. According to this sequence of vertices it follows that $v_i \in V_1$ if and only if i is even. Hence the length of the cycle must be even.
Now, assume that we have a graph G in which every cycle is even. We can also assume that G is connected. Now choose $v_0 \in V(G)$ and set

$$V_1 = \{v \in V(G) : \text{the shortest path between } v_0 \text{ and } v \text{ is even}\}, V_2 = V(G) \backslash V_1$$

It follows that no two vertices of V_i are adjacent for $i = 1, 2$, because there is no cycle in G which have odd length. Hence G is bipartite. $\square$

Definition 1.5.5. Let G be a graph on the edge set $[n]$ and let $S = K[x_1, \ldots, x_n]$ be the polynomial ring. For each edge $e = \{i, j\} \in E(G)$ we associate the monomial $x_e = x_i x_j \in S$. Let $E(G) = \{e_1, \ldots, e_m\}$ be the edge set of G. The semigroup ring $K[G] = K[x_{e_1}, \ldots, x_{e_m}]$ is called the *edge ring* of graph G.

Chapter 2

Polyominoes and Polyomino Ideals

2.1 Cells and their combinatorics

In this section, we define cells and some basic definitions related to cells. To do so, first we recall the natural partial order defined on $\mathbb{N}^2$ defined as: for any $c, d \in \mathbb{N}^2$ with $d = (k, l)$ and $c = (i, j)$, we say $c \leq d$ iff $j \leq l$ and $i \leq k$. With this notation, if $c \leq d$, then the set $[c, d] = \{e \in \mathbb{N}^2 | \ c \leq e \leq d\}$ is named as an interval of $\mathbb{N}^2$.
The interval $[c, d]$ is a proper if $j < l$ and $i < k$. The notches of this proper interval $[c, d]$ are the elements $(k, j), (k, l), (i, l), (i, j)$ and (i, l). We call the elements $(k, l), (i, j)$ the diagonal corners of $[c, d]$, whereas the elements $(k, j), (i, l)$ are named as the anti-diagonal corners of $[c, d]$. Also, c and d are in *horizontal* position if $j = l$, and c and d are in *vertical* position if $i = k$.
A *cell* D is a unit interval in $\mathbb{N}^2$, that is, $D = [c, d]$ is a cell if $d = c + (1, 1)$. The corners of D, denoted by $V(D)$ are called the *vertices* of D. If we name e and f, the anti-diagonal vertices of D, then the set $E(D) = \{\{c, e\}, \{f, d\}, \{e, d\}, \{c, f\}\}$ is the edge set of D. A cell D with lower left corner $e = (m, n)$ belong to a proper interval $[c, d]$ of $\mathbb{N}^2$ if

$$j \leq n \leq l - 1 \quad \text{and} \quad i \leq m \leq k - 1. \tag{2.1}$$

where $d = (k, l), c = (i, j)$. If one of the inequalities in (2.1) is an equality then the cell D is called a *border cell* of interval $[c, d]$. Let (k, l) and(i, j) be the lower left corners of two cells D and E correspondingly. Then the set,

$$[D, E] =: \{F \colon F \in \mathbb{N}^2 \text{ with lower left corner } (m, n), \text{ for } j \leq n \leq l,\ i \leq m \leq k\}$$

is called *cell interval.* The cell interval $[D, E]$ is called a *vertical* or *horizontal* cell interval if (k, l) and (i, j) are in vertical or horizontal position respectively.

Let we have a finite collection of cells $\mathcal{P}$ in $\mathbb{N}^2$. The set $V(\mathcal{P}) = \bigcup\limits_{C \in \mathcal{P}} V(C)$ is vertex set and $E(\mathcal{P}) = \bigcup\limits_{C \in \mathcal{P}} E(C)$ is the edge set of $\mathcal{P}$. If $v \in V(\mathcal{P})$ is a vertex of four different cells of $\mathcal{P}$, then v is named as an *interior vertex.* The set $\operatorname{int}(\mathcal{P}) = \{v \in V(\mathcal{P}) : v \text{ is interior vertex}\}$, is called *interior* of $\mathcal{P}$. The *boundary of* $\mathcal{P}$ is the set $V(\mathcal{P}) \setminus \operatorname{int}(\mathcal{P})$ and denoted by $\partial\mathcal{P}$. For a collection $\mathcal{P}$ as shown in Figure 2.1, the bold dots are interior while all other belong to int of $\mathcal{P}$.

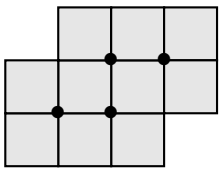

Figure 2.1:

$\mathcal{P}$ is called row convex if $[C, D] \subset \mathcal{P}$ for all $C, D \in \mathcal{P}$ such that $[C, D]$ is a horizontal cell interval. Similarly, $\mathcal{P}$ is column convex if $[E, F] \subset \mathcal{P}$ for all $E, F \in \mathcal{P}$ such that $[E, F]$ is a vertical cell interval. If $\mathcal{P}$ is row convex as well as column convex, then $\mathcal{P}$ is called *convex.* Figure 2.1 shows a convex collection of cells and the collection of cells $\mathcal{P}$ in Figure 2.3 is row convex but not column convex.

Two cells A and B in $\mathcal{P}$ are called *weakly connected* cells if there exist cells $A = J_1, \ldots, J_m = B$ such that $J_i \cap J_{i+1} \neq \emptyset$ for $i = 1, \ldots, m-1$, and they are said to be *connected* if $J_i \cap J_{i+1}$ is particularly an edge. If in a $\mathcal{P}$ all cells are connected, then $\mathcal{P}$ is a polyomino, e.g. see Figure 2.3.

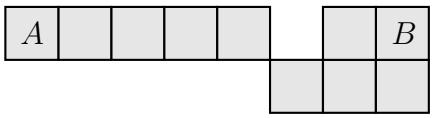

Figure 2.2: Weakly connected collection of cells

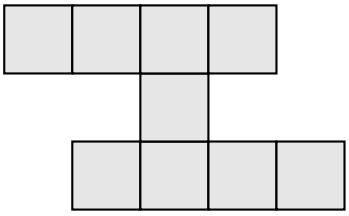

Figure 2.3: Polyomino

For any finite collection of cells, $\mathcal{I} = [c, d] \subset \mathbb{N}^2$ is a proper interval of $\mathbb{N}^2$ such that vertices of $\mathcal{P}$ are contained in interior of $\mathcal{I}$. A polyomino is *simple* if any cell C of $[a, b]$ which is not in $\mathcal{P}$ is connected to a border cell D of $[a, b]$ by a path $C = J_1, \ldots, J_m = D$ such that $J_i \notin \mathcal{P}$ for all $i = i, \ldots, m$. For example see Figure 2.4.

Non-simple Simple

Figure 2.4:

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ be connected components of a $\mathcal{P}$. We attach a graph H to $\mathcal{P}$ with $V(H) = [m]$ and $\{j, l\} \in E(H)$ if $\mathcal{P}_j \cap \mathcal{P}_l \neq \emptyset$. Here we list all combinatorial properties of finite collection of cells and in particularly polyomino given in [13].

1. Let $\mathcal{P}$ be a weakly connected and convex collection of cells, and let $a, b \in V(\mathcal{P})$ be two vertices which are in horizontal or vertical position. Then $[a, b] \subset V(\mathcal{P})$.

2. Let $\mathcal{P}$ be a weakly connected and convex, and $[g, h]$ be a proper interval in $\mathbb{N}^2$. If the corners of $[g, h]$ belong to $V(\mathcal{P})$, then the cells of $[g, h]$ belong to $\mathcal{P}$.

3. Let $\mathcal{P}$ be a simple polyomino and $\mathcal{P}_1$ and $\mathcal{P}_2$ be two connected components of $\mathcal{P}$. Then $|\mathcal{P}_1 \cap \mathcal{P}_2| \leq 1$.

4. If $\mathcal{P}$ is a weakly connected and simple collection of cells then the associated graph G is a tree graph.

2.2 Polyomino ideals

Let $\mathcal{P}$ be a polyomino and S be the ring of polynomials over a field K whose variables are indexed by vertices in $\mathcal{P}$, that is, $S = K[x_u : u \text{ is a vertex of } \mathcal{P}]$. For every proper interval $[c, d]$ of $\mathbb{N}^2$, there is a binomial $f_{c,d} = x_c x_d - x_e x_f$, where e and f are antidiagonal vertices of $[c, d]$. If all the cells of $[c, d] \in \mathcal{P}$, then $[c, d]$ is called an *inner interval* of $\mathcal{P}$ and the binomial $f_{c,d}$ associated to $[c, d]$ is called *inner 2-minor* of $\mathcal{P}$. The ideal $I_{\mathcal{P}} = \langle f_{a,b} | [a, b] \text{ is an inner interval} \rangle \subset S$ is called *polyomino ideal*. The quotient ring $S/I_{\mathcal{P}}$ attached to $\mathcal{P}$ is denoted by $K[\mathcal{P}]$.

The interesting question here is: For which classes of polyominoes, the associated ring $K[\mathcal{P}]$ is a domain? In other words, we want to find out the classes of polyominoes, the attached polyomino ideals are prime. The prime monomial ideals are well understood and have a precise characterization. They are the ones which are generated by variables. However, understanding the primality of binomial ideal

is a hard question. As mentioned in chapter 1, a binomial ideal is prime if and only if it is a toric ideal of a suitable toric ring. As a first attempt to study the primness of polyomino ideal, in [13] it was shown that for a convex polyomino, the polyomino ideal can be seen as a toric ideal. In fact it was shown thatfor a convex polyomino, the polyomino ideal is same as toric ideal of an edge ring of a bipartite graph. Later on, this concept was generalized to the case of a simple polyomino in [12]. In the following text, we explain that how one can develop a connection between a polyomino ideal and a suitable bipartite graph.

2.3 Convex polyominoes and their associated bipartite graphs

Assume that $\mathcal{P}$ is a polyomino and $[c, d] \subset \mathbb{N}^2$ is the smallest interval such that vertices of $\mathcal{P}$ are contained in $[c, d]$. We may assume that $c = (1, 1)$ and $d = (m, n)$. We attach a bipartite graph to $\mathcal{P}$ as follows. Let H be a bipartite graph with bipartition of vertices given as $\{s_1, \ldots, s_m\} \bigcup \{t_1, \ldots, t_n\}$. Then $\{s_j, t_l\} \in E(H)$ iff $(j, l) \in V(\mathcal{P})$. The edge ring $K[H]$ attached to H is given by

$$K[H] = K[s_j t_l : (j, l) \in E(H)] \subset K[s_1, \ldots, s_m, t_1, \ldots, t_n].$$

With the identification of each vertex of $\mathcal{P}$ as an edge H, the toric ring $K[H]$ can also be presented as,

$$K[H] = K[s_j t_l (j, l) \text{ is a vertex of } \mathcal{P}] \subset K[s_1, \ldots, s_m, t_1, \ldots, t_n].$$

Let $\varphi : S \to R$ be a K-algebra homomorphism such that $x_{jl} \to s_j t_l$, for all (j, l) from vertex set of $\mathcal{P}$. We denote the toric ideal $\mathrm{Ker}(\varphi)$ by $J_{\mathcal{P}}$. The ideal $J_{\mathcal{P}}$ is well studied, for example see [11], [16]. To be able to describe the generators of $J_{\mathcal{P}}$, we first give the following definitions.

$\mathcal{C} \subset V(H)$ is a cycle, if $\mathcal{C} = \{s_{i_1}, t_{j_1}, s_{i_2}, t_{j_2}, \ldots, s_{i_{r-1}}, t_{j_{r-1}}, s_{i_r}, t_{j_r}\}$ such that $\{s_{i_k}, t_{j_k}\}$ and $\{t_{j_k}, s_{i_{k+1}}\}$ belongs to $E(H)$ for each $k = 1, \ldots, r$. The cycle $\mathcal{C}$ is

called an *even* cycle if $|\mathcal{C}|$ is even. To a given cycle $\mathcal{C}$ in G, we associate a binomial $f_{\mathcal{C}} = x_{i_1j_1}x_{i_2j_2}\dots x_{i_{r-1}j_{r-1}}x_{i_rj_r} - x_{i_2j_1}x_{i_3j_2}\dots x_{i_rj_{r-1}}x_{i_1j_r}$. It is known that the toric ideal $J_{\mathcal{P}}$ attached to $K[H]$ is generated by those $f_{\mathcal{C}}$ for which $\mathcal{C}$ is an even cycles , see [11, Lemma 1.1] and [16, Proposition 8.1.2].

With the identification of each vertex of $\mathcal{P}$ with an edge of the bipartite graph G, we observe the following fact. Each inner minor of $\mathcal{P}$ correspond to a cycle of length 4 in G. From this fact, we see that $I_{\mathcal{P}} \subset J_{\mathcal{P}}$. Note that in general, $I_{\mathcal{P}} \neq J_{\mathcal{P}}$. The interesting question that arise here is that for which polyominoes do we have $I_{\mathcal{P}} = J_{\mathcal{P}}$, because this equality gives us the class of polyomino where the attached ideal is prime. In [13], it is shown that this equality hold if $\mathcal{P}$ is a convex polyomino. The details are given in following

Theorem 2.3.1. [13, Theorem 2.2] If $\mathcal{P}$ is a convex polyomino. Then $S/I_{\mathcal{P}}$ is a normal and Cohen-Macaulay domain of dimension $|V(\mathcal{P})| - |\mathcal{P}|$. In particular, if $\mathcal{P}$ is weakly connected and $[c, d]$ is the smallest interval such that $V(\mathcal{P}) \subset [a, b]$. Then $K[\mathcal{P}]$ is a Cohen-Macaulay domain with $\dim K[\mathcal{P}] = \text{size}([c, d]) + 1$.

Proof. Let $\mathcal{P}_1, \dots, \mathcal{P}_q$ be the components of $\mathcal{P}$. Then $V(\mathcal{P}) = \bigsqcup_{i=1}^{q} V(\mathcal{P}_i)$ and $I_{\mathcal{P}} = \sum_{i=1}^{q} I_{\mathcal{P}_i}$. So, $K[\mathcal{P}]$ is a Cohen-Macaulay and normal domain iff every $K[\mathcal{P}_i]$ is. We may consider that $\mathcal{P}$ is weakly connected. From [13] it follows that $I_{\mathcal{P}} = J_{\mathcal{P}}$. Hence, $K[\mathcal{P}] = S/I_{\mathcal{P}}$ is domain, since $I_{\mathcal{P}}$ is prime. We know from [16, Proposition 8.1.2] and [11, Lemma 1.1] that universal Gröbner basis of $J_{\mathcal{P}}$ are generated by binomials that corresponds to even cycles of graph G attached to $\mathcal{P}$. Which shows that $in_<(I_{\mathcal{P}})$ is square-free w.r.t. any monomial order $<$. Then by using the theorem by Strumfles [15] and [1, Theorem 6.3.5] we obtain that $K[\mathcal{P}]$ is normal and Cohen-Macaulay. Since, $|V(\mathcal{P})| - |\mathcal{P}| = \sum_{i=1}^{q} |V(\mathcal{P}_i)| - |\mathcal{P}|$ and $K[\mathcal{P}] \cong K[G]$, where $K[G]$ is edge ring of bipartite graph attached to $\mathcal{P}$. So we can use [16, Corollary 8.2.13], which states that $\dim K[G]$ is cardinality of vertex set of G difference 1 for connected bipartite graph G.

Let $I = [c, d] \subset \mathbb{N}^2$ be the smallest interval such that $V(\mathcal{P}) \subset I$. We may assume that $c = (1, 1)$ and $d = (m, n)$, then $\text{size}([c, d]) = m + n - 2$. So with the identification of edge ring of G with $K[\mathcal{P}]$ that $V(G) = \{s_1, \ldots, s_m\} \cup \{t_1, \ldots, t_n\}$. Therefore $\dim K[\mathcal{P}] = (m + n) - 1 = \text{size}([c, d]) + 1$.
Now only remain to show that $|V(\mathcal{P})| - |\mathcal{P}| = \text{size}([c, d]) + 1$. We will use mathematical induction to prove this. When $\mathcal{P}$ is consist of one column then the formula is true. Now let $\mathcal{R}$ be the cells after removing the right most column Q of $\mathcal{R}$, and $[c', d']$ be the smallest interval such that $V(\mathcal{R}) \subset [c', d']$. Suppose that Q has t number of cells which are sharing edge with the cells of $\mathcal{R}$ and r number of remaining cells in Q. Then $|V(\mathcal{P})| = |V(\mathcal{R})| + 2r + t + 1$, $|\mathcal{P}| = |\mathcal{R}| + r + t$ and $\text{size}([c, d]) = \text{size}([c', d']) + r + 1$.

2.4 Simple polyominoes are prime

Firstly, let's recall the definition for simple polyomino. Let $\mathcal{P}$ be a polyomino and let $I = [a, b]$ be an interval of $\mathbb{N}^2$ such that $\mathcal{P} \subset I$. A polyomino $\mathcal{P}$ is *simple* if for any cell C not belonging to $\mathcal{P}$ there exist a path of cells $C = C_1, C_2, \ldots, C_k = D$ with $C_i \notin \mathcal{P}$ for $i = 1, 2, \ldots, k$ such that D is not cell of I. Roughly speaking, a simple polyominoes are those polyomino which do not allow any holes. One can easily see that a convex polyomino is also a simple polyomino because the convexity of a convex polyomino does not not allow any holes in it. In [12] and [7], it is proved $I_{\mathcal{P}}$ is prime for simple polyominoes. In particular, in [7], the proof of primality of convex polyomino was extended to the cases of simple polyomino. Before stating this result, we give some related definitions.

Assume $\mathcal{P}$ is a polyomino. An interval $[c, d]$ with $d = (k, l)$ and $c = (i, j)$ is called a *horizontal edge interval* of $\mathcal{P}$ if the sets $\{r, j\}$ for $r = i, \ldots, j - 1$ are edges of $\mathcal{P}$ and $j = l$. A horizontal edge interval is maximal if it is not contained in any other horizontal edge interval of $\mathcal{P}$. Similarly, we define vertical edge interval and maximal vertical edge interval.

Let $\{V_1, \ldots, V_m\}$ and $\{H_1, \ldots, H_n\}$ be the set of maximal vertical and horizontal edge intervals of $\mathcal{P}$ respectively. We attach to $\mathcal{P}$ a bipartite graph $H(\mathcal{P})$, with set of vertices $\{v_1, v_2, \ldots, v_m\} \sqcup \{h_1, h_2, \ldots, h_n\}$ such that $\{v_i, h_j\}$ is an edge in $G(\mathcal{P})$ for $(i, j) = V_i \cap H_j \in V(\mathcal{P})$.

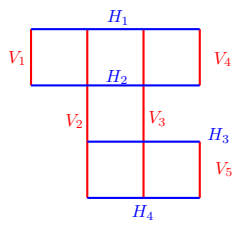

Figure 2.5: Maximal intervals of $\mathcal{P}$

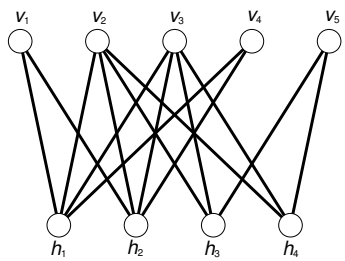

Figure 2.6: Associated bipartite graph of $\mathcal{P}$

Let $S = K[x_{jl}$ where (j, l) is a vertex of $\mathcal{P}]$ be a ring of polynomials over a field K as before. We will write $x_{j,l}$ for $x_{V_p \cap H_q}$, whenever $V_p \cap H_q = \{(j, l)\}$ To each cycle $\mathcal{C} : v_{j_1}, h_{l_1}, v_{j_2}, h_{l_2}, \ldots, v_{j_r}, h_{l_r}$ in $H(\mathcal{P})$ we associate the following binomial

$$f_{\mathcal{C}} = x_{j_1 l_1} x_{j_2 l_2} \cdots x_{j_r l_r} - x_{j_1 l_2} x_{j_2 l_3} \cdots x_{j_1 l_r}$$

A sequence of vertices $\mathcal{C}_{\mathcal{P}} = a_1, a_2, \ldots, a_p$ in $V(\mathcal{P})$ with $a_1 = a_p$ is called *cycle* of $\mathcal{P}$ if it satisfy:

- $[a_j, a_{j+1}]$ is a horizontal or vertical edge interval of $\mathcal{P} \forall j = 1, 2, ..., p-1$.
- for $j = 1, 2, ..., p$ if $[a_j, a_{j+1}]$ is a vertical edge interval then $[a_{j+2}, a_{j+3}]$ is a horizontal edge interval and vice versa. Here, $a_{p+1} = a_2$.

If each maximal interval of $\mathcal{P}$ has maximum two vertices of $\mathcal{C}_\mathcal{P}$, then $\mathcal{C}_\mathcal{P}$ is *primitive*. A cycle $C_\mathcal{P} : a_1, a_2, \ldots, a_m, a_1$ in $\mathcal{P}$ has a self crossing if there are j, l such that $a_j, a_{j+1} \in V_p$ and $a_l, a_{l+1} \in H_q$ with $V_p \cap H_q \neq \emptyset$. In this case v_p, h_q is edge of $G(\mathcal{P})$ which gives us a chord. Similarly if $C_\mathcal{P}$ is a cycle in $\mathcal{P}$ having no self crossing then area bounded by edge intervals $[a_i, a_{i+1}]$ and $[a_r, a_1]$ for $i = 1, 2, \ldots, r-1$. The following lemma is a main tool to prove that simple polyomino are prime. In order to prove this lemma we recall from chapter 1 that graph is *weakly chordal* if G has always chord in every cycle which has length greater than 4. We set,
$K[H(\mathcal{P})] = K[v_j h_l : (j, l) = V_p \cap H_q \in V(\mathcal{P})] \subset T = K[v_1, v_2, \ldots, v_m, h_1, h_2, \ldots, h_n]$.
Let $\psi : S \to T$ be a $K-$algebra surjective homomorphism as $x_{jl} \rightarrowtail v_p h_q$. The toric ideal of $K[H(\mathcal{P})]$ is denoted by $J_\mathcal{P}$. It is known from [11] that binomials associated with cycles in $G(\mathcal{P})$ generate $J_\mathcal{P}$.

Lemma 2.4.1. [12] Let $\mathcal{P}$ be a simple polyomino then the bipartite graph $G(\mathcal{P})$ associated to $\mathcal{P}$ is weakly chordal.

Proof. Let $G(\mathcal{P})$ is a bipartite graph associated to simple polyomino $\mathcal{P}$. Let $\mathcal{C} : v_{i_1}, h_{j_1}, v_{i_2}, h_{j_2}, \ldots, v_{i_r}, h_{j_r}$ be a cycle of length $2n$. If $n = 1, 2$ nothing is to prove, so consider $n \geq 3$. Also assume that $\mathcal{C}$ has no self crossing it implies that $\mathcal{C}$ has a chord.
Consider the associative primitive cycle $\mathcal{C}_\mathcal{P} : V_{i_1} \cap H_{j_1}, V_{i_2} \cap H_{j_1}, V_{i_2} \cap H_{j_2}, \ldots, V_{i_r} \cap H_{j_r}, V_{i_1} \cap H_{j_r}$. We can write $a_1 = V_{i_1} \cap H_{j_1}, a_2 = V_{i_1} \cap H_{j_2}, a_3 = V_{i_2} \cap H_{j_2}, \ldots, a_{2r-1} = V_{i_r} \cap H_{j_r}, a_{2r} = V_{i_1} \cap H_{j_r}$. By the definition of $\mathcal{C}_\mathcal{P}$ assume that a_1, a_2 are in same horizontal edge interval and a_m, a_1 are in a single vertical edge interval.
Firstly, we will show that $\text{int}(\mathcal{C}_\mathcal{P})$ is in $\mathcal{P}$. For this, assume we have a cell $D \in \mathcal{C}_\mathcal{P}$ such that $D \notin \mathcal{P}$. Let $\mathcal{L} \subset \mathbb{N}^2$ with $\mathcal{P} \subset \mathcal{L}$. Since $\mathcal{P}$ is simple, so there exists a path

of cells $C = C_1, C_2, \ldots, C_p$ with $C_j \notin \mathcal{P}$ for $j = 1, 2, \ldots, p$ and C_p is a boarder cell of $\mathcal{L}$. This gives us that,

$$V(C_1) \cup V(C_2) \cup \ldots \cup V(C_m)$$

intersect at least one of the $[a_i, a_{j+1}]$ for $j = 1, 2, \ldots, r-1$ or $[a_r, a_1]$, which contradicts that $\mathcal{C}_\mathcal{P}$ is a cycle in $\mathcal{P}$, hence D belongs to $\mathcal{P}$. This shows that interior intervals of $\mathcal{C}_\mathcal{P}$ are contained in inner intervals of $\mathcal{P}$.

Let $\mathcal{M}$ be a inner interval of $\mathcal{C}_\mathcal{P}$ which is maximal and $a_1, a_2 \in \mathcal{M}$. Consider c, b are the corner of $\mathcal{M}$ and assume that a_1, b are vertices on diagonal and a_2, c are antidiagonal vertices. Since $a_1, a_2 \in V(\mathcal{C}_\mathcal{P})$, and if $c, b \in V(\mathcal{C}_\mathcal{P})$ then length of cycle $\mathcal{C}$ is 4 since $\mathcal{C}_\mathcal{P}$ is primitive.

Suppose $c \notin V(\mathcal{C}_\mathcal{P})$ and H' be a maximal horizontal edge interval such that $c, d \in H'$. As $\mathcal{M}$ is maximal inner interval, so it gives that $H' \cap V(\mathcal{C}_\mathcal{P}) \neq \phi$. For example see Figure 2.7 and 2.8.Therefore, $\{v_{i_1}\}$ is a chord in $\mathcal{C}$. $\square$

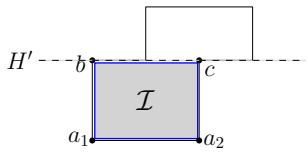

Figure 2.7: When $b, c \in V(\mathcal{C}_\mathcal{P})$

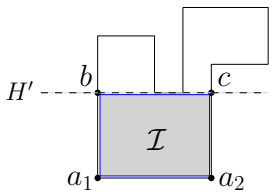

Figure 2.8: When $b \notin V(\mathcal{C}_\mathcal{P})$

Theorem 2.4.2. [12] Let $\mathcal{P}$ be a simple polyomino then, $I_\mathcal{P} = J_\mathcal{P}$.

Proof. First we will show that $I_\mathcal{P} \subset J_\mathcal{P}$. Let $f = x_{ij}x_{kl} - x_{il}x_{kj} \in I_\mathcal{P}$. Then we have V_p, V_q and H_r, H_s maximal horizontal and vertical edge intervals in such a way that

$(i,l),(i,j) \in V_p, (k,l),(k,j) \in V_q, (k,j),(i,j) \in H_r$ and $(k,l),(i,l) \in H_s$. It gives that $\psi(x_{ij}x_{kl} - x_{il}x_{kj}) = v_p h_r h_s v_q - v_p h_s h_r v_q$, which implies that $f \in \mathrm{Ker}(\psi) = J_{\mathcal{P}}$. Hence, $I_{\mathcal{P}} \subset J_{\mathcal{P}}$.

Now we have to show that $J_{\mathcal{P}} \subset I_{\mathcal{P}}$. It is known from [11] and [10] that $J_{\mathcal{P}}$ is generated by quadratic binomials associated to cycles of length 4 for a weakly-chordal graph. So it is sufficient to prove that $f_{\mathcal{C}} \in I_{\mathcal{P}}$ in order to prove $J_{\mathcal{P}} \subset I_{\mathcal{P}}$. Assume we have an interval $\mathcal{M}$ in such a way that all the cells of $\mathcal{P}$ are contained in $\mathcal{M}$ and $\mathcal{C} : h_1, v_1, h_2, v_2$. Then the associative primitive cycle in $\mathcal{P}$ is $\mathcal{C}_{\mathcal{P}} : H_1 \cap V_1, H_2 \cap V_1, H_2 \cap V_2, H_1 \cap V_2$. We may assume $a_{11} = H_1 \cap V_1, a_{21} = H_2 \cap V_1, a_{22} = H_2 \cap V_2$ and $a_{12} = H_1 \cap V_2$. As shown in the proof of above lemma $\mathcal{C}_{\mathcal{P}}$ determine an interval in $\mathcal{K}$. Let a_{11} and a_{22} be the diagonal vertices of $\mathcal{K}$. Now we need to show that $\mathcal{K} = [a_{11}, a_{22}]$ is an inner interval.

Contrarily, suppose that $\mathcal{K}$ is not an inner intrval, then there is cell E of $\mathcal{K}$ which do not belong to $\mathcal{P}$. So, we have a path of cells $E = E_1, E_2, \ldots, E_p$ with $E_j \notin \mathcal{P}$ for $j = 1, 2, \ldots, p$ and D_p is a border cell of $\mathcal{K}$. Then

$$V(D_1) \cup V(D_2) \cup \ldots \cup V(D_m)$$

cuts minimum one the maximal intervals V_1, V_2, H_1, H_2, say V_1 which contradicts the fact that V_1 is an edge interval. Hence $\mathcal{K}$ is an inner interval of $\mathcal{P}$ and $f_{\mathcal{C}} \in I_{\mathcal{P}}$. $\square$

Corollary 2.4.3. [12] Let $\mathcal{P}$ be a simple polyomino. Then $K[\mathcal{P}]$ is Koszul and a normal Cohen-Macaulay domain.

A polyomino ideal may be prime even when the polyomino is not simple. In the upcoming chapter we discuss some certain cases where polyomino ideal is prime for a non simple polyomino.

Chapter 3

The case of nonsimple polyominoes

3.1 polyomino with one hole

As we have seen in previous section that polyomino ideal associated to simple polyomino is prime. Here in this section we discuss a case where the ideal is prime but polyomino is non-simple. We will use some terminology in the following theorem. Let's define these terminology here. Let $\mathcal{I} \subset \mathbb{N}^2$ be an interval. Then cells of $\mathcal{I}$ form a polyomino and we write is as $\mathcal{P}_{\mathcal{I}}$. The $B(\mathcal{P})$ is the collection of all boundary cells of $\mathcal{P}$.

Theorem 3.1.1. Let $\mathcal{I} \subset \mathbb{N}^2$ be an interval and $\mathcal{P}$ a convex polyomino which is sub polyomino of $\mathcal{P}_{\mathcal{I}}$. Let $\mathcal{P}^c = \mathcal{P}_{\mathcal{I}} \backslash \mathcal{P}$ and suppose that $\mathcal{P}$ is a polyomino. Then the polyomino ideal $I_{\mathcal{P}^c}$ is a prime ideal.

Proof. Here we have two possibilities when $B(\mathcal{P}) \cap B(\mathcal{P}_{\mathcal{I}}) \neq \emptyset$, the polyomino $\mathcal{P}^c$ is a simple polyomino, see figure 3.2 and the ideal $I_{\mathcal{P}^c}$ is already a prime ideal. So we may assume that $B(\mathcal{P}) \cap B(\mathcal{P}_{\mathcal{I}}) = \emptyset$.
Let $\mathcal{I} = [a, b]$ with c and d be the anti-diagonal corners of $\mathcal{I}$, where b and c are in horizontal position. Then by theorem 4.1.1 x_c can not divide the initial monomial

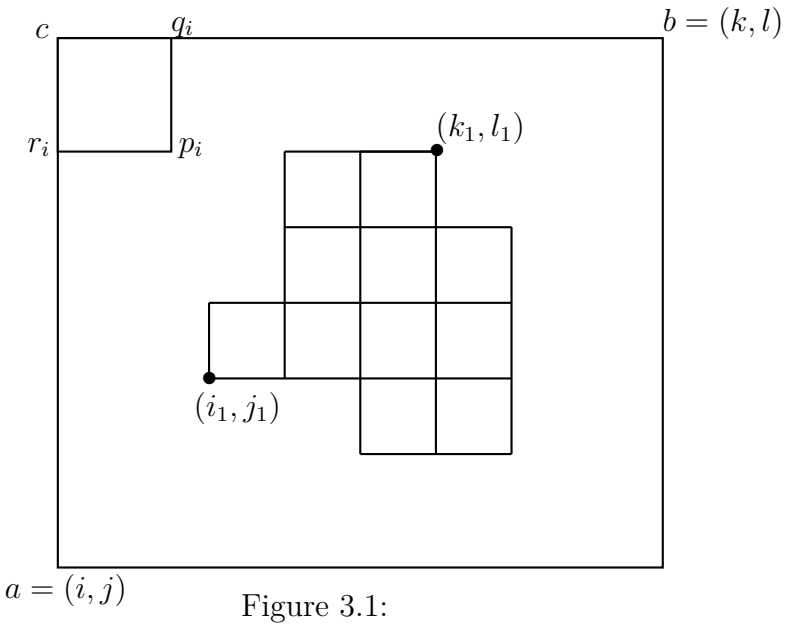

Figure 3.1:

of any binomial from the reduced Grö bner basis of $I_{\mathcal{P}^c}$ w.r.t. lexicographic order $<$. This implies that x_c is nonzero divisor in $S/in_< I_{\mathcal{P}^c}$ and hence in $S/I_{\mathcal{P}^c}$. To prove that $S/I_{\mathcal{P}^c}$ is integral domain we will prove that $(S/I_{\mathcal{P}^c})_{x_c}$ is integral domain. Here $(S/I_{\mathcal{P}^c})_{x_c}$ is the localization of $S/I_{\mathcal{P}^c}$ at the element x_c. To prove $(S/I_{\mathcal{P}^c})_{x_c} = S_{x_c}/(I_{\mathcal{P}^c})_{x_c}$ is integral domain, we show that

$$(I_{\mathcal{P}^c})_{x_c} = I_{\mathcal{P}'} S_{x_c}$$

where $\mathcal{P}'$ is a simple subpolyomino of $\mathcal{P}^c$ or we can say that $I_{\mathcal{P}'}$ is a prime ideal [4 and 8].

Let $\mathcal{A} = \{p_1, p_2, \ldots, p_m\}$ be the set of vertices of $\mathcal{P}^c$ for which there is an interval $[r_i, q_i]$ of $\mathcal{P}^c$ whose anti-diagonal corners are c and p_i. see Figure 3.1. Here we can see that $r_i \in [a, c]$ and $q_i \in [c, b]$, Now as $x_{r_i} x_{q_i} - x_c x_{p_i} \in I_{\mathcal{P}^c}$ and as we know x_c is invertible in S_{x_c} so $x_{p_i} = x_{q_i} x_{r_i} x_c^{-1}$ in $S_{x_c}/(I_{\mathcal{P}^c})_{x_c}$. Hence the variables x_{p_i} can be ignored in $S_{x_c}/(I_{\mathcal{P}^c})_{x_c}$, while $p_i \in \mathcal{A}$.

Let's take two elements p_i, P_j from $\mathcal{A}$ for which $[p_i, p_j]$ is an interval of $\mathcal{P}^c$, then the anti-diagonal corner of $[p_i, p_j]$ are also in $\mathcal{A}$. Thus $f_{p_i,p_j} = x_{p_i} x_{p_j} - x_{p_k} x_{p_l}$ is an inner binomial of $\mathcal{P}^c$, where p_k and p_l are anti-diagonal corner of $[p_i, p_j]$.

Let $[v, p_i]$ be an interval of $\mathcal{P}^c$ such that $v \notin \mathcal{A}$ and $p_i \in \mathcal{A}$. As $[r_i, p_i] \backslash \{r_i\} \subset \mathcal{A}$, this implies that the anti-diagonal corner p_i' of $[v, p_i]$ which is in horizontal position with p_i is also in $\mathcal{A}$. Let v' be the other anti-diagonal corner of $[v, p_i]$, then the inner binomial $x_v x_{p_i} - x_{v'} x'_{p_i}$ can be written as $x_{r_i}(x_v x_{q_i} - x_{v'} x'_{q_i})$ in $I_{\mathcal{P}^c}$. Similarly, if $[p_i, v]$ is an interval of $\mathcal{P}^c$ then $x_v x_{p_i} - x_{v'} x'_{p_i}$ is a multiple of $x_v x_{r_i} - x_{v'} x'_{r_i}$ in $I_{\mathcal{P}^c}$.

Let $\mathcal{P}'$ is a collection of cells of $\mathcal{P}^c$ obtained after the removing of all cells that appear in $\bigcup_{i=1}^{n} \mathcal{P}_{[r_i,q_i]}$ and $a = (i,j), b = (k,l)$, then $c = (i,l)$. Now we choose $m_1 = (i_1, j_1) \in V(\mathcal{P})$ such that for any $m \in V(\mathcal{P})$ we have $m_1 \leq m$. Similarly we choose $n_1 = (k_1, l_1) \in V(\mathcal{P})$ such that for any $n \in V(\mathcal{P})$ we have $n_1 \geq n$. In $\mathcal{P}'$ we identify the vertical interval $[a, (i, j_1)]$ with $[(i_1, j), (i_1, j_1)]$, and the horizontal interval $[(k_1, l_1), (k, l_1)]$ with $[(k_1, l), b]$. With this identification and using the above arguments we have $I_{\mathcal{P}'} S_{x_c} = (I_{\mathcal{P}^c})_{x_c}$. Now only we have to prove that $\mathcal{P}'$ is a simple polyomino. First we will show that in this construction $\mathcal{P}'$ is a polyomino indeed. Let $\mathcal{B}$ be the collection of border cells of $\mathcal{P}_{\mathcal{I}}$ belonging to $\mathcal{P}'$. Then $\mathcal{B}$ is connected. Using the convexity of $\mathcal{P}$, we can see that every cell of $\mathcal{P}'$ is connected to at least one cell of $\mathcal{B}$, hence $\mathcal{P}'$ is connected and is polyomino, as desired.

Second, we prove that $\mathcal{P}'$ is simple. Let $\mathcal{J}$ be an interval such that $\mathcal{P}' \subset \mathcal{P}_{\mathcal{I}} \subset \mathcal{P}_{\mathcal{J}}$. Contrarily suppose that $\mathcal{P}'$ is not simple. Then we have a cell $D \notin \mathcal{P}'$ for which every path connecting D to border cell of $\mathcal{P}_{\mathcal{J}}$ must contain at least one cell of $\mathcal{P}'$. Then the inclusion $\mathcal{P}' \subset \mathcal{P}^c$ shows that D must be cell of convex polyomino $\mathcal{P}$. Then by using this argument we can say that every cell of $\mathcal{P}^c$ whose edge intersect with $B(\mathcal{P})$ must be contained in $\mathcal{P}'$, which can not be possible by our construction of the $\mathcal{P}'$. Hence $\mathcal{P}'$ is simple polyomino. $\square$

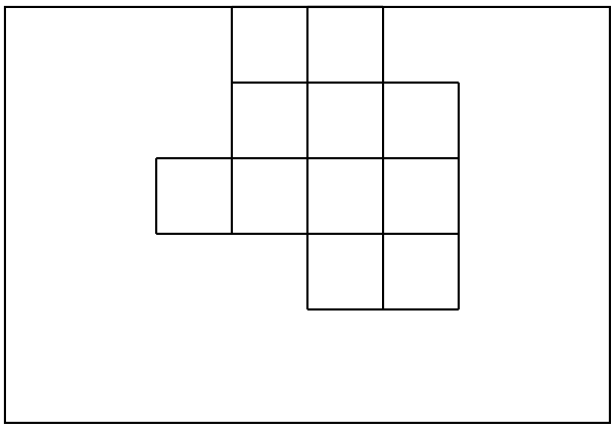

Figure 3.2: When $B(\mathcal{P}) \cap B(\mathcal{P}_{\mathcal{I}}) \neq \emptyset$

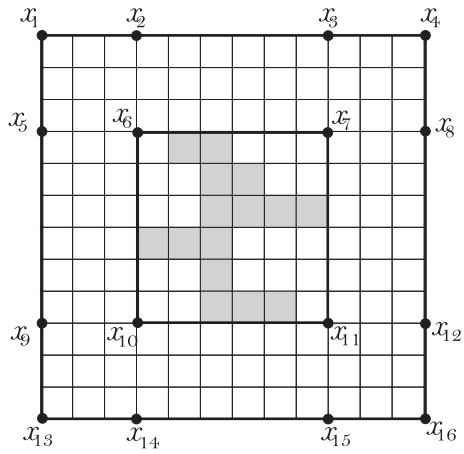

Figure 3.3: A polyomino $\mathcal{P}$

3.2 Toric ideals of finite graphs

We have seen that polyomino ideals for simple polyominoes are prime. In the above section we have discussed a case where polyomino ideal is prime but the polyomino is non-simple. However, these binomial ideals belong to a subclass of binomial ideals arising from Koszul bipartite graphs ([10]). Thus, this provides a view point for a new class of binomial ideals which are prime, the study of nonsimple polyomino ideals is indispensable.

We show that the ideals of Theorem 3.1.1 cannot come from finite simple graphs. The following result shows our views.

Theorem 3.2.1. [8] Let $\mathcal{I} \subset \mathbb{N}^2$ be an interval and $\mathcal{P}$ a simple polyomino which is subpolyomino of $\mathcal{P}_{\mathcal{I}}$. Let $\mathcal{P}^c = \mathcal{P}_{\mathcal{I}} \backslash \mathcal{P}$ and suppose that $\mathcal{P}^c$ is a polyomino. Then its polyomino ideal can not come from a finite simple graph.

Proof. Let $\mathcal{J} \subset \mathbb{N}^2$ be the smallest interval such that $V(\mathcal{P}) \subset \mathcal{J}$. Now we choose $x_1, \ldots, x_{16} \in V(\mathcal{P})$ as shown in Figure 3.3.

Assume that there exist a finite simple graph G with vertex set $V(G)$ and edge set $E(G)$ in such a way that the ideal $I_{\mathcal{P}}$ is equal to toric ideal I_G arising from graph G. Let $K[G] = K[t_i t_j | \{i, j\} \in E(G)]$ be the edge ring of G. Then we have an isomorphism $\phi : K[\mathcal{P}] \to K[G]$ defined as $\phi(x_a) = t_i t_j$ where $\{i, j\}$ is unique edge

of graph G.

The 2-minor $x_2x_7 - x_3x_6$ is an inner minor of $\mathcal{P}^c$, so $\phi(x_2x_7) = \phi(x_3x_6)$. Assume that $\phi(x_2) = t_it_j$, then $\phi(x_7) = t_kt_l$ where i, jk, l are pairwise distinct vertices of G and $\{i, j\}, \{k, l\} \in E(G)$. Then $\phi(x_3x_6) = t_it_jt_kt_l$ which gives us the following possibilities:

(i) $\phi(x_3) = t_it_k$ and $\phi(x_6) = t_jt_l$

(ii) $\phi(x_3) = t_it_l$ and $\phi(x_6) = t_jt_k$

(iii) $\phi(x_3) = t_jt_k$ and $\phi(x_6) = t_it_l$

(iv) $\phi(x_3) = t_jt_l$ and $\phi(x_6) = t_it_k$

We may assume that $\phi(x_3) = t_it_k$ and $\phi(x_6) = t_jt_l$ because the discussion is same for the remaining cases. By using the inclusion $x_1x_6 - x_2x_5 \in \mathcal{P}^c$ and $\phi(x_2) = t_it_j$, $\phi(x_6) = t_jt_l$. So we see that $\phi(x_1) = t_it_p$ and $\phi(x_5) = t_lt_p$ where $\{i, p\}, \{l, p\} \in E(G)$ for some $p \in V(G)\backslash\{i, j, k, l\}$. Note that $p = k$ because otherwise $\phi(x_5) = \phi(x_7) = t_kt_l$, which is not possible. Now from $x_5x_{10} - x_6x_9 \in I_{\mathcal{P}^c}$ and $\phi(x_5) = t_pt_l, \phi(x_6) = t_jt_l$, we obtain $\phi(x_{10}) = t_jt_q$ and $\phi(x_9) = t_pt_q$ for some $q \in V(\mathcal{P}^c)\backslash\{i, p, l, j\}$. Continuing in the same way, from $x_9x_{14} - x_{10}x_{13} \in I_{\mathcal{P}^c}$ and $\phi(x_9) = t_pt_q$ and $\phi(x_{10}) = t_jt_q$, we get $\phi(x_{14}) = t_rt_j$ and $\phi(x_{13}) = t_rt_p$ for some $r \in V(\mathcal{P}^c)\backslash\{i, j, l, p, q\}$. Then, by using $x_{10}x_{15} - x_{11}x_{14} \in I_{\mathcal{P}^c}, \phi(x_{10}) = t_jt_q$ and $\phi(x_{14}) = t_rt_j$, we get $\phi(x_{15}) = t_st_r$ and $\phi(x_{11}) = t_st_q$ for some $s \in V(\mathcal{P}^c)\backslash\{j, p, q, r\}$. Furthermore, by using $x_3x_8 - x_4x_7 \in I_{\mathcal{P}^c}, \phi(x_3) = t_it_k$ and $\phi(x_7) = t_kt_l$, we obtain $\phi(x_4) = t_it_y$ and $\phi(x_8) = t_lt_y$ for some $y \in V(G)\backslash\{i, k, l, j, p\}$. Similarly, from $x_7x_{12} - x_{11}x_8 \in I_{\mathcal{P}^c}, \phi(x_7) = t_kt_l, \phi(x_8) = t_yt_l$ and $\phi(x_{11}) = t_st_q$, it follows that $t_k|t_st_q$. Thus, one has either $k = s$ and $\phi(x_{12}) = t_qt_y$ or $k = q$ and $\phi(x_{12}) = t_st_y$.

Let $k = s$, then $\phi(x_6x_{11} - x_7x10) = (t_jt_l)(t_kt_q) - (t_kt_l)(t_jt_q) = 0$, which guarantees $x_6x_{11} - x_7x_{10} \in I_G$. However, one has $x_6x_{11} - x_7x_{10} \notin I_{\mathcal{P}^c}$, because it is not an inner minor of $\mathcal{P}^c$, and it gives us a contradiction to our assumption $I_G = I_{\mathcal{P}^c}$. Hence, $k = q$ and $(x_{12}) = t_st_y$. But then $x_{11}x_{16} - x_{12}x_{15} \in I_{\mathcal{P}^c} = I_G, \phi(x_{12}x_{15}) = (t_st_y)(t_st_r)$

and $\phi(x_{11}) = t_s t_k$. Thus, one has either $k = r$ or $k = s$, which is not possible; otherwise either $\phi(x_{11}) = t_s t_r = \phi(x_{15})$ or $\phi(x_{11}) = t_s^2$. As a result, we conclude that $I_G \neq I_{\mathcal{P}^c}$ for any finite simple graph G. $\square$

Chapter 4

Gröbner bases of polyomino ideal

4.1 Gröbner bases of polyomino ideal

Let $\mathcal{P}$ be a polyomino and $S = K[x_a : a \in V(\mathcal{P})]$ be a polynomial ring, as before. Here we will use the degree lexicographic order, such that for $a = (i, j), b = (k, l) \in \mathcal{P}$, $x_a < x_b$ if $a < b$. The basics about Gröbner basis are given in Chapter 1.

Theorem 4.1.1. [13] Let $\mathcal{P}$ be a polyomino. Then the set of inner 2-minors of $\mathcal{P}$ form a reduced (quadratic) Gröbner basis w.r.t. $<^1_{\text{lex}}$ if and only if for any two inner intervals $[a, b]$ and $[b, c]$ of $\mathcal{P}$, either $[e, c]$ or $[d, c]$ is an inner interval of $\mathcal{P}$, where d and e are the anti-diagonal corners of $[a, b]$, see Figure 4.1 .

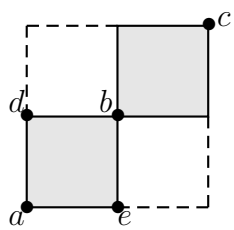

Figure 4.1:

Proof. Let $\mathcal{P}$ be a polyomino and $\mathcal{N}$ be the set of inner 2-minors of $I_{\mathcal{P}}$. We will consider firt term as a leading term for every binomial. The set $\mathcal{N}$ makes a reduced

Gröbner bases of $I_{\mathcal{P}}$ w.r.t. $<^1_{\text{lex}}$ iff all $S-$polynomials of $\mathcal{N}$ reduce to zero. Consider $f_{a,b}, f_{r,s} \in \mathcal{N}$ given by $f_{a,b} = x_b x_a - x_c x_d$ and $f_{r,s} = x_s x_r - x_p x_q$, where d, c are antidiagonal vertices of $[a, b]$, and p, q are antidiagonal vertices of $[r, s]$, as shown in Figure 4.2.

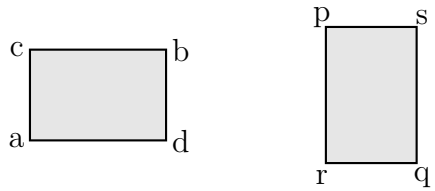

Figure 4.2:

We take the nontrivial case where $\gcd(\text{in}_<(f_{a,b}), \text{in}_<(f_{r,s})) \neq 1$. We can consider one of the followings: (i) $a = r$, (ii) $b = s$, (iii) $a = s$ (or $b = r$).

Take the first one when $a = r$. Let $x_b > x_s$. Then $f_{a,s} = x_s x_a - x_p x_q$ and $S(f_{a,b}, f_{a,s}) = x_b x_p x_q - x_s x_c x_d$. Also let $p \neq c$ and $q \neq d$, otherwise trivially, $S(f_{a,b}, f_{a,s})$ reduces to zero. Then possible situations are shown in Figure 4.3.

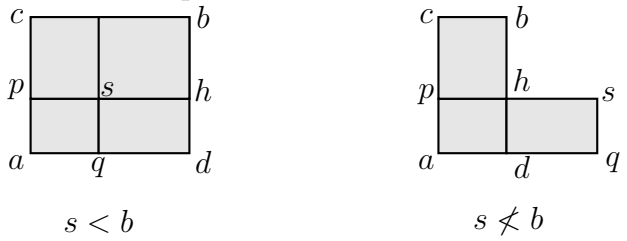

Figure 4.3:

When $s < b$, we have

$$S(f_{a,b}, f_{a,s}) = x_q(x_b x_p - x_c x_h) + x_c(x_h x_q - x_s x_d)$$

When $s \not\leq b$, we have

$$S(f_{a,b}, f_{a,s}) = x_q(x_b x_p - x_c x_h) - x_c(x_s x_d - x_h x_q)$$

It depicts that in all above situations $S(f_{a,b}, f_{a,s})$ reduces to zero w.r.t. the inner 2-minors $f_{p,b}$ and $f_{q,h}$ (or $f_{d,s}$) of $\mathcal{P}$, where $h \in [b, d]$ as shown in Figure 4.3.

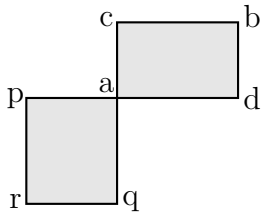

Figure 4.4:

Now we consider the special situation when $a = s$, see Figure 4.4.

Then $S(f_{a,b}, f_{r,a}) = x_b x_p x_q - x_c x_d x_r$ reduces to zero iff either $[p, b]$ or $[q, b]$ is an inner interval, which completes the proof. $\square$

Corollary 4.1.2. Let $\mathcal{I} \subset \mathbb{N}^2$ and $\mathcal{P}$ a convex polyomino which is subpolyomino of $\mathcal{P}_{\mathcal{I}}$. Let $\mathcal{P}^c = \mathcal{P}_{\mathcal{I}} \backslash \mathcal{P}$ be a polyomino. Then the inner 2-minors of $\mathcal{P}^c$ form a reduced Gröbner bases of $I_{\mathcal{P}^c}$ w.r.t. $<_{\text{lex}}$.

4.2 Gröbner bases of balanced polyominoes

Assume $\mathcal{P}$ be a polyomino and c, d are two vertices of $\mathcal{P}$ which are in vertical (horizontal) position, then the interval $[c, d]$ is called *vertical (horizontal) edge interval* of $\mathcal{P}$.

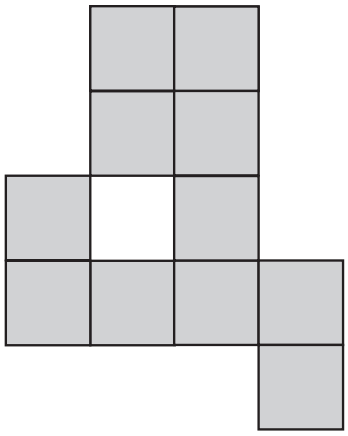

Figure 4.5:

In [13] it is defined that an integer value function α defined on vertices of polyomino is *admissible*, if $\forall$ maximal horizontal intervals or maximal vertical intervals $[a, b]$ one has

$$\sum_{c\in[a,b]} \alpha(c) = 0$$

In Figure 4.6 an admissible labeling of the polyomino displayed Figure 4.5 is shown.

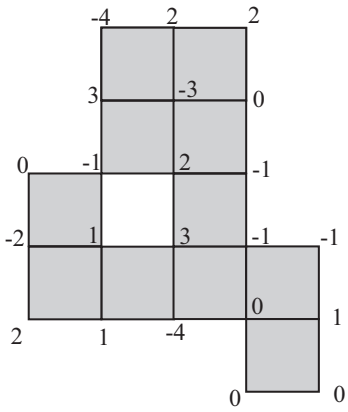

Figure 4.6: An admissible labeling

For an admissible labeling α we define a binomial in $S = K[x_a \colon a \in V(\mathcal{P})]$

$$f_\alpha = \prod_{\substack{a\in V(\mathcal{P})\\ \alpha(a)>0}} x_a^{\alpha(a)} - \prod_{\substack{a\in V(\mathcal{P})\\ \alpha(a)<0}} x_a^{-\alpha(a)}.$$

Let $J_\mathcal{P} = \langle f_\alpha | \alpha \text{ is an admissible labeling of } \mathcal{P}\rangle \subset S$. Clearly, the polyomino ideal $I_\mathcal{P}$ is contained in $J_\mathcal{P}$. A polyomino $\mathcal{P}$ is called *balanced polyomino* if $f_\alpha \in I_\mathcal{P}$ for any admissible labeling α, i.e. $I_\mathcal{P} = J_\mathcal{P}$.

Take the free-abelian group $G = \oplus_{(i,j)\in V(\mathcal{P})}\mathbb{Z}_{e_{ij}}$ with bases elements e_{ij}. To any cell $D = [(i, j), (i+1, j+1)]$ of $\mathcal{P}$ we attach a binomial $g_D = e_{ij}+e_{i+1,j+1}-e_{i+1,j}-e_{i,j+1} \in G$ and let $\wedge \subset G$ be the lattice spanned by these elements.

Lemma 4.2.1. [9] The elements g_D make a K-basis of $\wedge$ and hence $rank_\mathbb{Z}\wedge = |\mathcal{P}|$. Moreover, $\wedge$ is saturated, i.e. $G/\wedge$ is torsion-free .

We denote by $I_\wedge$, the lattice ideal generated by all binomials of the form

$$f_v = \prod_{\substack{c\in V(\mathcal{P})\\ v_c>0}} x_c^{v_c} - \prod_{\substack{c\in V(\mathcal{P})\\ v_c<0}} x_c^{-v_c)}$$

where $v \in \wedge$. In [9] it is proved that, if $\mathcal{P}$ is a balanced polyomino then $I_{\mathcal{P}} = I_{\wedge}$. And for a balanced polyomino $\mathcal{P}$, the $I_{\mathcal{P}}$ is a prime ideal of height $|\mathcal{P}|$.

4.2.1 Primitive binomials

We will identify the primitive binomials in $I_{\mathcal{P}}$ for a balanced polyomino $\mathcal{P}$. In this way we will be able to show that the initial ideal of $I_{\mathcal{P}}$ is squarefree monomial ideal w.r.t. any monomial ideal. The definition of a cycle in $\mathcal{P}$ is given in Chapter **??**. Given a cycle $\mathcal{C}$ of length m we attach a binomial

$$f_{\mathcal{C}} = \prod_{i=1}^{(m-1)/2} x_{a_{2i-1}} - \prod_{i=1}^{(m-1)/2} x_{a_{2i}}$$

Theorem 4.2.2. [9] Let $\mathcal{P}$ be a balanced polyomino.

1. Let $\mathcal{C}$ be a cycle in $\mathcal{P}$. Then $f_{\mathcal{C}} \in I_{\mathcal{P}}$.

2. Let $f \in I_{\mathcal{P}}$ be a primitive binomial. Then their exists a cycle $\mathcal{C}$ in $\mathcal{P}$ such that each maximal interval of $\mathcal{P}$ contains at most two vertices of $\mathcal{C}$ and $f = \pm f_{\mathcal{C}}$.

Corollary 4.2.3. Let $\mathcal{P}$ be a balanced polyomino. Then $I_{\mathcal{P}}$ admits a squarefree initial ideal for any monomial order.

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