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Generalized solutions for inextensible string equations

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Abstract

We study the system of equations of motion for inextensible strings. This system can be recast into a discontinuous system of conservation laws as well as into the total variation wave equation. We prove existence of generalized Young measure solutions with non-negative tension after transforming the problem into a system of conservation laws and approximating it with a regularized system for which we obtain uniform estimates of the energy and the tension. We also discuss sufficient conditions for non-negativity of the tension for strong solutions.

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1. Introduction

An inextensible string is defined (cf. [4]) to be the one for which the stretch is constrained to be equal to 1, whatever system of forces is applied to it. As in [30], some authors refer to it as a *chain* which is a long but very thin material that is inextensible but completely flexible, and hence mathematically described as a rectifiable curve of fixed length. Dynamics of pipes, flagella, chains, or ribbons of rhythmic gymnastics, mechanism of whips, and galactic motion

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are only a few phenomena and applications that can be related to inextensible strings (see [10, 21,18] for more details).

The motion executed by a homogeneous, inextensible string with unit length and density can be modeled by the system

$$\begin{cases} \eta_{tt}(t,s) = \big(\sigma(t,s)\,\eta_s(t,s)\big)_s + g, \ \ s \in [0,1], \\ |\eta_s| = 1, \end{cases} \tag{1.1}$$

where $g \in \mathbb{R}^3$ is the given gravity vector, $\eta \in \mathbb{R}^3$ is the unknown position vector for material point s at time t. The unknown scalar multiplier σ, which is called *tension*, satisfies the equation

$$\sigma_{ss}(t,s) - |\eta_{ss}(t,s)|^2\sigma(t,s) + |\eta_{st}(t,s)|^2 = 0 \tag{1.2}$$

(see Section 2.4 for the derivation of (1.2) from (1.1)). We are given the initial positions and velocities of the string as

$$\eta(0,s) = \alpha(s) \ \text{ and } \ \eta_t(0,s) = \beta(s). \tag{1.3}$$

There are several options for boundary conditions:

a) two fixed ends:

$$\eta(t,0) = \alpha(0) \ \text{ and } \ \eta(t,1) = \alpha(1) \tag{1.4}$$

b) two free ends:

$$\sigma(t,0) = \sigma(t,1) = 0 \tag{1.5}$$

c) the "ring" or periodic conditions (here it is convenient to consider $s \in \mathbb{R}$ instead of $s \in [0,1]$):

$$\eta(t,s) = \eta(t,s+1) \ \text{ and } \ \sigma(t,s) = \sigma(t,s+1) \tag{1.6}$$

d) the "whip" boundary conditions when one end is free and one is fixed:

$$\sigma(t,0) = 0 \ \text{ and } \ \eta(t,1) = 0. \tag{1.7}$$

We make the convention that $s = 0$ corresponds to the free end while the end $s = 1$ is fixed at the origin of the space.

Even though the analysis of the dynamics of inextensible strings subject to different kinds of boundary conditions is a notable problem which goes back to Galileo, Leibniz and Bernoulli (cf. [30,4,26]), and it has been investigated by many authors in various contexts (see e.g. [10,22,21, 33,42]), there are still very few results about general well-posedness. According to [26], V. Yudovich was interested in this problem (possibly because of its relation to the Euler equations, see our Section 2.6), and obtained some unpublished results. One of the available existence results is by Reeken [31,32] who proves well-posedness for an infinite string with gravity when the initial data is near the trivial (downwards vertical) stable stationary solution (close in H^{26}).

Another one is due to Preston [26] who considers (1.1) in the absence of gravity with the whip boundary conditions (1.7). He obtains local existence and uniqueness in weighted Sobolev spaces for which the energy is bounded. He uses the method of lines, approximating with a discrete system of chains. In his paper, he imagines that the graph of the whip extends smoothly through the origin (which corresponds to the fixed end), and hence the tension extends to an even smooth function. This evenness leads to what he calls the compatibility boundary condition given by

$$\sigma_s(t,1) = 0.$$

In the presence of gravity (and still assuming the boundary condition (1.7)) this condition looks like

$$\sigma_s(t,1) = -g\eta_s(t,1); \tag{1.8}$$

it is just a consequence of our formula (2.22) and therefore is not an independent boundary condition. These conditions are related to the delicate issue of non-negativity of the tension. We study this issue in Section 2.4, allowing for all possible boundary conditions and presence of gravity, and find pre-conditions which a priori guarantee that the tension is non-negative for strong solutions. Rather surprisingly, the generalized solutions which we will construct in the subsequent sections of this paper will always have non-negative tension.

Dickey [18] looks into the two-dimensional case, also ignoring the gravity. He defines a new variable as the angle the tangent to the string makes with the positive x-axis, and obtains a transformed system for which he discusses two asymptotic theories, one in which the amplitude of the angle is small and another in which the amplitude is large.

In [27], Preston studies the space of curves parametrized by the unit speed (with one fixed end) as a Hilbert submanifold of the Hilbert space $L^2(0,1;\mathbb{R}^3)$. He proves that the geodesics on his manifold are determined by the inextensible string system (1.1), (1.7) with $g=0$. For technical reasons, he extends the curves through the fixed point by oddness to get curves with two free endpoints. He notes that if periodic boundary conditions were used, the results of his paper would change, for example, he would work on ordinary Sobolev spaces on the circle, rather than weighted Sobolev spaces on the interval.

Thess et al. [37] observed that the motion of inextensible string has deep similarities with the one of an ideal incompressible fluid, which is governed by the Euler equations. The two objects were recently put into a common geometric framework in [9]. We discuss these issues in more detail in Remark 2.6. Accordingly, the studies of the “toy model” (1.1) may shed more light on the nature of turbulence [16].

After certain transformations of (1.1) (see Section 2.2) we obtain the hyperbolic system of conservation laws in (2.1). This kind of systems are mentioned in the book by Dafermos [17, Chapter 7] as examples of balance laws in one space dimension arising in the contexts of planar oscillations of thermoelastic medium and oscillations of flexible, extensible elastic strings. To our knowledge, there is no existence result in this context for conservation laws as well as for the 1-Laplacian wave equation (2.10) which is derived from (1.1) by certain transformations (see Section 2.2). The difficulty of the problem is not surprising since the system of conservation laws (2.7) is not strictly hyperbolic, and its flux is discontinuous at zero.

Scalar hyperbolic conservation laws with a discontinuous flux were recently considered in [14]. Although the authors of that paper notice that their procedures do not work in the case of systems, we managed to find a slightly similar approach in the case of our particular system (2.7). Note that a related but different class of problems concerns scalar conservation laws with a flux that is discontinuous in the spatial and not in the unknown variable [8,25].

In this paper we show global existence of solutions in the sense of Young measures for the equations of motion of the inextensible string without restrictions on the initial data. Our solutions always have non-negative tension. We introduce the approximate problem which gives the opportunity to numerically evaluate the Young measure solutions for the inextensible string. This seems to be the first treatment of well-posedness both for the systems of hyperbolic conservation laws with discontinuous flux and for the total variation wave equation.

We work with the most complex boundary conditions, namely the "whip" conditions (1.7), but the results of the paper remain valid for any of (1.4), (1.5) or (1.6). In some places throughout the paper we emphasize the technical differences of those cases with respect to (1.7). Moreover, the three-dimensional space was chosen due to the physical meaning of the problem, but, mathematically, everything presented in the paper is true in any dimension.

The paper is organized in the following way. In Section 2.1, we introduce the basic notation. In Section 2.2, we make a series of transformations of our problem and obtain a system of hyperbolic conservation laws with discontinuous flux and the total variation wave equation. In Section 2.3, we derive an equivalent system which is more tractable due to the lack of discontinuity. In Section 2.4, we discuss the non-negativity of the tension which is crucial in our considerations. In Section 2.5, we make some preliminary observations related to the energy. In Section 2.6, we show how our problem can be derived from the physical principle of least action, and justify its relation to the motion of an ideal incompressible fluid and to the optimal transport. In Section 3.1, we recall the main concepts of the theory of generalized Young measures. In Section 3.2, we define the generalized solutions to our system of conservation laws with discontinuous flux. In Section 4.1, we introduce an approximate problem and study its global well-posedness. In Section 4.2, we define the energy for the approximate problem, and show dissipativity of that problem. This allows us to derive, in Section 4.3, a crucial uniform L^1 bound for the tension. In Section 5.1, we prove the main result of the paper, which is the existence of generalized Young measure solutions with non-negative tension to the initial-boundary value problem for the equations of motion of the inextensible string, employing the equivalent continuous formulation introduced in Sections 2.3 and 3.2. Then, in Section 5.2, we touch on some examples which illustrate our key finding that even though strong solutions with non-negative tension do not exist for some initial data, generalized solutions with non-negative tension do exist and can be interpreted from the point of view of mechanics.

2. Preliminaries

2.1. Some conventions

Throughout the paper we will denote $\Omega = (0, T) \times (0, 1)$. The scalar product of any two vectors χ, ξ in $\mathbb{R}^3$ is simply denoted by $\chi\xi$, and $|\chi|$ is the Euclidean norm $\sqrt{\chi\chi}$. The notation $Lip_1([0, 1]; \mathbb{R}^3)$ stands for the set of continuous functions $f : [0, 1] \to \mathbb{R}^3$ satisfying

$$|f(s_1) - f(s_2)| \leq |s_1 - s_2|, \ s_1, s_2 \in [0, 1].$$

The symbol S^{n-1} stands for the unit sphere in $\mathbb{R}^n$, $n \in \mathbb{N}$. $M^+(U)$ and $M^1(U)$ are the spaces of positive finite and probability measures, respectively, on a closed set $U \subset \mathbb{R}^n$.

$L^\infty_w(U_1, \mu; \mathcal{M}^1(U))$ is the space of μ-weakly*-measurable maps (cf. [35]) from an open or closed set $U_1 \subset \mathbb{R}^m$ into $M^1(U)$ (the default measure μ on U_1 is the Lebesgue measure). Generic positive constants are denoted by C. Finally, by *regular solutions* in various contexts we mean sufficiently smooth functions so that all derivatives involved in the associated arguments are continuous.

2.2. Changes of variables and formal transformations

1. We make an ansatz that $\sigma \geq 0$ (cf. the discussion in Section 2.4). By putting $\kappa := \sigma\, \eta_s$ we get $\sigma = |\kappa|$ and $\eta_s = \frac{\kappa}{|\kappa|}$. We can then formally rewrite (1.1) as

$$\begin{cases} \eta_{tt} = \kappa_s + g, \\ \eta_s = \dfrac{\kappa}{|\kappa|} \end{cases} \iff \begin{cases} v_t = \kappa_s + g, \\ v_s = \left(\dfrac{\kappa}{|\kappa|}\right)_t \end{cases} \tag{2.1}$$

where for the second system we use $v := \eta_t$. From (1.7) we infer that the boundary conditions for κ take the form

$$\kappa(t,0) = 0 \qquad \text{and} \qquad \kappa_s(t,1) = -g. \tag{2.2}$$

The second condition follows from the fact that the velocity of the fixed end is zero, that is,

$$v(t,1) = 0, \tag{2.3}$$

which is easily obtained from the second condition in (1.7) since $v = \eta_t$. Note that we can find the initial conditions for σ (and thus for κ) using (1.2), (1.3), (1.8) and the first condition in (1.7), see (2.24), but we do not need them at this stage. (In the case when the boundary conditions are different from the whip ones, (1.8) and (1.7) are to be replaced by suitable corresponding conditions.) We also observe that

$$\frac{\kappa}{|\kappa|}(0,s) = \eta_s(0,s) = \alpha_s(s) \tag{2.4}$$

and

$$v(0,s) = \eta_t(0,s) = \beta(s). \tag{2.5}$$

2. If $\alpha(1) = 0$, then by using

$$\eta(t,s) = \alpha(s) + \int_0^t v(r,s)\, dr \quad \text{and} \quad \sigma = |\kappa|, \tag{2.6}$$

we can come back from the "velocity v – contact force κ" formulation (2.1)–(2.5) to the original "position η – tension σ" setting (1.1), (1.3), (1.7).

3. Let $\Upsilon = (v, \kappa) \in \mathbb{R}^6$, and define the map $F : \mathbb{R}^6 \times [0, T] \to \mathbb{R}^6$, $(v, \kappa, t) \mapsto \left(\frac{\kappa}{|\kappa|}, v - gt\right)$. Then (2.1) can be rewritten in the form

$$\Upsilon_s = [F(\Upsilon, t)]_t. \tag{2.7}$$

This is a system of conservation laws with discontinuous flux F, where s plays the role of time and t plays the role of space.

4. Let us now further define

$$\phi(t, s) := \int_0^t \kappa(z, s)\, dz. \tag{2.8}$$

From (2.1) we get

$$\int_0^t v_t\, dt = \int_0^t \kappa_s\, dt + \int_0^t g\, dt$$

which, by (2.5) and (2.8), gives

$$v = \phi_s + g\,t + \beta. \tag{2.9}$$

Together with (2.1) this leads to

$$\phi_{ss}(t, s) + \beta_s = \left(\frac{\phi_t}{|\phi_t|}\right)_t =: \Delta_1\phi(t, s). \tag{2.10}$$

The initial/boundary conditions for ϕ are

$$\phi(t, 0) = 0 \text{ and } \phi_s(t, 1) = -g\,t, \tag{2.11a}$$

$$\phi(0, s) = 0 \text{ and } \phi_t(0, s) = \kappa(0, s). \tag{2.11b}$$

Remark 2.1. The nonlinear hyperbolic equation (2.10) can be referred to as the vectorial 1-Laplacian wave equation because it involves the vectorial 1-Laplacian operator (cf. [36]). Here, once again, s plays the role of time and t plays the role of space. It can also be called the (vectorial) total variation wave equation because its parabolic analogue is the total variation flow [3]. We surmise that the total variation wave equation might be relevant in image processing, since its parabolic and elliptic counterparts play an important role there, we refer to [3,15] for more information; see also [34] and the references therein for the vectorial case (color images).

5. Since $|\eta_s| = 1$, a necessary assumption for existence of regular solutions is

$$|\alpha_s| = 1. \tag{2.12}$$

Differentiating the equation $|\eta_s|^2 = 1$ with respect to time we get $\eta_s \eta_{st} = 0$, yielding the second necessary condition

$$\alpha_s \beta_s = 0. \tag{2.13}$$

2.3. Removing the discontinuity

As a result of the transformation $u = \eta_s \sqrt{\sigma}$, $v = \eta_t$ we can rewrite the second system in (2.1) as

$$\begin{cases} v_t = \big(u\,|u|\big)_s + g, \\ v_s = \left(\dfrac{u}{|u|}\right)_t. \end{cases} \tag{2.14}$$

Defining $\xi := (v, u) \in \mathbb{R}^6$ we can further put this in the form

$$\Phi(\xi)_t = \Psi(\xi)_s + (g, 0) \quad \text{where} \quad \begin{cases} \Phi(\xi) = \left(v, \dfrac{u}{|u|}\right) \\ \Psi(\xi) = \big(u\,|u|, v\big). \end{cases}$$

Let $\mathcal{P} : \mathbb{R}^6 \to \mathbb{R}^6$ be the projection $\xi \mapsto (0, u)$. Then, inspired by the implicit constitutive theory (cf. [14]), we formally define the map $\Gamma : \mathbb{R}^6 \to \mathbb{R}^6$ as

$$\Gamma(\xi) := \left(v, \frac{u}{|u|} + u\right)$$

and, in order to patch the discontinuity of the function Γ in zero, consider its continuous inverse: for $\gamma = (v, w) \in \mathbb{R}^6$,

$$\Gamma^{-1}(\gamma) = (v, \mathcal{M}(w)),$$

where

$$\mathcal{M}(w) := \begin{cases} 0 \quad \text{for} \quad |w| \le 1, \\ w - \frac{w}{|w|} \quad \text{for} \quad |w| \ge 1. \end{cases}$$

Taking the derivative of $\Phi(\xi)$ with respect to time, we find

$$\Gamma(\xi)_t - \mathcal{P}\xi_t = \Psi(\xi)_s + (g, 0),$$

whence

$$\Gamma(\Gamma^{-1}(\gamma))_t - [\mathcal{P}(\Gamma^{-1}(\gamma))]_t = \Psi(\Gamma^{-1}(\gamma))_s + (g, 0).$$

We formally conclude that

$$\gamma_t - [\mathcal{P}(\Gamma^{-1}(\gamma))]_t = \Psi(\Gamma^{-1}(\gamma))_s + (g, 0).$$

Defining the operators

$$
\begin{aligned}
\mathcal{A}(\gamma) &= \gamma - \mathcal{P}(\Gamma^{-1}(\gamma)), \\
\mathcal{B}(\gamma) &= \Psi(\Gamma^{-1}(\gamma))
\end{aligned}
\tag{2.15}
$$

we obtain

$$
\mathcal{A}(\gamma)_t = \mathcal{B}(\gamma)_s + (g, 0). \tag{2.16}
$$

Observation 2.2. The new equation (2.16) is equivalent to the original system (1.1) coupled with the additional restriction

$$
\sigma \geq 0, \tag{2.17}
$$

provided the solutions are regular and some natural compatibility conditions hold. Indeed, it is straightforward to check that for any solution (η, σ) of the system (1.1), (2.17), the corresponding vector function

$$
\gamma = (v, w) = (\eta_t, \eta_s(1 + \sqrt{\sigma}))
$$

satisfies (2.16). Conversely, take any regular solution $\gamma = (v, w)$ to (2.16). We now assume that

$$
|w(0, s)| \geq 1, \ \ w(0, s)v_s(0, s) = 0 \tag{2.18}
$$

for all $s \in [0, 1]$. In Section 3.2 we will realize that this is a necessary and legitimate assumption. At the relative interior of the set $\{(t, s) \in \overline{\Omega} : \ |w(t, s)| \geq 1\}$, letting $\kappa = \frac{w}{|w|}(|w| - 1)^2$, we obtain

$$
\begin{cases}
v_t = \kappa_s + g, \\
v_s = \left(\dfrac{w}{|w|}\right)_t
\end{cases}
\tag{2.19}
$$

Since

$$
1 = \left|\frac{w}{|w|}\right|^2,
$$

differentiating it with respect to time gives

$$
0 = \left(\frac{w}{|w|}\right)_t \frac{w}{|w|} = v_s \frac{w}{|w|}. \tag{2.20}
$$

Assume that there is a point $(t_0, s_0) \in \overline{\Omega}$ such that $|w(t_0, s_0)| < 1$. Without loss of generality, it does not lie on the boundary of Ω. Let K be the connected component of the set $\{(t, s) \in \Omega : |w(t, s)| < 1, s = s_0\}$ containing (t_0, s_0), and let $t_1 = \inf\{t \geq 0 : (t, s_0) \in K\}$. If $t_1 = 0$ then

$$
|w(t_1, s_0)| \geq 1, \ \ w(t_1, s_0)v_s(t_1, s_0) = 0 \tag{2.21}
$$

due to (2.18), and if $t_1 > 0$ then (2.21) follows from (2.20) by continuity. For $(t, s) = (t, s_0) \in K$ we have $\mathcal{A}(\gamma(t, s)) = \gamma(t, s)$, $\mathcal{B}(\gamma(t, s)) = (0, v(t, s))$. Hence, the solution to (2.16) on K can be written explicitly as

$$v(t, s_0) = (t - t_1)g + v(t_1, s_0),\ w(t, s_0) = (t - t_1)v_s(t_1, s_0) + w(t_1, s_0).$$

By the Pythagorean theorem,

$$|w(t, s_0)| \geq |w(t_1, s_0)| \geq 1,$$

arriving at a contradiction. Consequently, $|w| \geq 1$ on $\overline{\Omega}$, and thus (2.19) holds everywhere. By (2.19), there exists a vector function η such that $\eta_s = \frac{w}{|w|}$ and $\eta_t = v$. This function η solves the system (1.1), (2.17) with $\sigma = (|w| - 1)^2$. Note that η is determined up to a constant unless initial or boundary conditions are specified.

By the above analysis, we have killed the discontinuity since $\mathcal{A}$ and $\mathcal{B}$ are both continuous with $\mathcal{A}$ being sublinear and $\mathcal{B}$ having at most quadratic growth. We will therefore proceed in the same way amid the weak formulation of our problem in Section 3.2.

Remark 2.3. Observation 2.2 and considerations from Sections 2.2 and 3.2 imply that the original (η, σ)-setting (1.1) coupled with (2.17), the (v, κ)-setting (2.1) and the γ-setting (2.16) are all equivalent (at least formally) provided proper compatibility assumptions on the initial and boundary data are met.

2.4. The equation for the tension

Differentiating the constraint $|\eta_s|^2 = 1$ with respect to s shows that $\eta_s \eta_{ss} = 0$. Hence, multiplying the first equation in (1.1) by η_s we get

$$\eta_s\, \eta_{tt} = \sigma_s + g\eta_s. \tag{2.22}$$

Now, differentiating $|\eta_s|^2 = 1$ twice with respect to time we obtain

$$\eta_s\, \eta_{stt} + \eta_{st}\, \eta_{st} = 0.$$

Due to (2.22),

$$\eta_{ss}\, \eta_{tt} + \eta_s\, \eta_{stt} = \sigma_{ss} + g\eta_{ss}.$$

Combining these two equations we get

$$\sigma_{ss} - (\eta_{tt} - g)\eta_{ss} + |\eta_{st}|^2 = 0.$$

Expressing $(\eta_{tt} - g)$ by (1.1), we end up with

$$\sigma_{ss} - |\eta_{ss}|^2 \sigma + |\eta_{st}|^2 = 0. \tag{2.23}$$

Proposition 2.4. *Let (η, σ) be a regular solution to (1.1), (1.3) with one of the boundary conditions (1.4)–(1.7). Assume that one of the following assumptions holds:*

(i) *the boundary condition is (1.5) or (1.6);*
(ii) *the boundary condition is (1.7) and $g = 0$;*
(iii) *the boundary condition is (1.4), $|\alpha(0) - \alpha(1)| < 1$ and $g = 0$.*

Then $\sigma \geq 0$ for all times.

Proof. Assume that, for some t, the minimum of $\sigma(t, \cdot)$ is negative. Note that from (2.23) we have

$$\sigma\,|\eta_{ss}|^2 - \sigma_{ss} \geq 0.$$

By the maximum principle [28], either $\sigma(t, \cdot)$ is a negative constant, or the minimum is achieved at $s = 0$ or 1.

The first alternative is impossible for (1.5) and (1.7), and in the remaining cases it implies $|\eta_{ss}(t, \cdot)| \equiv 0$, so the string should be straight, and thus

$$|\eta(t, 0) - \eta(t, 1)| = 1.$$

This obviously contradicts (1.6), whereas (1.4) would yield $|\alpha(0) - \alpha(1)| = 1$.

The second alternative can only hold [28] provided $\sigma_s(t, 0) > 0$ (if the minimum is at 0) or $\sigma_s(t, 1) < 0$ (if the minimum is at 1). This immediately rules out the periodic case, so the negative minimum can only be achieved at fixed ends. But (2.22) implies that at such points $\sigma_s = -g\eta_s$, and we again arrive at a contradiction. □

This proof implies that, for the “whip” boundary condition (1.7), instead of assuming that the gravity is zero, it suffices to know a priori that $g\eta_s(t, 1) \leq 0$, whereas, for two fixed ends (1.4), it suffices to know a priori that $g\eta_s(t, 0) \geq 0$ and $g\eta_s(t, 1) \leq 0$. We believe that there exist much weaker hypotheses which guarantee non-negativity of the tension. Our conjecture is that, for both (1.4) and (1.7), if $\sigma_0(s) := \sigma(0, s) \geq 0$ for all $s \in [0, 1]$, then $\sigma \geq 0$ on $\overline{\Omega}$. Remember that σ_0 is determined by α, β and the boundary conditions. For example, in the “whip” case (1.7) it is the solution of the problem

$$(\sigma_0)_{ss} - |\alpha_{ss}|^2 \sigma_0 + |\beta_s|^2 = 0,\ \ \sigma_0(0) = 0,\ \ (\sigma_0)_s(1) = -g\alpha_s(1). \tag{2.24}$$

However, for non-zero gravity, σ can be negative at the initial moment of time and even for all times. For instance, (1.1) has an unstable stationary solution

$$\eta_u(s) = \alpha_u(s) = (s - 1)\frac{g}{|g|},\ \ \sigma_u(s) = -|g|s, \tag{2.25}$$

which satisfies both (1.4) and (1.7).

Nevertheless, our ansatz $\sigma \geq 0$ is meaningful even for such “unstable” problems as (1.1), (1.3), (1.7) with the initial data

$$\alpha = \alpha_u,\ \beta = 0. \tag{2.26}$$

There exist objects which can be interpreted as generalized solutions to this problem with non-negative tension. We will get back to this example in Section 5.2.

2.5. Conservation of energy

We define the kinetic and potential energies as

$$K(t) = \frac{1}{2}\int_0^1 |\eta_t|^2\,ds \quad \text{and} \quad P(t) = -\int_0^1 g\,\eta\,ds. \tag{2.27}$$

Proposition 2.5. *Let (η, σ) be a regular solution to (1.1), (1.3) with one of the boundary conditions (1.4)–(1.7). Then the total energy $E(t) := K(t) + P(t)$ is conserved.*

Proof. From (1.1) we have

$$\begin{aligned}\frac{d}{dt}(K(t) + P(t)) &= \frac{1}{2}\frac{d}{dt}\int_0^1 |\eta_t|^2\,ds - \int_0^1 g\,\eta_t\,ds = \\ &= \int_0^1 \eta_{tt}\,\eta_t\,ds - \int_0^1 g\,\eta_t\,ds = \int_0^1 (\sigma\eta_s)_s\,\eta_t\,ds \\ &= \sigma(t,1)\,\eta_s(t,1)\eta_t(t,1) - \sigma(t,0)\,\eta_s(t,0)\eta_t(t,0) - \int_0^1 \sigma\,\eta_s\eta_{ts}\,ds.\end{aligned}$$

The third term is identically zero as observed in the end of Section 2.2. The first two terms vanish if 0 and 1 are either free or fixed ends. In the periodic case their difference is still zero. □

In the absence of the gravity, as also mentioned in [18], the energy of the whip is entirely kinetic. In this case from Proposition 2.5 we obtain that

$$\int_0^1 |\eta_t(t,s)|^2\,ds = \int_0^1 |\eta_t(0,s)|^2\,ds = 2E(0),\ t > 0.$$

In the general case, we have

$$\begin{aligned}\frac{1}{2}\int_0^1 |\eta_t(t,s)|^2\,ds &= \frac{1}{2}\int_0^1 |\eta_t(0,s)|^2\,ds + \int_0^1 g(\eta - \alpha)\,ds \\ &= E(0) + \int_0^1 g\eta\,ds.\end{aligned} \tag{2.28}$$

If the initial energy is finite, with the help of Grönwall's lemma, (2.28) implies

$$\int_0^1 |\eta_t(t,s)|^2\,ds \le C,\ \ t\in[0,T] \tag{2.29}$$

(cf. the reasoning in Section 4.2). When at least one end is fixed, the potential energy is a priori bounded because of $|\eta_s|=1$, and thus C in (2.29) does not depend on T.

2.6. *The least action principle*

In this section we show that system (1.1) can be viewed as a manifestation of the celebrated physical principle of least action [6,20]. Although our reasoning here is formal and rather sloppy, this claim can be rigorously justified at least for $g=0$ by the methods of infinite-dimensional Riemannian geometry (see e.g. [27]). The main objective of this section is to advocate the relation of the inextensible string problem to the Euler equations for ideal incompressible fluids and to the optimal transport, and the results of this section are never used in the rest of the paper.

Being guided by the physical principle of least action, we define the *action functional* for the inextensible string as the time integral of the difference between the kinetic and potential energies (cf. [20]):

$$S(\eta)=\int_0^T K(t)-P(t)\,dt=\int_\Omega\left(\frac{1}{2}|\eta_t|^2+g\eta\right)ds\,dt. \tag{2.30}$$

Consider the set of inextensible strings with one fixed end and with fixed initial and final configurations:

$$\begin{aligned}\mathcal{W}:=\{\eta\in C^1(\overline{\Omega};\mathbb{R}^3):&\\ |\eta_s(t,s)|^2=1;\ \eta(t,1)=0;\ \eta(0,s)&=\eta_0(s),\ \eta(T,s)=\eta_T(s)\},\end{aligned} \tag{2.31}$$

and let us look for minimizers of the functional S within the constraint set $\mathcal{W}$. We claim that for each local constrained minimizer η there is a scalar function σ such that the pair (η,σ) is a solution to (1.1), (1.7).

Indeed, take any local constrained minimizer η. Let h be an arbitrary element of the unit sphere in $C^1(\overline{\Omega};\mathbb{R}^3)$, satisfying

$$h(t,1)=0,\ h(0,s)=0,\ h(T,s)=0, \tag{2.32}$$

and let ϵ be a small parameter. Then

$$\int_\Omega\left(\frac{1}{2}|\eta_t+\epsilon h_t|^2+g(\eta+\epsilon h)\right)ds\,dt\ge\int_\Omega\left(\frac{1}{2}|\eta_t|^2+g\eta\right)ds\,dt \tag{2.33}$$

provided $\eta+\epsilon h\in\mathcal{W}$. Dividing by ϵ, we can recast this in the form

$$\int_{\Omega} \left(\eta_t h_t + \frac{1}{2}\epsilon |h_t|^2 + gh \right) ds\, dt \geq 0 \tag{2.34}$$

provided

$$2h_s \eta_s + \epsilon |h_s|^2 = 0. \tag{2.35}$$

Letting $\epsilon \to 0$, we deduce

$$\int_{\Omega} (\eta_t h_t + gh)\, ds\, dt = 0 \tag{2.36}$$

provided

$$h_s \eta_s = 0. \tag{2.37}$$

Observe that we have the equality sign in (2.36) instead of the inequality sign in (2.34) since we can replace h by $-h$ in (2.36) without violating the constraints (2.32), (2.37). Integrating by parts in (2.36), we see that

$$\int_{\Omega} (\eta_{tt} - g) h\, ds\, dt = 0$$

for all h satisfying (2.32), (2.37). Denote

$$Z(t,s) := \int_0^s (\eta_{tt}(t,\xi) - g(t,\xi))\, d\xi.$$

Then

$$\int_{\Omega} Z_s h\, ds\, dt = 0,$$

and integration by parts gives

$$\int_{\Omega} Z h_s\, ds\, dt = 0 \tag{2.38}$$

for all h satisfying (2.32), (2.37). By a Hilbertian duality argument, it is possible to deduce from (2.38) that there exists a measurable scalar function $\sigma(t,s)$ such that $Z = \sigma \eta_s$. Since $Z(t,0) = 0$ by construction, we necessarily have $\sigma(t,0) = 0$, whence (η, σ) solves (1.1), (1.7).

Remark 2.6. In the gravity-free case $g = 0$, the action is purely kinetic, and the least action principle can be recast in a geometric way: (1.1), (1.2), (1.7) are the geodesic equations for the Hilbert manifold of the unit speed curves equipped with the L^2 Riemannian metric [27]. A similar interpretation of the Euler equations of motion of an ideal incompressible fluid, both homogeneous and inhomogeneous, goes back to [5] (see also [7,11,38]). A common framework for the two models was proposed in [9]; they are examples of geodesic equations on infinite-dimensional manifolds of volume preserving immersions endowed with the L^2 metric. The analogy between the two objects was also promoted in [37], where the "vorticity" for the inextensible string is introduced and a blow-up simulation is provided. When the external forces are present, the geodesic formulation for the motion of ideal fluid is replaced by the minimization of the Lagrangian action [11], hence the analogy with (1.1) is preserved. From the perspective of the optimal transport theory [38,39], the solutions of the inextensible string problem (as well as the trajectories of a moving ideal fluid) perform the dynamical transportation of material objects which optimizes some relevant cost functional. Other examples of this nature are discussed in [12]; they include the celebrated Monge–Kantorovich optimal transport with quadratic transportation cost, hydrostatic Boussinesq equations and the Born–Infeld electromagnetism.

3. Setting in the context of Young measures

3.1. Introduction

We will essentially follow [41] for a basic introduction to the generalized Young measures.

Let $m, l, d \in \mathbb{N}$, $p \in [1, +\infty)$, $\Gamma \subset \mathbb{R}^m$ be an open set. We define $\mathcal{F}_p$ as the collection of continuous functions $f : \Gamma \times \mathbb{R}^l \to \mathbb{R}^d$ for which the limit

$$f^\infty(x, z) := \lim_{\substack{x' \to x \\ z' \to z \\ s \to \infty}} \frac{f(x', sz')}{s^p}$$

exists for all $(x, z) \in \overline{\Gamma} \times \mathbb{R}^l$ and is continuous in (x, z). The function f^∞ is called the *L^p-recession function* of f. Note that it is p-homogeneous in z, i.e., $f^\infty(x, rz) = r^p f^\infty(x, z)$ for all $r \geq 0$.

A generalized Young measure on $\mathbb{R}^l$ with parameters in Γ is defined as a triple $(\nu, \lambda, \nu^\infty)$ such that

$$\nu \in L^\infty_w(\Gamma; \mathcal{M}^1(\mathbb{R}^l)),$$

$$\lambda \in \mathcal{M}^+(\overline{\Gamma}),$$

$$\nu^\infty \in L^\infty_w(\overline{\Gamma}, \lambda; \mathcal{M}^1(S^{l-1})).$$

Note that ν is defined Lebesgue-a.e. on Γ, and ν^∞ is defined λ-a.e. on $\overline{\Gamma}$; ν is called the *oscillation measure*, λ is the *concentration measure* and ν^∞ is the *concentration-angle measure*.

Now, we can state the fundamental theorem on generalized Young measures (see [1,19,23,41]):

Theorem 3.1. *Let $\{w_n\} \subset L^p(\Gamma; \mathbb{R}^l)$ be an L^p-bounded sequence of maps. Then there exists a subsequence (not relabeled) and a generalized Young measure $(\nu, \lambda, \nu^\infty)$ such that, for every $f \in \mathcal{F}_p$,*

$$\int_\Gamma f(x, w_n(x))dx \to \int_\Gamma \langle \nu_x, f(x,\xi)\rangle dx + \int_{\overline{\Gamma}} \langle \nu_x^\infty, f^\infty(x,\theta)\rangle \lambda(dx),$$

where

$$\langle \nu_x, f(x,\xi)\rangle = \int_{\mathbb{R}^l} f(x,\xi)\nu_x(d\xi),\ \ \langle \nu_x^\infty, f^\infty(x,\theta)\rangle = \int_{S^{l-1}} f^\infty(x,\theta)\nu^\infty(d\theta).$$

Remark 3.2. In particular, for $f(x,\xi) = |\xi|^p$ we infer that

$$\|w_n\|^p_{L^p(\Gamma)^l} \to \int_\Gamma \langle \nu_x, |\xi|^p\rangle dx + \lambda(\overline{\Gamma}) < +\infty$$

in view of $f^\infty \equiv 1$ on S^{l-1}.

3.2. *Weak setting of the inextensible string problem*

Consider the problem of finding a velocity field v and a contact force κ, which was derived in Section 2.2 from the original problem (1.1), (1.3), (1.7):

$$v_t = \kappa_s + g, \tag{3.1a}$$

$$v_s = \left(\frac{\kappa}{|\kappa|}\right)_t, \tag{3.1b}$$

$$\kappa|_{s=0} = 0, \tag{3.1c}$$

$$\left.\frac{\kappa}{|\kappa|}\right|_{t=0} = \alpha_s, \tag{3.1d}$$

$$v|_{s=1} = 0, \tag{3.1e}$$

$$v|_{t=0} = \beta. \tag{3.1f}$$

Let us define the auxiliary function $h_0 \colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ as

$$h_0(r) = 1 + \sqrt{r}. \tag{3.2}$$

Then we have $h_0^{-1}(r) = (r-1)^2, r \geq 1$, and we can continue h_0^{-1} by zero for $r \leq 1$. We also define $H_0, H_0^* \colon \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\begin{aligned} H_0(\chi) &= \frac{\chi}{|\chi|} h_0^{-1}(|\chi|), \quad H_0(0) = 0, \\ H_0^*(\chi) &= \frac{\chi}{|\chi|}\sqrt{h_0^{-1}(|\chi|)}, \quad H_0^*(0) = 0. \end{aligned} \tag{3.3}$$

Let

$$w = \frac{\kappa}{|\kappa|} + \frac{\kappa}{\sqrt{|\kappa|}} = h_0(|\kappa|)\frac{\kappa}{|\kappa|}.$$

Then $\kappa = H_0(w)$ and

$$\frac{\kappa}{|\kappa|} = \frac{\kappa}{|\kappa|} + \frac{\kappa}{\sqrt{|\kappa|}} - \frac{w}{|w|}\sqrt{|\kappa|} = w - \frac{w}{|w|}\sqrt{h_0^{-1}(|w|)} = w - H_0^*(w),$$

so we can rewrite (3.1a) and (3.1b) as

$$v_t = (H_0(w))_s + g, \tag{3.4a}$$

$$v_s = \left(w - H_0^*(w)\right)_t. \tag{3.4b}$$

In Section 2.3 we showed that this system was equivalent to (1.1), (2.17). Observe that, in the current setting, (2.18) is a consequence of the compatibility conditions (2.12) and (2.13). Indeed,

$$|w(0,s)| \geq |w(0,s) - H_0^*(w(0,s))| = |\alpha_s(s)| = 1,$$
$$w(0,s)v_s(0,s) = |w(0,s)|\alpha_s(s)\beta_s(s) = 0.$$

Define the space $\tilde{C}^\infty(\overline{\Omega})$ of test functions to be the set of pairs $\varphi = (\phi, \psi)$, $\phi, \psi \in C^\infty(\overline{\Omega}; \mathbb{R}^3)$ such that

$$\begin{aligned} &\phi|_{s=1} = 0, \quad \phi_s|_{s=0} = 0, \quad \phi|_{t=T} = 0, \\ &\psi|_{s=0} = 0, \quad \psi_s|_{s=1} = 0, \quad \psi|_{t=T} = 0. \end{aligned} \tag{3.5}$$

Take any $\varphi = (\phi, \psi) \in \tilde{C}^\infty(\overline{\Omega})$. Multiplying (3.1a) (or (3.4a)) by ϕ and integrating in space and time gives

$$\int_\Omega v\phi_t\,ds\,dt = \int_\Omega H_0(w)\phi_s\,ds\,dt - \int_0^1 \beta\phi|_{t=0}\,ds - \int_\Omega g\phi\,ds\,dt. \tag{3.6}$$

Doing the same with (3.1b) (or (3.4b)) and ψ gives

$$\int_\Omega [w - H_0^*(w)]\psi_t\,ds\,dt = \int_\Omega v\psi_s\,ds\,dt + \int_0^1 \alpha\psi_s|_{t=0}\,ds. \tag{3.7}$$

Observe that we have taken into account (3.1c)–(3.1f), and the setting (3.6)–(3.7) already incorporates the initial and boundary conditions. We also used the assumption

$$\alpha(1) = 0. \tag{3.8}$$

Denote $\gamma = (v, w) \in \mathbb{R}^6$, and define functions $\mathcal{A}, \mathcal{B}\colon \Omega \times \mathbb{R}^6 \to \mathbb{R}^6$ (by recalling (2.15)) as follows:

$$\mathcal{A}(t,s,\gamma) = \mathcal{A}(t,s,v,w) = (v, w - H_0^*(w)), \tag{3.9}$$

$$\mathcal{B}(t,s,\gamma) = \mathcal{B}(t,s,v,w) = (H_0(w), v). \tag{3.10}$$

Since $\mathcal{A}$ does not depend on t, s, we will often abuse the notation and simply write $\mathcal{A}(\gamma)$ instead of $\mathcal{A}(t,s,\gamma)$, similarly with $\mathcal{B}$. We also introduce the operator

$$\begin{aligned}\Xi_0(\alpha,\beta,\varphi) &= \Xi_0(\alpha,\beta,\phi,\psi)\\ &= -\int_0^1 \beta\phi|_{t=0}\,ds + \int_0^1 \alpha\psi_s|_{t=0}\,ds - \int_\Omega g\phi\,ds\,dt.\end{aligned} \tag{3.11}$$

Then (3.6) and (3.7) can be merged to get

$$\int_\Omega \mathcal{A}(\gamma)\varphi_t\,ds\,dt = \int_\Omega \mathcal{B}(\gamma)\varphi_s\,ds\,dt + \Xi_0(\alpha,\beta,\varphi). \tag{3.12}$$

Observe that $\mathcal{A}$ and $\mathcal{B}$ are in the class $\mathcal{F}_2$ (with $\Gamma = \Omega$). Moreover, since $\mathcal{A}$ is sublinear, $\mathcal{A}^\infty \equiv 0$, whereas it can be checked that $\mathcal{B}^\infty(v,w) = (w|w|, 0)$.

These considerations and analogy with [13,19,35,41] suggest:

Definition 3.3. A triple $(\nu, \lambda, \nu^\infty)$ with

$$\nu \in L_w^\infty(\Omega; \mathcal{M}^1(\mathbb{R}^6)), \tag{3.13}$$

$$\lambda \in \mathcal{M}^+(\overline{\Omega}), \tag{3.14}$$

$$\nu^\infty \in L_w^\infty(\overline{\Omega}, \lambda; \mathcal{M}^1(S^5)), \tag{3.15}$$

is an admissible Young measure solution to (3.1) provided the energy-tension bound

$$\int_\Omega \langle \nu_{t,s}, |\xi|^2\rangle\,ds\,dt + \lambda(\overline{\Omega}) \le \Theta \tag{3.16}$$

holds, where Θ is a certain constant depending only on T, g, and the L^2-norms of α and β, and

$$\begin{aligned}\int_\Omega \langle \nu_{t,s}, \mathcal{A}(\xi)\rangle\varphi_t(t,s)\,ds\,dt &= \int_\Omega \langle \nu_{t,s}, \mathcal{B}(\xi)\rangle\varphi_s(t,s)\,ds\,dt\\ &\quad + \int_{\overline{\Omega}} \langle \nu_{t,s}^\infty, \mathcal{B}^\infty(\theta)\rangle\varphi_s(t,s)\lambda(dt,ds) + \Xi_0(\alpha,\beta,\varphi)\end{aligned} \tag{3.17}$$

for every $\varphi \in \tilde{C}^\infty(\overline{\Omega})$.

Remark 3.4. If an admissible Young measure solution $(\nu, \lambda, \nu^\infty)$ satisfies $\nu_{t,s} = \delta_{\gamma(t,s)}$ a.e. in Ω, where $\gamma : \Omega \to \mathbb{R}^6$ is a measurable function and δ is the Dirac delta, and $\lambda = 0$, then γ belongs to $L^2(\Omega; \mathbb{R}^6)$ and is a weak solution in the sense of (3.12). Assume now that γ is a regular function on $\overline{\Omega}$ and the compatibility conditions (2.12), (2.13) and (3.8) hold. Then (3.12) yields (2.16), and, as in Section 2.3, the vector function γ generates a pair (η, σ) satisfying (1.1), (2.17). Since η is determined up to a constant, we can choose it to satisfy $\eta(0, 1) = 0$. Then, similarly to our previous considerations, we can check that the initial and boundary conditions (1.3), (1.7) are met.

Remark 3.5. The arguments in Sections 4 and 5.1 provide a rigorous expression for Θ.

Remark 3.6. An important open problem is the one of uniqueness of regular solutions to (1.1). The upward whip anomaly (see Sections 2.4 and 5.2) hints that it should be more rational to study the issue of uniqueness for (1.1) coupled with (2.17) (equipped with suitable initial and boundary conditions, either in a strong form, e.g., (1.3), (1.7), or in a weak form, e.g., (3.12)). A positive answer to this question is the cornerstone for such possible developments in the studies of the inextensible string equations as existence of dissipative solutions [24,40] and their relation with the Young measure ones, or discovery of additional admissibility constraints in the definition of Young measure solutions which would secure weak-strong uniqueness [13] for (2.16).

4. Well-posedness and uniform bounds for the approximate problem

4.1. Global regularity

Let $\epsilon \in (0, 1]$ be a constant and consider the auxiliary problem

$$v_t = \epsilon v_{ss} + \kappa_s + g, \tag{4.1a}$$

$$v_s = \left(\epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\right)_t - \epsilon\left(\epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\right)_{ss}, \tag{4.1b}$$

$$\kappa|_{s=0} = 0, \tag{4.1c}$$

$$\left(\epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\right)\Bigg|_{t=0} = \alpha_s, \tag{4.1d}$$

$$v|_{s=1} = 0, \tag{4.1e}$$

$$v|_{t=0} = \beta, \tag{4.1f}$$

$$\left(\epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\right)_s\Bigg|_{s=1} = 0, \tag{4.1g}$$

$$v_s|_{s=0} = 0. \tag{4.1h}$$

Remark 4.1. The last two boundary conditions were added for purely technical reasons, since the order of the new system (4.1) is higher than that of the original system. These two restrictions will completely disappear after we will have passed to the limit as $\epsilon \to 0$.

Denote $\tau = \epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon+|\kappa|^2}}$. Then, $\kappa = G(\tau)$, where G is a function with positive-semidefinite Jacobian matrix, and $G(0) = 0$. Moreover, observe that the eigenvalues of $\nabla G^{-1}(\kappa)$ are $\epsilon + \frac{\epsilon}{(\epsilon+|\kappa|^2)^{3/2}}$ and $\epsilon + \frac{1}{(\epsilon+|\kappa|^2)^{1/2}}$ (the latter is a double eigenvalue). Thus, the eigenvalues of $\nabla G(\tau)$ are

$$\begin{aligned}\frac{1}{\epsilon+\epsilon^{-1/2}} \le \Lambda_1(\tau) &:= \frac{1}{\epsilon+(\epsilon+|G(\tau)|^2)^{-1/2}} \\ \le \Lambda_2(\tau) &:= \frac{\epsilon^{-1}}{1+(\epsilon+|G(\tau)|^2)^{-3/2}} \le \epsilon^{-1}.\end{aligned} \tag{4.2}$$

In particular, G is globally Lipschitz. Observe also that

$$|\kappa| \ge 1 \quad \Rightarrow \quad |\tau| \ge \epsilon + (1+\epsilon)^{-1/2} > 1,$$

and, consequently,

$$|\tau| \le 1 \quad \Rightarrow \quad |G(\tau)| < 1. \tag{4.3}$$

We can rewrite the problem (4.1) as

$$v_t = \epsilon v_{ss} + (G(\tau))_s + g, \tag{4.4a}$$

$$\tau_t = v_s + \epsilon\tau_{ss}, \tag{4.4b}$$

$$\tau|_{s=0} = 0, \quad \tau_s|_{s=1} = 0, \tag{4.4c}$$

$$v|_{s=1} = 0, \quad v_s|_{s=0} = 0, \tag{4.4d}$$

$$\tau|_{t=0} = \alpha_s, \quad v|_{t=0} = \beta. \tag{4.4e}$$

Theorem 4.2. *Let $\alpha, \beta \in C^3([0,1];\mathbb{R}^3)$, $\alpha_s(0) = 0$, $\alpha_{ss}(1) = 0$, $\beta_s(0) = 0$, $\beta(1) = 0$. Then there exists a unique solution (v, τ) to (4.4) in the class $C^\infty((0,T]\times[0,1];\mathbb{R}^6) \times C(\overline{\Omega};\mathbb{R}^6)$.*

Proof. *(Sketch)* The well-posedness of the semilinear problem (4.4) fits into the classical theory of Amann. Indeed, by [2, Theorem 14.6, Corollary 14.7], a smooth solution exists locally in time. By [2, Theorem 15.5], the solution can be continued in time as long as it remains bounded in L^∞. But the term $(G(\tau))_s$ can be rewritten as $\nabla G(\tau)\tau_s = \tilde{G}(t,x)\tau_s$, where $\tilde{G}$ is a bounded matrix-valued function, hence [29, Theorem 2] provides the required L^∞ bound. □

4.2. Uniform energy estimates

Hereafter in Section 4 we assume that

$$|\alpha_s(s)| \le 1 \quad \text{for} \quad 0 \le s \le 1, \tag{4.5}$$

that

$$\alpha|_{s=1} = 0,$$

and that there is a constant C_* such that

$$\int_0^1 |\alpha|^2(s)\,ds + \int_0^1 |\beta|^2(s)\,ds \le C_*. \tag{4.6}$$

Multiplying (4.4a) by v and integrating with respect to s gives

$$\begin{aligned}\int_0^1 v_t v\,ds &= \epsilon \int_0^1 v_{ss} v\,ds + \int_0^1 (G(\tau))_s v\,ds + \int_0^1 g v\,ds \\ &= -\epsilon \int_0^1 v_s v_s\,ds - \int_0^1 G(\tau) v_s\,ds + \int_0^1 g v\,ds \\ &= -\epsilon \int_0^1 v_s v_s\,ds + \epsilon \int_0^1 G(\tau)\tau_{ss}\,ds - \int_0^1 G(\tau)\tau_t\,ds + \int_0^1 g v\,ds.\end{aligned}$$

Hence,

$$-\epsilon \int_0^1 v_s v_s\,ds = \int_0^1 v_t v\,ds - \int_0^1 g v\,ds + \epsilon \int_0^1 \nabla G(\tau)\tau_s\tau_s\,ds + \int_0^1 G(\tau)\tau_t\,ds. \tag{4.7}$$

Considering the last term,

$$\begin{aligned}\int_0^1 G(\tau)\tau_t\,ds &= \int_0^1 \kappa \left(\epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\right)_t ds \\ &= \epsilon \int_0^1 \kappa\kappa_t\,ds + \int_0^1 \kappa \left(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\right)_t ds \\ &= \epsilon \int_0^1 \kappa\kappa_t d\,s + \int_0^1 \kappa \frac{\kappa_t}{\sqrt{\epsilon + |\kappa|^2}} ds - \int_0^1 \frac{|\kappa|^2\kappa}{(\sqrt{\epsilon + |\kappa|^2})^3}\kappa_t\,ds \\ &= \epsilon \int_0^1 \kappa\kappa_t ds + \epsilon \int_0^1 \frac{\kappa\kappa_t}{(\sqrt{\epsilon + |\kappa|^2})^3}\,ds \\ &= \epsilon \frac{d}{dt} \int_0^1 \Big(\frac{|\kappa|^2}{2} - \frac{1}{\sqrt{\epsilon + |\kappa|^2}}\Big)\,ds.\end{aligned}$$

Let

$$\eta(t,s) = \alpha(s) + \int_0^t v(r,s)\,dr \tag{4.8}$$

and define the energy as

$$E_\epsilon(t) = \frac{1}{2}\int_0^1 |v|^2\,ds - \int_0^1 g\eta\,ds + \frac{\epsilon}{2}\int_0^1 |\kappa|^2\,ds$$
$$+ \sqrt{\epsilon} - \epsilon\int_0^1 \frac{1}{\sqrt{\epsilon + |\kappa|^2}}\,ds + \epsilon\int_0^t\int_0^1 \nabla G(\tau)\tau_s\tau_s\,ds\,dt.$$

Then (4.7) yields

$$(E_\epsilon)_t = -\epsilon\int_0^1 v_s v_s\,ds \le 0.$$

The initial energy

$$E_\epsilon(0) = \frac{1}{2}\int_0^1 |\beta|^2\,ds - \int_0^1 g\alpha\,ds + \frac{\epsilon}{2}\int_0^1 |G(\alpha_s)|^2\,ds$$
$$+ \sqrt{\epsilon} - \epsilon\int_0^1 \frac{1}{\sqrt{\epsilon + |G(\alpha_s)|^2}}\,ds$$

is bounded due to (4.3), (4.5), (4.6). Therefore,

$$\frac{1}{2}\int_0^1 |v|^2\,ds + \frac{\epsilon}{2}\int_0^1 |\kappa|^2\,ds + \epsilon\int_0^t\int_0^1 \nabla G(\tau)\tau_s\tau_s\,ds\,dt \le C + \int_0^1 g\eta\,ds \le C. \tag{4.9}$$

Note that the second inequality follows from the first one and the Grönwall's lemma since

$$\frac{d}{dt}\int_0^1 g\eta\,ds = \int_0^1 gv\,ds \le \frac{1}{2}\int_0^1 |v|^2\,ds + \frac{1}{2}\int_0^1 |g|^2\,ds \le \int_0^1 g\eta\,ds + C.$$

Finally, using (4.2) we deduce that

$$\frac{1}{1+\epsilon^{-3/2}}\int_0^T\int_0^1 |\tau_s|^2\,ds\,dt \le \epsilon\int_0^T\int_0^1 \Lambda_1(\tau)|\tau_s|^2\,ds\,dt$$
$$\le \epsilon\int_0^T\int_0^1 \nabla G(\tau)\tau_s\tau_s\,ds\,dt \le C. \tag{4.10}$$

4.3. Estimate for the tension

The estimate obtained in this section, together with the one for kinetic energy, is crucial for the rest of the analysis. We let

$$\zeta(t,s)=\int_1^s \tau(t,w)\,dw. \tag{4.11}$$

From (4.4b) we find

$$\tau(t,s)=\eta_s(t,s)+\epsilon\int_0^t \tau_{ss}(r,s)\,dr.$$

Consequently,

$$\zeta(t,s)=\eta(t,s)+\epsilon\int_0^t \tau_s(r,s)\,dr. \tag{4.12}$$

From (4.8) we get

$$(|\eta|^2)_{tt}=2\eta_{tt}\eta+2\eta_t\eta_t=2v_t\eta+2|v|^2, \tag{4.13}$$

and from (4.4a) we obtain

$$\int_0^1 v_t\zeta\,ds=\epsilon\int_0^1 v_{ss}\zeta\,ds+\int_0^1 (G(\tau))_s\zeta\,ds+\int_0^1 g\zeta\,ds$$
$$=-\epsilon\int_0^1 v_s\tau\,ds-\int_0^1 G(\tau)\tau\,ds+\int_0^1 g\zeta\,ds. \tag{4.14}$$

Combining (4.11)–(4.14), we infer

$$
\begin{aligned}
\int_0^1 G(\tau)\tau\, ds &= -\epsilon \int_0^1 v_t \left[\int_0^t \tau_s(r,s)\, dr\right] ds - \int_0^1 \left(\frac{|\eta|^2}{2}\right)_{tt} ds + \int_0^1 |v|^2\, ds \\
&\quad - \epsilon \int_0^1 v_s \tau\, ds + \int_0^1 g\eta\, ds + \epsilon \int_0^1 g \left[\int_0^t \tau_s(r,s)\, dr\right] ds \\
&=: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t).
\end{aligned} \tag{4.15}
$$

The time integral of the first integral is

$$
\begin{aligned}
\int_0^T I_1(t)\, dt &= -\epsilon \int_0^T \int_0^1 v_t \left[\int_0^t \tau_s(r,s)\, dr\right] ds\, dt \\
&= -\epsilon \int_0^1 v\Big|_{t=T} \left[\int_0^T \tau_s(r,s)\, dr\right] ds + \epsilon \int_0^T \int_0^1 v\tau_s\, ds\, dt \\
&\le \frac{1}{2}\int_0^1 |v|^2\Big|_{t=T} ds + \frac{\epsilon^2}{2}\int_0^1 \left[\int_0^T \tau_s(r,s)\, dr\right]^2 ds \\
&\quad + \frac{1}{2}\int_0^T \int_0^1 |v|^2\, ds\, dt + \frac{\epsilon^2}{2}\int_0^T \int_0^1 |\tau_s|^2\, ds\, dt.
\end{aligned} \tag{4.16}
$$

The first and third terms are bounded by the energy estimate (4.9), and the second and the fourth ones are bounded by $C\epsilon^2(1+\epsilon^{-3/2})$ due to (4.10).

For the second integral in (4.15) we have

$$
\begin{aligned}
\int_0^T I_2(t)\, dt &= -\int_0^T \int_0^1 \left(\frac{|\eta|^2}{2}\right)_{tt} ds\, dt \\
&= -\int_0^1 \left(\frac{|\eta|^2}{2}\right)_t \Big|_{t=T} ds + \int_0^1 \left(\frac{|\eta|^2}{2}\right)_t \Big|_{t=0} ds \\
&= -\int_0^1 \eta\eta_t\Big|_{t=T} ds + \int_0^1 \alpha\beta\, ds \\
&\le \frac{1}{2}\int_0^1 |\eta_t|^2\Big|_{t=T} ds + \frac{1}{2}\int_0^1 |\eta|^2\Big|_{t=T} ds + \int_0^1 \alpha\beta\, ds.
\end{aligned}
$$

Here, the first integral is bounded by (4.9); the second integral is bounded since the linear operator $v \mapsto \eta$, i.e., $v(t) \mapsto \alpha + \int_0^t v(r)\,dr$, is bounded in the Banach space $L^\infty(0, T; L^2(0, 1; \mathbb{R}^3))$; the third integral is bounded due to (4.6).

Continuing from (4.15), I_3 and I_5 are bounded by the energy bound (4.9), and

$$\int_0^T I_4(t)\,dt = \epsilon \int_0^T \int_0^1 v\tau_s\,ds\,dt \le C + C\epsilon^2(1 + \epsilon^{-3/2})$$

as in (4.16). Finally,

$$\begin{aligned}
\int_0^T I_6(t)\,dt &= \epsilon \int_0^T \int_0^1 g \left[\int_0^t \tau_s(r, s)\,dr \right] ds\,dt \\
&= -\epsilon \int_0^T \int_0^1 (t - T) g \tau_s\,ds\,dt \\
&\le \frac{1}{2} \int_0^T \int_0^1 |(t - T)g|^2\,ds\,dt + \frac{\epsilon^2}{2} \int_0^T \int_0^1 |\tau_s|^2\,ds\,dt \\
&\le C + C\epsilon^2(1 + \epsilon^{-3/2}).
\end{aligned} \tag{4.17}$$

Therefore, from (4.15) we conclude that

$$\int_0^T \int_0^1 G(\tau)\tau\,ds \le C,$$

whence

$$\int_0^T \int_0^1 \kappa \left(\epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}} \right) ds\,dt \le C.$$

Thus,

$$\begin{aligned}
\int_\Omega |\kappa(t, s)|\,ds\,dt &\le C + \int_{\Omega,\ |\kappa| \ge 1} |\kappa|\,ds\,dt \\
&\le C + \int_{\Omega,\ |\kappa| \ge 1} (\epsilon + (1 + \epsilon)^{-1/2})|\kappa|\,ds\,dt
\end{aligned}$$

$$\leq C + \int\limits_{\Omega,\ |\kappa|\geq 1} \left(\epsilon|\kappa| + \frac{|\kappa|}{\sqrt{\epsilon + |\kappa|^2}} \right) |\kappa|\, ds\, dt$$

$$\leq C + \int\limits_{\Omega} \kappa \left(\epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}} \right) ds\, dt \leq C. \tag{4.18}$$

5. Existence of the Young measure solution

5.1. Main theorem

Theorem 5.1. *Given a pair $\alpha \in Lip_1([0,1]; \mathbb{R}^3)$, $\beta \in L^2(0,1; \mathbb{R}^3)$ with $\alpha(1) = 0$, there exists an admissible Young measure solution to (3.1).*

Proof. Take any sequence $\varepsilon_n \to 0$. The data (α, β) can be approximated in $L^2(0,1; \mathbb{R}^6)$ by a sequence of C^3-functions (α_n, β_n) such that $|(\alpha_n)_s(s)| \leq 1$, $(\alpha_n)_s(0) = 0$, $(\alpha_n)_{ss}(1) = 0$, $\alpha_n(1) = 0$, $(\beta_n)_s(0) = 0$, $\beta_n(1) = 0$. By Theorem 4.2 there exist smooth solutions (v_n, τ_n) to (4.4) with $\epsilon = \varepsilon_n$, $\alpha = \alpha_n$, $\beta = \beta_n$. Then (v_n, κ_n) where $\kappa_n = G(\tau_n)$ is a smooth solution to (4.1) with $\epsilon = \varepsilon_n$, $\alpha = \alpha_n$, and $\beta = \beta_n$. The uniform energy and tension bounds imply

$$\|v_n\|_{L^\infty(0,T;L^2(0,1))} \leq C, \tag{5.1}$$

$$\|\kappa_n\|_{L^1(\Omega)} \leq C. \tag{5.2}$$

Let

$$w_n = \frac{\kappa_n}{\sqrt{\varepsilon_n + |\kappa_n|^2}} + \frac{\kappa_n}{\sqrt{|\kappa_n|}}. \tag{5.3}$$

Then

$$\|w_n\|_{L^2(\Omega)} \leq C. \tag{5.4}$$

Consider the function $h_{\varepsilon_n} : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ defined as

$$h_{\varepsilon_n}(r) = \frac{r}{\sqrt{\varepsilon_n + r^2}} + \sqrt{r},$$

which becomes h_0 in (3.2) when $\epsilon = 0$. We can easily check that this function is strictly increasing. Thus, there exists the inverse function $h_{\varepsilon_n}^{-1} : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ which is continuous. Observe that $h_{\varepsilon_n}^{-1}(0) = 0$. Let us also introduce the functions $H_{\varepsilon_n}, H_{\varepsilon_n}^* : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$H_{\varepsilon_n}(\chi) = \frac{\chi}{|\chi|} h_{\varepsilon_n}^{-1}(|\chi|),\ H_{\varepsilon_n}^*(\chi) = \frac{\chi}{|\chi|}\sqrt{h_{\varepsilon_n}^{-1}(|\chi|)},\ H_{\varepsilon_n}(0) = H_{\varepsilon_n}^*(0) = 0,$$

which, similarly, become (3.3) when $\epsilon = 0$. Observe that these functions are continuous at zero (in fact everywhere). From (5.3) we find that

$$\kappa_n = H_{\varepsilon_n}(w_n) \quad \text{and} \quad \frac{\kappa_n}{\sqrt{\varepsilon_n + |\kappa_n|^2}} = w_n - H^*_{\varepsilon_n}(w_n).$$

Now, (4.1a) and (4.1b) imply

$$(v_n)_t = \varepsilon_n (v_n)_{ss} + (H_{\varepsilon_n}(w_n))_s + g, \tag{5.5}$$

and

$$(v_n)_s = (\varepsilon_n H_{\varepsilon_n}(w_n) + w_n - H^*_{\varepsilon_n}(w_n))_t - \varepsilon_n(\varepsilon_n H_{\varepsilon_n}(w_n) + w_n - H^*_{\varepsilon_n}(w_n))_{ss}. \tag{5.6}$$

We need the following result to proceed.

Lemma 5.2. *We have*

$$H_{\varepsilon_n}(\chi) \to H_0(\chi), \quad H^*_{\varepsilon_n}(\chi) \to H^*_0(\chi)$$

uniformly on $\mathbb{R}^3$.

Proof. Suppose there exists sequences ε_{n_k} and χ_k such that

$$|H_{\varepsilon_{n_k}}(\chi_k) - H_0(\chi_k)| \geq \delta$$

for some $\delta > 0$. In the sequel we write ε_k instead of ε_{n_k}. Due to the above inequality, we get

$$|h^{-1}_{\varepsilon_k}(|\chi_k|) - h^{-1}_0(|\chi_k|)| \geq \delta. \tag{5.7}$$

Without loss of generality, there exists $\overline{\chi} = \lim_{k\to\infty} |\chi_k|$, which can be equal to $+\infty$. Assume first that $\overline{\chi} \leq 1$. Then $h^{-1}_0(\overline{\chi}) = 0$, and, since $h^{-1}_0(|\chi_k|)$ is non-negative, we must have $d_k := h^{-1}_{\varepsilon_k}(|\chi_k|) \geq \delta$ for k large enough. Therefore,

$$\begin{aligned} |\chi_k| = h_{\varepsilon_k}(d_k) &= \frac{d_k}{\sqrt{\varepsilon_k + d_k^2}} + \sqrt{d_k} \\ &\geq \frac{\delta}{\sqrt{\varepsilon_k + \delta^2}} + \sqrt{\delta} \to 1 + \delta \end{aligned}$$

which contradicts the assumption $\overline{\chi} \leq 1$. Now, consider the case $\overline{\chi} > 1$. Then without loss of generality $|\chi_k| > 1$ for all k. Denote $r_k = h^{-1}_0(|\chi_k|)$. Then, there exist numbers k_l for $l = 1, 2, \ldots,$ such that either $r_{k_l} \geq d_{k_l}$ or $r_{k_l} \leq d_{k_l}$ for all l. To simplify the notation, we write r_k and d_k instead of r_{k_l} and d_{k_l}. Due to (5.7) we either have $r_k \geq d_k + \delta$ or $d_k \geq r_k + \delta$. In the first case, we have

$$\begin{aligned} \frac{d_k}{\sqrt{\varepsilon_k + d_k^2}} + \sqrt{d_k} = 1 + \sqrt{r_k} &\geq 1 + \sqrt{d_k + \delta} \\ &\geq \frac{d_k}{\sqrt{\varepsilon_k + d_k^2}} + \sqrt{d_k + \delta} \end{aligned}$$

$$> \frac{d_k}{\sqrt{\varepsilon_k + d_k^2}} + \sqrt{d_k}$$

which gives a contradiction. In the second case we have

$$\begin{aligned} 1 + \sqrt{r_k} &= \frac{d_k}{\sqrt{\varepsilon_k + d_k^2}} + \sqrt{d_k} \\ &\geq \frac{r_k + \delta}{\sqrt{\varepsilon_k + (r_k + \delta)^2}} + \sqrt{r_k + \delta}. \end{aligned}$$

However, the last inequality cannot hold for all k since by Lagrange's mean value theorem we have

$$\sqrt{r_k + \delta} - \sqrt{r_k} = \frac{1}{2\sqrt{r_k + c_0}}\delta \geq \frac{\delta}{2\sqrt{r_k + \delta}}$$

for some $c_0 \in (0, \delta)$, and

$$\begin{aligned} 1 - \frac{r_k + \delta}{\sqrt{\varepsilon_k + (r_k + \delta)^2}} &= \frac{\sqrt{\varepsilon_k + (r_k + \delta)^2} - \sqrt{(r_k + \delta)^2}}{\sqrt{\varepsilon_k + (r_k + \delta)^2}} \\ &= \frac{\frac{\varepsilon_k}{2\sqrt{(r_k+\delta)^2 + c_k}}}{\sqrt{\varepsilon_k + (r_k + \delta)^2}} \leq \frac{\varepsilon_k}{2(r_k + \delta)^2} \\ &< \frac{\delta}{2\sqrt{r_k + \delta}} \end{aligned}$$

for k large enough and some $c_k \in (0, \varepsilon_k)$. Similarly, one shows that the inequality

$$|H^*_{\varepsilon_{n_k}}(\chi_k) - H^*_0(\chi_k)| \geq \delta$$

cannot hold. $\square$

We now return to the proof of the theorem and introduce the functions $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon, \mathcal{D}_\varepsilon, \mathcal{D} \colon \mathbb{R}^6 \to \mathbb{R}^6$ as

$$\begin{aligned} &\mathcal{A}_\varepsilon(\gamma) = \mathcal{A}_\varepsilon(v, w) = (v, w - H^*_\varepsilon(w)), \\ &\mathcal{B}_\varepsilon(\gamma) = (H_\varepsilon(w), v), \\ &\mathcal{D}_\varepsilon(\gamma) = (0, H_\varepsilon(w)), \\ &\mathcal{D}(\gamma) = (0, H_0(w)). \end{aligned}$$

Note that when $\varepsilon = 0$ operators $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ reduce to the ones we defined earlier in (2.15). Let $\gamma_n = (v_n, w_n)$. Then, (5.5) and (5.6) may be rewritten as

$$(\mathcal{A}_{\varepsilon_n})_t(\gamma_n)+\varepsilon_n(\mathcal{D}_{\varepsilon_n})_t(\gamma_n)= \\ =(\mathcal{B}_{\varepsilon_n})_s(\gamma_n)+\varepsilon_n(\mathcal{A}_{\varepsilon_n})_{ss}(\gamma_n)+\varepsilon_n^2(\mathcal{D}_{\varepsilon_n})_{ss}(\gamma_n)+(g,0). \tag{5.8}$$

Moreover, by (5.5) and (5.6), the initial and boundary conditions (4.1c)–(4.1h), and the restriction $\alpha_n(1)=0$, we find that for any $\varphi=(\phi,\psi)\in\tilde{C}^\infty(\Omega)$, which was defined by (3.5), we have

$$\int_\Omega v_n\phi_t\,ds\,dt=\int_\Omega H_{\varepsilon_n}(w_n)\phi_s\,ds\,dt-\int_0^1\beta_n\phi|_{t=0}\,ds \\ -\int_\Omega g\phi\,ds\,dt-\varepsilon_n\int_\Omega v_n\phi_{ss}\,ds\,dt,$$

$$\int_\Omega [w_n-H^*_{\varepsilon_n}(w_n)+\varepsilon_n H_{\varepsilon_n}(w_n)]\psi_t\,ds\,dt \\ =\int_\Omega v_n\psi_s\,ds\,dt+\int_0^1\alpha_n\psi_s|_{t=0}\,ds-\varepsilon_n\int_\Omega[w_n-H^*_{\varepsilon_n}(w_n)+\varepsilon_n H_{\varepsilon_n}(w_n)]\psi_{ss}\,ds\,dt.$$

These can be merged to give

$$\int_\Omega \mathcal{A}_{\varepsilon_n}(\gamma_n)\varphi_t\,ds\,dt+\varepsilon_n\int_\Omega\mathcal{D}_{\varepsilon_n}(\gamma_n)\varphi_t\,ds\,dt= \\ =\int_\Omega\mathcal{B}_{\varepsilon_n}(\gamma_n)\varphi_s\,ds\,dt-\varepsilon_n\int_\Omega\mathcal{A}_{\varepsilon_n}(\gamma_n)\varphi_{ss}\,ds\,dt \\ -\varepsilon_n^2\int_\Omega\mathcal{D}_{\varepsilon_n}(\gamma_n)\varphi_{ss}\,ds\,dt+\Xi_0(\alpha_n,\beta_n,\varphi), \tag{5.9}$$

where we used the operator Ξ_0 defined in (3.11). Due to (5.1) and (5.4) we have

$$\|\gamma_n\|_{L^2(\Omega;\mathbb{R}^6)}\le C. \tag{5.10}$$

Observe that this constant merely depends on T, g, and the L^2-norms of α and β.

By Lemma 5.2 we obtain

$$\mathcal{A}_{\varepsilon_n}(\gamma)\to\mathcal{A}(\gamma), \tag{5.11}$$

$$\mathcal{B}_{\varepsilon_n}(\gamma)\to\mathcal{B}(\gamma), \tag{5.12}$$

$$\mathcal{D}_{\varepsilon_n}(\gamma)\to\mathcal{D}(\gamma), \tag{5.13}$$

uniformly in $\gamma\in\mathbb{R}^6$. From (5.9) we infer

$$
\begin{aligned}
&\int_\Omega \mathcal{A}(\gamma_n)\varphi_t\,ds\,dt - \int_\Omega \mathcal{B}(\gamma_n)\varphi_s\,ds\,dt - \Xi_0(\alpha,\beta,\varphi) = \\
&\quad = \int_\Omega [\mathcal{A}(\gamma_n) - \mathcal{A}_{\varepsilon_n}(\gamma_n)]\varphi_t\,ds\,dt + \int_\Omega [\mathcal{B}_{\varepsilon_n}(\gamma_n) - \mathcal{B}(\gamma_n)]\varphi_s\,ds\,dt \\
&\qquad - \varepsilon_n \int_\Omega [\mathcal{D}_{\varepsilon_n}(\gamma_n) - \mathcal{D}(\gamma_n)]\varphi_t\,ds\,dt \\
&\qquad - \varepsilon_n \int_\Omega [\mathcal{A}_{\varepsilon_n}(\gamma_n) - \mathcal{A}(\gamma_n)]\varphi_{ss}\,ds\,dt \\
&\qquad - \varepsilon_n^2 \int_\Omega [\mathcal{D}_{\varepsilon_n}(\gamma_n) - \mathcal{D}(\gamma_n)]\varphi_{ss}\,ds\,dt \\
&\qquad - \varepsilon_n \int_\Omega \mathcal{D}(\gamma_n)\varphi_t\,ds\,dt - \varepsilon_n \int_\Omega \mathcal{A}(\gamma_n)\varphi_{ss}\,ds\,dt \\
&\qquad - \varepsilon_n^2 \int_\Omega \mathcal{D}(\gamma_n)\varphi_{ss}\,ds\,dt + \Xi_0(\alpha_n - \alpha, \beta_n - \beta, \varphi).
\end{aligned}
\tag{5.14}
$$

The first five terms on the right-hand side tend to zero by (5.11)–(5.13). Since $\mathcal{A}$ and $\mathcal{D}$ are sublinear and subquadratic, respectively, (5.10) gives

$$
\begin{aligned}
\|\mathcal{A}(\gamma_n)\|_{L^2(\Omega;\mathbb{R}^6)} &\le C, \\
\|\mathcal{D}(\gamma_n)\|_{L^1(\Omega;\mathbb{R}^6)} &\le C.
\end{aligned}
$$

Recall that $\alpha_n \to \alpha$, $\beta_n \to \beta$ in $L^2(0,1;\mathbb{R}^3)$. Hence, we conclude that the remaining terms on the right-hand side of (5.14) go to zero. Consider the functions

$$
\begin{aligned}
\tilde{\mathcal{A}}(t,s,\xi) &= \mathcal{A}(t,s,\xi)\varphi_t(t,s), \\
\tilde{\mathcal{B}}(t,s,\xi) &= \mathcal{B}(t,s,\xi)\varphi_s(t,s).
\end{aligned}
$$

It is easy to see that $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are in the class $\mathcal{F}_2$ (with $\Gamma = \Omega$), $\tilde{\mathcal{A}}^\infty \equiv 0$, and $\tilde{\mathcal{B}}^\infty(t,s,\xi) = \mathcal{B}^\infty(\xi)\varphi_s(t,s)$. By Theorem 3.1, we can pass to the limit in (5.14) (passing to a subsequence, if necessary) and obtain

$$
\begin{aligned}
&\int_\Omega \langle \nu_{t,s}, \tilde{\mathcal{A}}(t,s,\xi)\rangle\,ds\,dt - \int_\Omega \langle \nu_{t,s}, \tilde{\mathcal{B}}(t,s,\xi)\rangle\,ds\,dt \\
&\qquad - \int_{\overline{\Omega}} \langle \nu^\infty_{t,s}, \tilde{\mathcal{B}}^\infty(t,s,\theta)\rangle\,\lambda(dt,ds) - \Xi_0(\alpha,\beta,\varphi) = 0,
\end{aligned}
\tag{5.15}
$$

which yields (3.17). Remark 3.2 and (5.10) imply (3.16). $\square$

5.2. Examples

Let us briefly examine the implications of Theorem 5.1 for some particular cases of chain dynamics with the "whip" boundary conditions and non-zero gravity g. In the case of the initial data (2.26), we get existence of a generalized solution which is a priori different from the stationary solution (2.25) plainly because the latter one does not admit non-negative tension. A qualitative glimpse at the auxiliary problems (4.1) and (4.4) implies that the "approximate strings" start to evolve close to the upright position (2.26) but eventually with the course of time they approach their steady-states. As ϵ goes to zero, these steady-states approach the downwards vertical orientation with

$$v(s) = 0, \ \kappa(s) = -gs. \tag{5.16}$$

Hence, our solution must be relevant in connection with the problem of falling of a chain which is initially in an upright position and then its upper end is released and the lower one remains fixed.

On the other hand, there are many physical and mechanical works dealing with a problem of falling of a chain which initially has two ends together and then one of them is released (see [42] for a review). In this case, the initial data are

$$\alpha(s) = g\left(\frac{1}{2|g|} - \left|\frac{s}{|g|} - \frac{1}{2|g|}\right|\right), \ \beta(s) = 0. \tag{5.17}$$

Although the compatibility condition (2.12) is violated for $s = \frac{1}{2}$, the hypothesis of Theorem 5.1 is met. Thus, the Young measure solution exists, providing a new framework for a correct description of this mechanical system.

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